Cauchy problem for operators with triple effectively hyperbolic characteristics
– Ivrii's conjecture –

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Abstract

Ivrii's conjecture asserts that the Cauchy problem is C^∞ well-posed for any lower order term if every critical point of the principal symbol is effectively hyperbolic. Effectively hyperbolic critical point is at most triple characteristic. If every characteristic is at most double this conjecture has been proved in 1980'. In this paper we prove the conjecture for the remaining cases, that is for operators with triple effectively hyperbolic characteristics.

Keywords: Ivrii's conjecture, Bézout matrix, triple characteristic, Tricomi type, effectively hyperbolic, Cauchy problem, Weyl calculus.

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1 Introduction

This paper is devoted to the Cauchy problem

$$(1.1)\qquad \begin{cases} Pu = D_t^m u + \sum_{j=0}^{m-1} \sum_{|\alpha|+j\le m} a_{j,\alpha}(t,x) D_x^\alpha D_t^j u = 0, \\ D_t^j u(0,x) = u_j(x), \quad j = 0,\dots,m-1 \end{cases}$$

where $t \ge 0$, $x \in \mathbb{R}^d$ and the coefficients $a_{j,\alpha}(t,x)$ are real valued C^∞ functions in a neighborhood of the origin of $\mathbb{R}^{1+d}$ and $D_x = (D_{x_1},\dots,D_{x_d})$, $D_{x_j} = (1/i)(\partial/\partial x_j)$ and $D_t = (1/i)(\partial/\partial t)$. The Cauchy problem (1.1) is C^∞ well-posed at the origin *for* $t \ge 0$ if one can find a $\delta > 0$ and a neighborhood U of the origin of $\mathbb{R}^d$ such that (1.1) has a unique solution $u \in C^\infty([0,\delta) \times U)$ for any $u_j(x) \in C^\infty(\mathbb{R}^d)$. We assume that the principal symbol of P

$$p(t,x,\tau,\xi) = \tau^m + \sum_{j=0}^{m-1} \sum_{|\alpha|+j=m} a_{j,\alpha}(t,x)\xi^\alpha \tau^j$$

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is hyperbolic *for* $t \geq 0$, that is there exist $\delta' > 0$ and a neighborhood U' of the origin such that

$$p = 0 \text{ has only real roots in } \tau \text{ for } (t,x) \in [0,\delta') \times U' \text{ and } \xi \in \mathbb{R}^d \tag{1.2}$$

which is indeed necessary in order that the Cauchy problem (1.1) is C^∞ well-posed near the origin for $t \geq 0$ ([17], [19]).

In [7], Ivrii and Petkov proved that if the Cauchy problem (1.1) is C^∞ well posed for any lower order term then the Hamilton map F_p has a pair of non-zero real eigenvalues at every critical point ([7, Theorem 3]). With $X = (t,x)$, $\Xi = (\tau,\xi)$ the Hamilton map F_p is defined by

$$F_p(X,\Xi) = \begin{bmatrix} \dfrac{\partial^2 p}{\partial X \partial \Xi} & \dfrac{\partial^2 p}{\partial \Xi \partial \Xi} \\ -\dfrac{\partial^2 p}{\partial X \partial X} & -\dfrac{\partial^2 p}{\partial \Xi \partial X} \end{bmatrix}$$

and a critical point (X,Ξ) is a point where $\partial p/\partial X = \partial p/\partial \Xi = 0$. Note that $p(X,\Xi) = 0$ at critical points by the homogeneity in Ξ so that (X,Ξ) is a multiple characteristic and τ is a multiple characteristic root of p. A critical point where the Hamilton map F_p has a pair of non-zero real eigenvalues is called *effectively hyperbolic* ([4], [10]). In [8], Ivrii has proved that if every critical point is effectively hyperbolic, and p admits a decomposition $p = q_1 q_2$ with real smooth symbols q_i near the critical point, then the Cauchy problem is C^∞ well-posed for every lower order term. In this case the critical point is effectively hyperbolic if and only if the Poisson bracket $\{q_1, q_2\}$ does not vanish there. He has conjectured that the assertion would hold without any additional condition.

If a critical point (X,Ξ) is effectively hyperbolic then τ is a characteristic root of multiplicity at most 3 ([7, Lemma 8.1]). If every multiple characteristic root is at most double, the conjecture has been proved in [8], [20], [11, 12, 13], [21, 23, 22]. When there exists an effectively hyperbolic critical point (X,Ξ) such that τ is a triple characteristic root where p cannot be factorized, several partial results are obtained in [2], [27], [28], [26]. Note that if there is a triple characteristic which is not effectively hyperbolic the Cauchy problem is not well-posed in the Gevrey class of order $s > 2$ in general, even the subprincipal symbol vanishes identically ([3]).

In this paper we prove

Theorem 1.1. *Assume* (1.2). *If every critical point* $(0,0,\tau,\xi)$, $\xi \neq 0$ *is effectively hyperbolic then for any* $a_{j,\alpha}(t,x)$ *with* $j + |\alpha| \leq m-1$, *which are* C^∞ *in a neighborhood of* $(0,0)$, *there exist* $\delta > 0$, *a neighborhood* U *of the origin and* $n > 0$ *such that for any* $s \in \mathbb{R}$ *and any* f *with* $t^{-n+1/2}\langle D\rangle^s f \in L^2((0,\delta) \times \mathbb{R}^d)$ *there exists* u *with* $t^{-n}\langle D\rangle^{-n-2+s+m-j} D_t^j u \in L^2((0,\delta) \times \mathbb{R}^d)$, $j = 0,1,\ldots,m-1$ *satisfying*

$$Pu = f \quad \text{in} \quad (0,\delta) \times U.$$

Here $\langle D\rangle$ stands for $\sqrt{1+|D_x|^2}$. For some more detailed information about the constant n, see (10.1) below.

Theorem 1.2. *Under the same assumption as in Theorem 1.1, for any $a_{j,\alpha}(t,x)$ with $j+|\alpha|\le m-1$, which are C^∞ in a neighborhood of $(0,0)$, there exist $\delta>0$ and a neighborhood U of the origin such that for any $u_j(x)\in C_0^\infty(\mathbb{R}^d)$, $j=0,1,\ldots,m-1$, there exists $u(t,x)\in C^\infty([0,\delta)\times U)$ satisfying* (1.1) *in $[0,\delta)\times U$. If $u(t,x)\in C^\infty([0,\delta)\times U)$ with $\partial_t^j u(0,x)=0$, $j=0,1,\ldots,m-1$, satisfies $Pu=0$ in $[0,\delta)\times U$ then $u=0$ in a neighborhood of* $(0,0)$.

Proof. Compute $u_j(x)=D_t^ju(0,x)$ for $j=m,m+1,\ldots$ from $u_j(x)$, $j=0,1,\ldots,m-1$ and the equation $Pu=0$. By a Borel's lemma there is $w(t,x)\in C_0^\infty(\mathbb{R}^{1+d})$ such that $D_t^jw(0,x)=u_j(x)$ for all $j\in\mathbb{N}$. Since $(D_t^jPw)(0,x)=0$ for all $j\in\mathbb{N}$ it is clear that $t^{-n+1/2}\langle D\rangle^sPw\in L^2((0,\delta)\times\mathbb{R}^d)$ for any s. Thanks to Theorem 1.1 there exists v with $t^{-n}\langle D\rangle^{-2n-2+s+m-j}D_t^jv\in L^2((0,\delta)\times\mathbb{R}^d)$, $j=0,1,\ldots,m-1$ satisfying $Pv=-Pw$ in $(0,\delta)\times U$. Since we may assume $n\ge 1$ hence $D_t^jv(0,x)=0$, $j=0,1,\ldots,m-1$ we conclude that $u=v+w$ is a desired solution. Local uniqueness follows from Theorem 13.4 because $\partial_t^ku(0,x)=0$ for any $k\in\mathbb{N}$ by $Pu=0$. □

Remark 1.1. Under the assumption of Theorem 1.1 we see that p has necessarily *non-real* characteristic roots in the $t<0$ side near $(0,0,\xi)$ if $p(0,0,\tau,\xi)=0$ has a triple characteristic root. Therefore P would be a *Tricomi type* operator in this case. In fact from [7, Lemma 8.1] it follows that if τ is a triple characteristic root at $(0,0,\xi)$ and all characteristic roots are real in a *full* neighborhood of $(0,0,\xi)$ then $F_p(0,0,\tau,\xi)=O$.

Remark 1.2. For any characteristic root τ of multiplicity $r\ge 3$ at (t,x,ξ) with $t\ge 0$ the point (t,x,τ,ξ) is a critical point, where $F_p(t,x,\tau,\xi)=O$ *unless* $r=3$ and $t=0$ ([7, Lemma 8.1]). For any double characteristic root τ at (t,x,ξ) with $t\ge 0$ the point (t,x,τ,ξ) is a critical point if $t>0$ while it is not necessarily critical point if $t=0$. Here is a simple example

$$P=(D_t^2-t^\ell D_x^2)(D_t+c\,D_x),\qquad \ell\in\mathbb{N},\quad x\in\mathbb{R},\quad t\ge 0$$

where $c\in\mathbb{R}$. Let $c\neq 0$ then it is clear that $\tau=0$ is a double characteristic root at $(0,0,1)$. If $\ell=1$ then $\partial_tp(0,0,0,1)=-c\neq 0$ and hence $(0,0,0,1)$ is not a critical point. If $\ell\ge 2$ then $(0,0,0,1)$ is a critical point and F_p has non-zero real eigenvalues there if and only if $\ell=2$. Let $c=0$ then $\tau=0$ is a triple characteristic root at $(0,0,1)$ hence $(0,0,0,1)$ is a critical point. At $(0,0,0,1)$, F_p has non-zero real eigenvalues if and only if $\ell=1$.

2 Outline of the proof of Theorem 1.1

As noted in Introduction, if a critical point (X,Ξ) is effectively hyperbolic then τ is a characteristic root of multiplicity at most 3. This implies that it is essential to study operators P of third order

$$P = D_t^3 + \sum_{j=1}^{3} a_j(t,x,D)\langle D\rangle^j D_t^{3-j} \tag{2.1}$$

which is differential operator in t with coefficients $a_j \in S^0$, classical pseudodifferential operator of order 0, where $\langle D\rangle = \mathrm{op}((1+|\xi|^2)^{1/2})$. One can reduce P to the case with $a_1(t,x,D)=0$ and hence the principal symbol is

$$p(t,x,\tau,\xi) = \tau^3 - a(t,x,\xi)|\xi|^2\tau - b(t,x,\xi)|\xi|^3. \tag{2.2}$$

All characteristic roots are real for $t \geq 0$ implies that

$$\Delta = 4\,a(t,x,\xi)^3 - 27\,b(t,x,\xi)^2 \geq 0, \quad (t,x,\xi) \in [0,T) \times U \times \mathbb{R}^d. \tag{2.3}$$

Assume that $p(0,0,\tau,\bar{\xi})=0$ has a triple characteristic root $\bar{\tau}$, which is necessarily $\bar{\tau}=0$. The critical point $(0,0,\bar{\tau},\bar{\xi})$ is effectively hyperbolic if and only if

$$\partial_t a(0,0,\bar{\xi}) \neq 0. \tag{2.4}$$

So we can assume that $a = e(t+\alpha(x,\xi))$ with $e>0$. Add to P a second order term $Me\langle D\rangle D_t$ with a large parameter $M>0$ which is irrelevant because eventually it is proved that any lower order term can be controlled. The coefficient $a(t,x,\xi)$ changes to $e(t+\alpha+M\langle\xi\rangle^{-1})$ which we still denote by the same a.

With $U = {}^t(D_t^2u, \langle D\rangle D_t u, \langle D\rangle^2 u)$ the equation $Pu=f$ is reduced to

$$D_t U = A(t,x,D)\langle D\rangle U + B(t,x,D)U + F \tag{2.5}$$

where $A, B \in S^0$, $F = {}^t(f,0,0)$ and

$$A(t,x,\xi) = \begin{bmatrix} 0 & a & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let S be the Bézout matrix of p and $\partial p/\partial\tau$, that is

$$S(t,x,\xi) = \begin{bmatrix} 3 & 0 & -a \\ 0 & 2a & 3b \\ -a & 3b & a^2 \end{bmatrix}$$

then S is nonnegative definite and symmetrizes A, that is SA is symmetric which is easily examined directly, though this is a special case of a general fact (see [14], [25]). We now diagonalize S by an orthogonal matrix T so that $T^{-1}ST = \Lambda = \mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3)$ where $0<\lambda_1<\lambda_2<\lambda_3$ are the eigenvalues of S which satisfies

$$\left|\partial_x^\alpha \partial_\xi^\beta \lambda_j\right| \precsim a^{3-j-|\alpha+\beta|/2}\langle\xi\rangle^{-|\beta|}, \quad j=1,2,3.$$

Then the equation is reduced to a 3×3 first order system of $V=T^{-1}U$; roughly

$$D_tV=A^T\langle D\rangle V+B^TV,\quad A^T=T^{-1}AT \tag{2.6}$$

where Λ symmetrizes A^T. A significant feature of λ_j is that

$$\frac{\Delta}{a}\precsim\lambda_1\precsim a^2,\quad \lambda_2\simeq a,\quad \lambda_3\simeq 1.$$

From the conditions (2.3) and (2.4) the discriminant Δ is essentially a third order polynomial in t and we can find a smooth $\psi(x,\xi)$ and $c>0$ such that

$$\frac{\Delta}{a}\geq c\min\big\{t^2,(t-\psi)^2+M\rho\langle\xi\rangle^{-1}\big\} \tag{2.7}$$

where $\rho=\alpha+M\langle\xi\rangle^{-1}$ and ψ satisfies

$$\big|\partial_x^\alpha\partial_\xi^\beta\psi\big|\precsim\rho^{1-|\alpha+\beta|/2}\langle\xi\rangle^{-|\beta|}.$$

Since (2.6) is a symmetrizable system with a diagonal symmetrizer Λ, a natural energy will be

$$\big(\mathrm{op}(\Lambda)V,V\big)=\sum_{j=1}^{3}\big(\mathrm{op}(\lambda_j)V_j,V_j\big)$$

and (2.7) suggests that a weighted energy with a scalar weight $\mathrm{op}(t^{-n}\phi^{-n})$

$$\phi=\omega+t-\psi,\quad \omega=\sqrt{(t-\psi)^2+M\rho\langle\xi\rangle^{-1}}$$

would work, which is essentially the same weight as the weight employed for studying double effectively hyperbolic characteristics in [23] (see also [24]), there $M\langle\xi\rangle^{-1}$ was used in place of $M\rho\langle\xi\rangle^{-1}$. A main feature of the weight function $t\phi$ is

$$\partial_t(t\phi)=\kappa\,(t\phi),\quad \kappa=\frac{1}{t}+\frac{1}{\omega}.$$

Our task is now to show the weighted energy

$$\mathsf{Re}\,e^{-\theta t}(\mathrm{op}(\Lambda)\mathrm{op}(t^{-n}\phi^{-n})V,\mathrm{op}(t^{-n}\phi^{-n})V)$$

works well and can control any lower order term, yielding weighted energy estimates for P. In doing so it is crucial that λ_j, ω, ρ and ϕ are admissible weights for the metric

$$g=M^{-1}\big(\langle\xi\rangle|dx|^2+\langle\xi\rangle^{-1}|d\xi|^2\big)$$

and $\lambda_j\in S(\lambda_j,g)$, $\phi\in S(\phi,g)$ so on. This fact enables us to apply the Weyl calculus to $\mathrm{op}(\lambda_j)$, $\mathrm{op}(\phi^{-n})$ and so on. One of main points to derive energy estimates is the following inequalities

$$\begin{aligned}\mathsf{Re}\,(\mathrm{op}(\Lambda)\mathrm{op}(\partial_t(t^{-n}\phi^{-n}))V,W)-\theta\,(\mathrm{op}(\Lambda)W,W)&\\ \leq -n(1-CM^{-1})\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2&\\ -c\,\theta\big(\|\mathrm{op}(\Lambda^{1/2})W\|^2+\sum_{j=1}^{3}M^{3-j}\|\langle D\rangle^{-(3-j)/2}W_j\|^2\big)&\end{aligned}$$

where $W = \mathrm{op}(t^{-n}\phi^{-n})V$, while for $B = (b_{ij})$ with $b_{ij} \in S(1, g)$ we see

$$\big|(\mathrm{op}(\Lambda)\mathrm{op}(B)W, W)\big| \le C\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2$$
$$+C\big(\|\mathrm{op}(\Lambda^{1/2})W\|^2 + \sum_{j=1}^{3} M^{3-j}\|\langle D\rangle^{-(3-j)/2}W_j\|^2\big)$$

which are proved applying the Weyl calculus of pseudodifferential operators.

3 Lower bound of discriminant

Study third order operators P of the form (2.1) with $a_1(t, x, D) = 0$, hence the principal symbol has the form (2.2) where $a(t, x, \xi)$ and $b(t, x, \xi)$ are homogeneous of degree 0 in ξ and assumed to satisfy (2.3) with some $T > 0$ and some neighborhood U of the origin of $\mathbb{R}^d$. Assume that $p(t, x, \tau, \xi)$ has a triple characteristic root $\bar{\tau}$ at $(0, 0, \bar{\xi})$, $|\bar{\xi}| = 1$ and $(0, 0, \bar{\tau}, \bar{\xi})$ is effectively hyperbolic. It is clear that

$$\bar{\tau} = 0, \quad a(0, 0, \bar{\xi}) = 0, \quad b(0, 0, \bar{\xi}) = 0.$$

Since $\partial_x^\alpha \partial_\xi^\beta a(0, 0, \bar{\xi}) = 0$ for $|\alpha + \beta| = 1$ and $\partial_x^\alpha \partial_\xi^\beta b(0, 0, \bar{\xi}) = 0$ for $|\alpha + \beta| \le 2$ by (2.3) (see Lemma 4.2 below) it is easy to see

$$\det\big(\lambda - F_p(0, 0, 0, \bar{\xi})\big) = \lambda^{2d}\big(\lambda^2 - \{\partial_t a(0, 0, \bar{\xi})\}^2\big) \tag{3.1}$$

hence $(0, 0, 0, \bar{\xi})$ is effectively hyperbolic if and only if

$$\partial_t a(0, 0, \bar{\xi}) \neq 0.$$

Since $a(0, 0, \bar{\xi}) = 0$ and $\partial_t a(0, 0, \bar{\xi}) \neq 0$ there is a neighborhood $\mathcal{U}$ of $(0, 0, \bar{\xi})$ in which one can write

$$a(t, x, \xi) = e(t, x, \xi)(t + \alpha(x, \xi))$$

where $e > 0$ in $\mathcal{U}$. Note that $\alpha(x, \xi) \ge 0$ near $\bar{\xi}$ because $a(t, x, \xi) \ge 0$ in $[0, T) \times U \times \mathbb{R}^d$.

3.1 A perturbed discriminant

Introducing a small parameter ϵ we consider

$$\begin{aligned} &\tau^3 - e(t, x, \xi)(t + \alpha(x, \xi) + \epsilon^2)|\xi|^2\tau - b(t, x, \xi)\,|\xi|^3 \\ &\qquad = \tau^3 - a(t, x, \xi, \epsilon)|\xi|^2 - b(t, x, \xi)|\xi|^3. \end{aligned} \tag{3.2}$$

From now on we write $b(X)$ or $a(X, \epsilon)$ and so on to make clearer that these symbols are defined in some neighborhood of $\bar{X} = (0, \bar{\xi})$ or $(\bar{X}, 0)$. Consider the discriminant of (3.2);

$$\Delta(t, X, \epsilon) = 4\,a^3(t, X, \epsilon) - 27\,b^2(t, X).$$

Lemma 3.1. *One can write*

$$\Delta = \tilde{e}(t, X, \epsilon)\big(t^3 + a_1(X, \epsilon)t^2 + a_2(X, \epsilon)t + a_3(X, \epsilon)\big)$$

in a neighborhood of $(0, \bar{X}, 0)$ *where* $a_j(\bar{X}, 0) = 0$, $j = 1, 2, 3$ *and* $\tilde{e} > 0$.

Proof. It is clear that $\partial_t^k a^3(0, \bar{X}, 0) = 0$ for $k = 0, 1, 2$ and $\partial_t^3 a^3(0, \bar{X}, 0) \neq 0$. Show $\partial_t b(0, \bar{X}, 0) = 0$. Suppose the contrary and hence

$$b(t, \bar{X}, 0) = t\big(b_1 + tb_2(t)\big)$$

with $b_1 \neq 0$. Since $a(t, \bar{X}, 0) = c\,t$ with $c > 0$ then $\Delta(t, \bar{X}, 0) = 4\,c^3\,t^3 - 27\,b(t, \bar{X}, 0)^2 \geq 0$ leads to a contradiction. Thus $\partial_t^k \Delta(0, \bar{X}, 0) = 0$ for $k = 0, 1, 2$ and $\partial_t^3 \Delta(0, \bar{X}, 0) \neq 0$. Then from the Malgrange preparation theorem (e.g. [5, Theorem 7.5.5]) one can conclude the assertion. □

Introducing

$$\rho(X, \epsilon) = \alpha(X) + \epsilon^2 \tag{3.3}$$

one can also write

$$\Delta = 4e^3(t + \rho)^3 - 27b^2 = 4e^3\Big\{(t + \rho)^3 - \frac{27}{4e^3}b^2\Big\} = 4e^3\big\{(t + \rho)^3 - \hat{b}^2\big\}$$

with $\hat{b} = 3\sqrt{3}\,b/2e^{3/2}$. Denoting

$$\hat{b}(t, X) = \sum_{j=0}^{2} \hat{b}_j(X)t^j + \hat{b}_3(t, X)t^3$$

where $\hat{b}_0(\bar{X}) = \hat{b}_1(\bar{X}) = 0$ which follows from the proof of Lemma 3.1, one can write

$$\begin{aligned} \Delta/\tilde{e} = \bar{\Delta} &= t^3 + a_1(X, \epsilon)t^2 + a_2(X, \epsilon)t + a_3(X, \epsilon) \\ &= E\,\Big\{(t + \rho)^3 - \big(\sum_{j=0}^{2} \hat{b}_j(X)t^j + \hat{b}_3(t, X)t^3\big)^2\Big\} \end{aligned} \tag{3.4}$$

with $E(t, X, \epsilon) = 4e^3/\tilde{e}$. Here note that $E(0, \bar{X}, 0) = 1$ since $e(0, \bar{X}, 0) = \partial_t a(0, \bar{X}, 0)$ and $\tilde{e}(0, \bar{X}, 0) = 4\partial_t a(0, \bar{X}, 0)^3$.

Lemma 3.2. *There is a neighborhood* V *of* $\bar{X}$ *such that*

$$\big|\hat{b}_1(X)\big| \leq 4\alpha^{1/2}(X) \quad (X \in V).$$

Proof. It is clear that $|\hat{b}_0(X)| \leq \alpha^{3/2}(X)$. If $\alpha(X) = 0$ then the assertion is obvious. Assume $\alpha(X) \neq 0$. Since

$$(t + \alpha(X))^3 \geq \big(\sum_{j=0}^{2} \hat{b}_j(X)t^j + \hat{b}_3(t, X)t^3\big)^2 \quad (0 \leq t \leq T) \tag{3.5}$$

choosing $t = 3\alpha(X) \le T$, writing $\alpha = \alpha(X)$, it follows from (3.5) that

$$8\alpha^{3/2} \ge \big|\hat{b}_0(X) + 3\hat{b}_1(X)\alpha\big| - C\alpha^2 \ge 3|\hat{b}_1(X)|\alpha - C\alpha^2 - \alpha^{3/2}$$

hence the assertion is clear because $\alpha(\bar{X}) = 0$. □

Lemma 3.3. *In a neighborhood of* $(\bar{X}, 0)$ *we have* $a_j(X,\epsilon) = O\big(\rho(X,\epsilon)^j\big)$ *for* $j = 1, 2, 3$. *More precisely*

$$\begin{aligned} a_1(X,\epsilon) &= E(0,X,\epsilon)\big(3\rho(X,\epsilon) - \hat{b}_1^2(X)\big) + O(\rho^{3/2}),\\ a_2(X,\epsilon) &= E(0,X,\epsilon)\big(3\rho^2(X,\epsilon) - 2\hat{b}_0(X)\hat{b}_1(X)\big) + O(\rho^{3/2}),\\ a_3(X,\epsilon) &= E(0,X,\epsilon)\big(\rho^3(X,\epsilon) - \hat{b}_0^2(X)\big). \end{aligned}$$

Proof. Since $\bar{\Delta}(0,X,\epsilon) \ge 0$ it follows from (3.4) that $a_3(X,\epsilon) = E(0,X,\epsilon)\big(\rho(X,\epsilon)^3 - \hat{b}_0(X)^2\big) \ge 0$ hence $\hat{b}_0 = O(\rho^{3/2})$ and consequently $a_3(X,\epsilon) = O(\rho^3)$. Since

$$\begin{aligned} \partial_t\bar{\Delta}\Big|_{t=0} = a_2(X,\epsilon) &= \partial_t E(0,X,\epsilon)a_3(X,\epsilon)\\ &+E(0,X,\epsilon)\big(3\rho^2(X,\epsilon) - 2\hat{b}_0(X)\hat{b}_1(X)\big) \end{aligned}$$

it follows that $\hat{b}_0(X)\hat{b}_1(X) = O(\rho^2)$ by Lemma 3.2 and hence the above equality shows the assertion for $a_2(X,\epsilon)$. Finally from

$$\begin{aligned} \partial_t^2\bar{\Delta}\Big|_{t=0} = 2a_1(X,\epsilon) &= \partial_t^2 E(0,X,\epsilon)a_3(X,\epsilon)\\ &+2\partial_t E(0,X,\epsilon)\big(3\rho^2(X,\epsilon) - 2\hat{b}_0(X)\hat{b}_1(X)\big)\\ &+2E(0,X,\epsilon)\big(3\rho(X,\epsilon) - \hat{b}_1(X)^2 - 2\hat{b}_0(X)\hat{b}_2(X)\big) \end{aligned}$$

and Lemma 3.2 one concludes the assertion for $a_1(X,\epsilon)$. □

3.2 Construction of $\psi(x,\xi)$

Denote

$$\nu(X,\epsilon) = \inf\{t \mid \bar{\Delta}(t,X,\epsilon) > 0\} \tag{3.6}$$

and hence $\bar{\Delta}(\nu,X,\epsilon) = 0$. First check that $\nu(X,\epsilon) \le 0$. Suppose the contrary. Since $\bar{\Delta}(t,X,\epsilon) \ge 0$ for $t \ge 0$ it follows that $\nu(X,\epsilon)$ is a double root, that is one can write $\bar{\Delta}(t) = (t-\nu)^2(t-\tilde{\nu})$ with a real $\tilde{\nu}$. It is clear that $\tilde{\nu} \ne \nu$ and $\tilde{\nu} \le 0$ because $\bar{\Delta}(t) \ge 0$ for $t \ge 0$. Therefore we have $\tilde{\nu} < \nu$ and $\bar{\Delta}(t) > 0$ in $\tilde{\nu} < t < \nu$ which is incompatible with the definition of ν. Write

$$\bar{\Delta}(t,X,\epsilon) = (t - \nu(X,\epsilon))\big(t^2 + A_1(X,\epsilon)t + A_2(X,\epsilon)\big)$$

where $A_1 = \nu + a_1$. Here we prepare following lemma.

Lemma 3.4. *One can find a neighborhood* $\mathcal{U}$ *of* $(\bar{X}, 0)$ *such that for any* $(X, \epsilon) \in \mathcal{U}$ *there is* $j \in \{1, 2, 3\}$ *such that*

$$|\nu_j(X, \epsilon)| \geq \rho(X, \epsilon)/9$$

where $\bar{\Delta}(t, X, \epsilon) = \prod_{j=1}^{3}(t - \nu_j(X, \epsilon))$.

Proof. First show that there is $1/3 < \delta < 1/2$ such that

$$\max\Big\{|\rho^3 - \hat{b}_0^2|^{1/3},\ |\rho^2 - 2\hat{b}_0\hat{b}_1/3|^{1/2},\ |\rho - \hat{b}_1^2/3|\Big\} \geq \delta^2\rho. \tag{3.7}$$

In fact denoting $f(\delta) = 2(1 - \delta^6)^{1/2}(1 - \delta^2)^{1/2}/\sqrt{3} - 1 - \delta^4$ it is easy to check that $f(1/3) > 0$ and $f(1/2) < 0$. Take $1/3 < \delta < 1/2$ such that $f(\delta) = 0$. If $|\rho^3 - \hat{b}_0^2|^{1/2} < \delta^2\rho$ and $|\rho - \hat{b}_1^2/3| < \delta^2\rho$ then $|\hat{b}_0| \geq (1 - \delta^6)^{1/2}\rho^{3/2}$ and $|\hat{b}_1| \geq \sqrt{3}(1 - \delta^2)^{1/2}\rho^{1/2}$ hence

$$|\rho^2 - 2\hat{b}_0\hat{b}_1/3| \geq 2|\hat{b}_0\hat{b}_1|/3 - \rho^2 \geq \big(f(\delta) + \delta^4\big)\rho^2 = \delta^4\rho^2$$

which shows that $|\rho^2 - 2\hat{b}_0\hat{b}_1/3|^{1/2} \geq \delta^2\rho$. Thus (3.7) is proved. Thanks to Lemma 3.3, taking $E(0, \bar{X}, 0) = 1$ and $1/3 < \delta$, one can find a neighborhood $\mathcal{U}$ of $(\bar{X}, 0)$ such that

$$|a_1(X, \epsilon)| \geq \rho/3, \quad |a_2(X, \epsilon)| \geq \rho^2/3^3, \quad |a_3(X, \epsilon)| \geq \rho^3/3^6, \quad (X, \epsilon) \in \mathcal{U}.$$

Then the assertion follows from the relations between $\{\nu_i\}$ and $\{a_i\}$. □

Lemma 3.5. *Denote* ν *defined in* (3.6) *by* ν_1 *and by* ν_j, $j = 2, 3$ *the other roots of* $\bar{\Delta} = 0$ *in* t. *Then one can find a neighborhood* $\mathcal{U}$ *of* $(\bar{X}, 0)$ *and* $c_i > 0$ *such that*

$$\text{if} \quad \nu_1 + a_1 < 2\,c_1\,\rho, \;\; (X, \epsilon) \in \mathcal{U} \quad \text{then} \quad |\nu_1 - \nu_j| \geq c_2\,\rho, \;\; j = 2, 3. \tag{3.8}$$

In particular $\nu_1(X, \epsilon)$ *is smooth in* $(X, \epsilon) \in \mathcal{U} \cap \{\nu_1 + a_1 < 2\,c_1\rho\}$.

Proof. Set $\delta = 1/9$ and take $c_1 < \delta/4$. First note that if $\mathsf{Re}\,\nu_j \geq c_1\delta$, $j = 2, 3$ it is clear that $|\nu_1 - \nu_j| \geq |\nu_1 - \mathsf{Re}\,\nu_j| \geq \mathsf{Re}\,\nu_j \geq c_1\delta$ because $\nu_1 \leq 0$ then we may assume

$$\mathsf{Re}\,\nu_j < c_1\delta, \quad j = 2, 3. \tag{3.9}$$

Write

$$\bar{\Delta}(t) = \prod_{j=1}^{3}(t - \nu_j) = (t - \nu_1)\big((t + A_1/2)^2 - D\big)$$

and recall $\nu_1 + a_1 = A_1$. Consider the case that both ν_2, ν_3 are real so that $D \geq 0$ and $\nu_2, \nu_3 = -A_1/2 \pm \sqrt{D}$. If $D = 0$ and hence

$$-c_1\delta < \mathsf{Re}\,\nu_j = -A_1/2 < c_1\delta$$

in view of (3.8) and (3.9). Then we see $|\nu_1| \geq \delta\,\rho$ thanks to Lemma 3.4 and hence

$$|\nu_1 - \nu_j| \geq |\nu_1| - |\nu_j| \geq (\delta - c_1)\rho \geq 3\delta\rho/4.$$

If $D > 0$ then one has $-A_1/2 + \sqrt{D} \leq 0$. Otherwise $\bar{\Delta}(t)$ would be negative for some $t > 0$ near $-A_1/2 + \sqrt{D}$ which is a contradiction. Thus $\sqrt{D} \leq A_1/2 \leq c_1\,\delta$ which shows that

$$|\nu_2|,\ |\nu_3| \leq |A_1|/2 + \sqrt{D} \leq 2c_1\,\rho$$

and hence $|\nu_1| \geq \delta\,\rho$ by Lemma 3.4 again. Therefore

$$|\nu_1 - \nu_j| \geq |\nu_1| - |\nu_j| \geq (\delta - 2c_1)\,\rho \geq \delta\rho/2.$$

Turn to the case $D < 0$ such that $\nu_2,\,\nu_3 = -A_1/2 \pm i\sqrt{|D|}$. As observed above one may assume $|\mathsf{Re}\,\nu_j| = |A_1/2| < c_1\delta$. Thanks to Lemma 3.4 either $|\nu_1| \geq \delta\rho$ or $|\nu_2| = |\nu_3| \geq \delta\rho$. If $|\nu_1| \geq \delta\rho$ then it follows that

$$|\nu_1 - \nu_j| \geq |\nu_1 + A_1/2| \geq |\nu_1| - |A_1|/2 \geq (\delta - c_1)\rho \geq 3\delta\rho/4.$$

If $|\nu_2| = |\nu_3| \geq \delta\rho$ so that $|A_1|/2 + \sqrt{|D|} \geq \delta\rho$ hence $\sqrt{|D|} \geq \delta\rho - |A_1|/2 \geq (\delta - c_1)\rho$ which proves

$$|\nu_1 - \nu_j| \geq \sqrt{|D|} \geq (\delta - c_1)\rho \geq \delta\rho/2.$$

Thus $\nu_1(X,\epsilon)$ is a simple root and hence smooth provided that $\nu_1 + a_1 < c_1\rho$. □

Now define $\psi(X,\epsilon)$ which plays a crucial role in our arguments deriving weighted energy estimates. Choose $\chi(s) \in C^\infty(\mathbb{R})$ such that $0 \leq \chi(s) \leq 1$ with $\chi(s) = 1$ if $s \leq 0$ and $\chi(s) = 0$ for $s \geq 1$. Define

$$\psi(X,\epsilon) = -\chi\Big(\frac{\nu_1 + a_1}{2c_1\rho}\Big)\frac{\nu_1 + a_1}{2} \quad (\epsilon \neq 0).$$

We now prove

Proposition 3.1. *One can find a neighborhood $\mathcal{U}$ of $(\bar{X}, 0)$ such that*

$$\bar{\Delta}(t,X,\epsilon) \geq \upsilon \min\big\{t^2, (t - \psi(X,\epsilon))^2\big\}\,(t + \rho(X,\epsilon)) \tag{3.10}$$

holds for $(X,\epsilon) \in \mathcal{U}$, $\epsilon \neq 0$ and $t \in [0,T]$ where $\upsilon = \big(2(18\sqrt{2}+1)\big)^{-1}$.

Proof. Set $\delta = 1/9$ again. First check that one can find $c \geq \upsilon$ such that

$$\bar{\Delta}(t,X,\epsilon) \geq c\,t^2(t+\rho) \quad \text{if} \quad A_1 = \nu_1 + a_1 \geq 0. \tag{3.11}$$

Write $\bar{\Delta}(t) = (t - \nu_1)\big((t + A_1/2)^2 - D\big)$. Consider the case $D = 0$. From Lemma 3.4 either $|\nu_1| \geq \delta\rho$ or $|A_1/2| = A_1/2 \geq \delta\rho$. If $|\nu_1| \geq \delta\rho$ then $t - \nu_1 = t + |\nu_1| \geq t + \delta\rho$ hence $\delta^{-1}(t - \nu_1) \geq t + \rho$. Since $(t + A_1/2)^2 \geq t^2$ it is clear that (3.11) holds with $c = \delta$. If $A_1/2 \geq \delta\rho$ then $t + A_1/2 \geq t + \delta\rho$ and $t + A_1/2 \geq t$,

$t-\nu_1=t+|\nu_1|\geq t$ gives (3.11) with $c=\delta$. Next consider the case $D>0$. Since $\bar\Delta(t)\geq 0$ for $t\geq 0$ if follows that $-A_1/2+\sqrt{D}\leq 0$. Write

$$\bar\Delta(t)=(t-\nu_1)(t-\nu_2)(t-\nu_3)$$

where $\nu_2,\nu_3=-A_1/2\pm\sqrt{D}\leq 0$. If $|\nu_1|\geq\delta\rho$ then $\delta^{-1}(t-\nu_1)\geq t+\rho$ as above and $t-\nu_i=t+|\nu_i|\geq t$ then (3.11) with $c=\delta$. Consider the case $D<0$ so that $\nu_2,\nu_3=-A_1/2\pm i\sqrt{|D|}$. If $|\nu_1|\geq\delta\rho$ then (3.11) holds because $|t-\nu_i|\geq|t+A_1/2|\geq t$. If $|\nu_2|=|\nu_3|\geq\delta\rho$ then $A_1/2+\sqrt{|D|}\geq\delta\rho$. Since

$$\begin{aligned}(t-\nu_2)(t-\nu_3)=(t+A_1/2)^2+|D|&\geq\big(t+A_1/2+\sqrt{|D|}\,\big)^2/2\\&\geq(t+\delta\rho)^2/2\geq\delta\,t\,(t+\rho)/2\end{aligned}$$

(3.11) holds with $c=\delta/2$.

Turn to the case $A_1<0$. In this case, using $\psi=-(\nu_1+a_1)/2>0$, one can write

$$\bar\Delta(t)=(t-\nu_1)\big((t-\psi)^2-D\big).$$

Consider the case $|\nu_1|\geq\delta\rho$. Note that $D\leq 0$ otherwise $\psi+\sqrt{D}>0$ would be a positive simple root of $\bar\Delta(t)$ and a contradiction. Then

$$(t-\psi)^2-D=(t-\psi)^2+|D|\geq(t-\psi)^2.$$

Recalling $t-\nu_1=t+|\nu_1|\geq\delta(t+\rho)$ we get

$$\bar\Delta(t,X,\epsilon)\geq c\,(t-\psi)^2(t+\rho) \tag{3.12}$$

with $c=\delta$. Consider the case $|\nu_2|=|\nu_3|=\big|\psi\pm i\sqrt{|D|}\,\big|=\sqrt{\psi^2+|D|}\geq\delta\rho$ so that

$$(t-\nu_2)(t-\nu_3)=(t-\psi)^2+|D|\geq\big(|t-\psi|+\sqrt{|D|}\,\big)^2/2.$$

Assume $\psi\geq\sqrt{|D|}$ so that $\sqrt{2}\,\psi\geq\delta\rho$. For $0\leq t\leq\psi/2$ hence $t\leq|t-\psi|$ and $\psi/2\leq|t-\psi|$ one has

$$\begin{aligned}|t-\psi|=(1-\gamma)|t-\psi|+\gamma|t-\psi|&\geq(1-\gamma)t+\gamma\psi/2\\&\geq(1-\gamma)t+(\gamma\delta/2\sqrt{2})\rho\geq\delta(2\sqrt{2}+\delta)^{-1}(t+\rho)\end{aligned}$$

with $\gamma=2\sqrt{2}/(2\sqrt{2}+\delta)$. Since $|t-\psi|+\sqrt{|D|}\geq|t-\psi|\geq t$ and $|t-\nu_1|=t+|\nu_1|\geq t$ it is clear that (3.11) holds with $c=\delta/2(2\sqrt{2}+\delta)$. For $\psi/2\leq t$ such that $|t-\psi|\leq t$ one sees

$$t-\nu_1\geq t=(1-\gamma)t+\gamma t\geq(1-\gamma)t+\gamma\psi/2\geq\delta(2\sqrt{2}+\delta)^{-1}(t+\rho)$$

and hence $(t-\nu_1)\big((t-\psi)^2+|D|\big)\geq c\,(t+\rho)(t-\psi)^2$ which is (3.12) with $c=\delta/2(2\sqrt{2}+\delta)$. Next assume $\sqrt{|D|}\geq\psi$ so that $\sqrt{2}\sqrt{|D|}\geq\delta\rho$. For $0\leq t\leq\psi/2$ one has $|t-\psi|\geq t$ and hence

$$|t-\psi|+\sqrt{|D|}\geq t+\delta\rho/\sqrt{2}\geq(\delta/\sqrt{2})(t+\rho).$$

Noting $|t-\nu_1| = t + |\nu_1| \geq t$ it is clear that (3.11) holds with $c = \delta/2\sqrt{2}$. For $\psi/2 \leq t$ we see that

$$|t-\psi| + \sqrt{|D|} \geq t - |\psi| + \sqrt{|D|} \geq t, \quad |t-\psi| + \sqrt{|D|} \geq \sqrt{|D|} \geq \delta\rho/\sqrt{2}$$

which shows that $|t-\psi| + \sqrt{|D|} \geq \delta(\sqrt{2}+\delta)^{-1}(t+\rho)$. Recalling $|t-\nu_1| = t + |\nu_1| \geq t$ again one has (3.11) with $c = \delta/2(\sqrt{2}+\delta)$. Thus the proof is completed. □

Lemma 3.6. *One can find a neighborhood $\mathcal{U}$ of $(\bar{X}, 0)$ and $\epsilon_0 > 0$, $C^* > 0$ such that*

$$\frac{|\partial_t \Delta(t,X,\epsilon)|}{\Delta(t,X,\epsilon)} \leq C^*\Big(\frac{1}{t} + \frac{1}{|t-\psi| + \sqrt{a}\,\epsilon}\Big), \quad (X,\epsilon) \in \mathcal{U} \tag{3.13}$$

holds for $t \in (0, T]$ and $0 < \epsilon \leq \epsilon_0$.

Proof. It will suffice to show (3.13) for $\Delta(t, X, \sqrt{2}\epsilon)$ which we denote by $\tilde{\Delta}(t, X, \epsilon)$. It is clear that

$$\tilde{\Delta} = \Delta + 4e^3\big(3(t+\rho)^2\epsilon^2 + 3(t+\rho)\epsilon^4 + \epsilon^6\big) = \Delta + \Delta_r.$$

Writing $\tilde{\Delta} = \tilde{e}\big(\bar{\Delta} + \bar{\Delta}_r\big)$ it suffices to show the assertion for $\bar{\Delta} + \bar{\Delta}_r$ instead of $\tilde{\Delta}$. Note that

$$\frac{|\partial_t \bar{\Delta}_r|}{\bar{\Delta}_r} \leq C\big(1 + \frac{1}{t+\rho}\big) \leq C'\frac{1}{t} \tag{3.14}$$

always holds. Write $\bar{\Delta} = (t-\nu_1)(t-\nu_2)(t-\nu_3)$ and note that

$$\frac{\partial_t \bar{\Delta}}{\bar{\Delta}} = \sum_{j=1}^{3} \frac{1}{t-\nu_j}.$$

Checking the proof of Proposition 3.1 it is easy to see that

$$\big|\partial_t \bar{\Delta}/\bar{\Delta}\big| \leq C/t$$

when $A_1 \geq 0$. Therefore

$$\frac{|\partial_t \tilde{\Delta}|}{\tilde{\Delta}} \leq \frac{|\partial_t \bar{\Delta}|}{\bar{\Delta} + \bar{\Delta}_r} + \frac{|\partial_t \bar{\Delta}_r|}{\bar{\Delta} + \bar{\Delta}_r} \leq \frac{|\partial_t \bar{\Delta}|}{\bar{\Delta}} + \frac{|\partial_t \bar{\Delta}_r|}{\bar{\Delta}_r}$$

proves the assertion. Study the case that $A_1 < 0$. From the proof of Proposition 3.1 one can write

$$\bar{\Delta} = (t-\nu_1)\big((t-\psi)^2 - D\big)$$

where $\psi > 0$ and $D \leq 0$. If $|D| \geq a\epsilon^2$ then

$$\frac{|t-\psi|}{(t-\psi)^2 + |D|} \leq \frac{1}{((t-\psi)^2 + a\epsilon^2)^{1/2}} \leq \frac{\sqrt{2}}{|t-\psi| + \sqrt{a}\,\epsilon}$$

which proves the assertion since $|t-\nu_1| = t+|\nu_1| \geq t$. Similarly if $|t-\psi| \geq \sqrt{a}\,\epsilon$ one has

$$\frac{|t-\psi|}{(t-\psi)^2+|D|} \leq \frac{2}{|t-\psi|+\sqrt{a}\,\epsilon}.$$

If $|D| < a\epsilon^2$ and $|t-\psi| < \sqrt{a}\,\epsilon$ it follows that

$$|\partial_t\bar{\Delta}| \leq (t-\psi)^2 + |D| + 2|t-\nu_1||t-\psi| \leq 2a\epsilon^2 + Ca^{3/2}\epsilon$$

because $|t-\nu_1| \leq Ca$. In view of $C\bar{\Delta}_r \geq a^2\epsilon^2$ one concludes that

$$\frac{|\partial_t\bar{\Delta}|}{\bar{\Delta}+\bar{\Delta}_r} \leq \frac{|\partial_t\bar{\Delta}|}{\bar{\Delta}_r} \leq C\,\frac{2a\epsilon^2+Ca^{3/2}\epsilon}{a^2\epsilon^2}$$
$$\leq C\Big(\frac{1}{a}+\frac{1}{\sqrt{a}\,\epsilon}\Big) \leq C'\Big(\frac{1}{t}+\frac{1}{|t-\psi|+\sqrt{a}\,\epsilon}\Big)$$

which together with (3.14) proves the assertion. □

4 Extension of symbols

In the preceding Sections 3.1 and 3.2 all symbols we have studied are defined in some conic (in ξ) neighborhood of $(X,\epsilon) = (\bar{X},0)$ or $X = \bar{X}$. In this section we extend such symbols to those on $\mathbb{R}^d \times \mathbb{R}^d$ following [23] (also [24]).

4.1 Extension of symbols

Let $\bar{X} = (0,\bar{\xi})$ with $|\bar{\xi}| = 1$. Let $\chi(s) \in C^\infty(\mathbb{R})$ be equal to 1 in $|s| \leq 1$, vanishes in $|s| \geq 2$ such that $0 \leq \chi(s) \leq 1$. Define $y(x)$ and $\eta(\xi)$ by

$$y_j(x) = \chi(M^2 x_j)x_j, \quad \eta_j(\xi) = \chi(M^2(\xi_j\langle\xi\rangle_\gamma^{-1} - \bar{\xi}_j))(\xi_j - \bar{\xi}_j\langle\xi\rangle_\gamma) + \bar{\xi}_j\langle\xi\rangle_\gamma$$

for $j = 1,2,\ldots,d$ with

$$\langle\xi\rangle_\gamma = (\gamma^2+|\xi|^2)^{1/2}$$

where M and γ are large positive parameters constrained

$$\gamma \geq M^5. \tag{4.1}$$

It is easy to see that $(1-CM^{-2})\langle\xi\rangle_\gamma \leq |\eta| \leq (1+CM^{-2})\langle\xi\rangle_\gamma$ and

$$|y| \leq CM^{-2}, \quad |\eta/|\eta| - \bar{\xi}| \leq CM^{-2} \tag{4.2}$$

with some $C > 0$ so that (y,η) is contained in a conic neighborhood of $(0,\bar{\xi})$, shrinking with M. Note that $(y,\eta) = (x,\xi)$ on the conic neighborhood of $(0,\bar{\xi})$

$$W_M = \big\{(x,\xi) \mid |x| \leq M^{-2},\ |\xi_j/|\xi| - \bar{\xi}_j| \leq M^{-2}/2,\ |\xi| \geq \gamma M\big\} \tag{4.3}$$

since

$$\Big|\frac{\xi_j}{\langle\xi\rangle_\gamma}-\bar{\xi}_j\Big|\le\Big|\frac{\xi_j}{\langle\xi\rangle_\gamma}-\frac{\xi_j}{|\xi|}\Big|+\Big|\frac{\xi_j}{|\xi|}-\bar{\xi}_j\Big|\le\frac{M^{-1}}{2}+\frac{|\langle\xi\rangle_\gamma-|\xi||}{\langle\xi\rangle_\gamma}$$
$$\le\frac{M^{-2}}{2}+\frac{\gamma^2}{\langle\xi\rangle_\gamma(\langle\xi\rangle_\gamma+|\xi|)}\le M^{-2}$$

if $(x,\xi)\in W_M$ where δ_{ij} is the Kronecker's delta. Define extensions $\alpha(x,\xi)$, $a(t,x,\xi)$, $b(t,x,\xi)$, $\Delta(t,x,\xi)$, $\bar{\Delta}(t,x,\xi)$, ... of $\alpha(X)$, $a(t,X,\epsilon)$, $b(t,X)$, $\Delta(t,X,\epsilon)$, $\bar{\Delta}(t,X,\epsilon)$, ... by

$$\begin{aligned}&\alpha(x,\xi)=\alpha(y(x),\eta(\xi)),\quad a(t,x,\xi)=a(t,y(x),\eta(\xi),\epsilon(\xi)),\\&b(t,x,\xi)=b(t,y(x),\eta(\xi)),\quad \Delta(t,x,\xi)=\Delta(t,y(x),\eta(\xi),\epsilon(\xi)),\\&\bar{\Delta}(t,x,\xi)=\bar{\Delta}(t,y(x),\eta(\xi),\epsilon(\xi))\end{aligned}$$

so on with

$$\epsilon(\xi)=M^{1/2}\langle\xi\rangle_\gamma^{-1/2}. \tag{4.4}$$

In view of (4.1) and (4.2) such extended symbols are defined in $\mathbb{R}^d\times\mathbb{R}^d$. Let

$$G=M^4(|dx|^2+\langle\xi\rangle_\gamma^{-2}|d\xi|^2).$$

Then it is easy to see

$$y_j\in S(M^{-2},G),\quad \eta_j-\bar{\xi}_j\langle\xi\rangle_\gamma\in S(M^{-2}\langle\xi\rangle_\gamma,G),\quad \epsilon(\xi)\in S(M^{-2},G) \tag{4.5}$$

for $j=1,\ldots,d$. To avoid confusions we denote $\langle\eta(\xi)\rangle_\gamma$ by $[\xi]$ hence

$$[\xi]\in S(\langle\xi\rangle_\gamma,G),\quad [\xi]\langle\xi\rangle_\gamma^{-1}-1\in S(M^{-2},G). \tag{4.6}$$

Lemma 4.1. *Let $f(X,\epsilon)$ be a symbol defined in a conic neighborhood of $(\bar{X},0)$ which is homogeneous of degree 0 in ξ. If $\partial_x^\alpha\partial_\xi^\beta\partial_\epsilon^k f(\bar{X},0)=0$ for $0\le|\alpha+\beta|+k<r$ then $f(x,\xi)=f(y(x),\eta(\xi),\epsilon(\xi))\in S(M^{-2r},G)$. Let $h(X)$ be a symbol defined in a conic neighborhood of $\bar{X}$ which is homogeneous of degree 0 in ξ. Then*

$$h(x,\xi)-h(0,\bar{\xi})\in S(M^{-2},G).$$

Proof. We prove the first assertion. By the Taylor formula one can write

$$\begin{aligned}f(y,\eta,\epsilon)=&\sum_{|\alpha+\beta|+k=r}\frac{1}{\alpha!\beta!k!}y^\alpha(\eta-\bar{\xi}\langle\xi\rangle_\gamma)^\beta\epsilon^k\partial_x^\alpha\partial_\xi^\beta\partial_\epsilon^k f(0,\bar{\xi}\langle\xi\rangle_\gamma,0)\\&+(r+1)\sum_{|\alpha+\beta|+k=r+1}\Big[\frac{1}{\alpha!\beta!k!}y^\alpha(\eta-\bar{\xi}\langle\xi\rangle_\gamma)^\beta\epsilon^k\\&\times\int_0^1(1-\theta)^r\partial_x^\alpha\partial_\xi^\beta\partial_\epsilon^k f(\theta y,\theta(\eta-\bar{\xi}\langle\xi\rangle_\gamma)+\bar{\xi}\langle\xi\rangle_\gamma,\theta\epsilon)d\theta\Big].\end{aligned}$$

It is clear that

$$y^\alpha(\eta-\bar\xi\langle\xi\rangle_\gamma)^\beta\epsilon^k\partial_x^\alpha\partial_\xi^\beta\partial_\epsilon^k f(0,\bar\xi,0)\langle\xi\rangle_\gamma^{-|\beta|}\in S(M^{-2r},G)$$

for $|\alpha+\beta|+k=r$ in view of (4.5). Since $\langle\xi\rangle_\gamma/C\le|\theta(\eta-\bar\xi\langle\xi\rangle_\gamma)+\bar\xi\langle\xi\rangle_\gamma|\le C\langle\xi\rangle_\gamma$ the integral belongs to $S(\langle\xi\rangle_\gamma^{-|\beta|},G)$ hence the second term on the right-hand side is in $S(M^{-2r-2},G)$ thus the assertion. □

4.2 Estimate of extended symbols

From now on it is assumed that all constants are independent of M and γ if otherwise stated. We write $A\precsim B$ if A is bounded by constant, independent of M and γ, times B. Recall $\rho(X,\epsilon)=\alpha(X)+\epsilon^2$ so that

$$\rho(x,\xi)=\alpha(x,\xi)+M\langle\xi\rangle_\gamma^{-1}. \tag{4.7}$$

From Lemma 4.1 we see $\rho\in S(M^{-4},G)$ hence $|\partial_x^\alpha\partial_\xi^\beta\rho|\precsim\langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta|=2$. Since $\rho\ge 0$ it follows from the Glaeser inequality that

$$\big|\partial_x^\alpha\partial_\xi^\beta\rho\big|\precsim\sqrt{\rho}\,\langle\xi\rangle_\gamma^{-|\beta|}\qquad(|\alpha+\beta|=1). \tag{4.8}$$

Lemma 4.2. *Assume $|a(X,\epsilon)|\le C\rho(X,\epsilon)^n$ with some $n>0$ in a conic neighborhood of $(\bar X,0)$ and $a(X,\epsilon)$ is of homogeneous of degree 0 in ξ. Then there exists $C_{\alpha\beta}>0$ such that*

$$\big|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\big|\le C_{\alpha\beta}\rho(x,\xi)^{n-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}. \tag{4.9}$$

Proof. From the assumption it follows that $\partial_x^\alpha\partial_\xi^\beta\partial_\epsilon^k a(0,\bar\xi,0)=0$ for $|\alpha+\beta|+k<2n$ and hence Lemma 4.1 shows that $a(x,\xi)\in S(M^{-4n},G)$. Therefore for $|\alpha+\beta|\ge 2n$ one sees

$$\begin{aligned}\big|\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\big|&\le CM^{2|\alpha+\beta|-4n}\le C\big(C_0\rho^{-1}\big)^{|\alpha+\beta|/2-n}\\&=CC_0^{|\alpha+\beta|/2-n}\rho^{n-|\alpha+\beta|/2}\end{aligned}$$

because $M^4\le C_0\rho^{-1}$. Hence (4.9) holds for $|\alpha+\beta|\ge 2n$. The case $|\alpha+\beta|\le 2n-1$ remains to be checked. Writing $X=(x,\xi)$, $Y=(y,\eta\langle\xi\rangle_\gamma)$ and applying the Taylor formula to obtain

$$\begin{aligned}\big|a(X+sY)\big|&=\Big|\sum_{j=0}^{2n-1}\frac{s^j}{j!}d^ja(X;Y)+\frac{s^{2n}}{(2n)!}d^{2n}a(X+s\theta Y;Y)\Big|\\&\le C\Big(\sum_{j=0}^{2n-1}\frac{s^j}{j!}d^j\rho(X;Y)+\frac{s^{2n}}{(2n)!}d^{2n}\rho(X+s\theta'Y;Y)\Big)^n\end{aligned} \tag{4.10}$$

with some $0<\theta,\theta'<1$ where

$$d^ja(X;Y)=\sum_{|\alpha+\beta|=j}\frac{j!}{\alpha!\beta!}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)y^\alpha\eta^\beta\langle\xi\rangle_\gamma^{|\beta|}.$$

If $\rho(x,\xi)=0$ then $\partial_x^\alpha\partial_\xi^\beta\rho(x,\xi)=0$ for $|\alpha+\beta|=1$ because $\rho\geq 0$ and then it follows that $\partial_x^\alpha\partial_\xi^\beta a(x,\xi)=0$ for $|\alpha+\beta|\leq 2n-1$ from (4.10) hence (4.9) is obvious. We fix a small $s_0>0$ and assume $\rho(x,\xi)\neq 0$. If $\rho(x,\xi)\geq s_0$ then one has

$$\begin{aligned}\big|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\langle\xi\rangle_\gamma^{|\beta|}\big|&\leq C_{\alpha\beta}M^{-4n+2|\alpha+\beta|}\leq C_{\alpha\beta}\\ \leq C_{\alpha\beta}s_0^{-n+|\alpha+\beta|/2}s_0^{n-|\alpha+\beta|/2}&\leq C_{\alpha\beta}s_0^{-n+|\alpha+\beta|/2}\rho^{n-|\alpha+\beta|/2}\end{aligned}$$

which proves (4.9). Assume $0<\rho(x,\xi)<s_0$. Note that

$$\big|d^{2n}a(X+s\theta Y;Y)\big|\leq C,\quad d^{2n}\rho(X+s\theta' Y;Y)\leq C\rho(X)^{1-n}$$

for any $|(y,\eta)|\leq 1/2$. Indeed the first one is clear from $a(x,\xi)\in S(M^{-4n},G)$. To check the second inequality it is enough to note that for $|\alpha+\beta|=2n$

$$\big|\partial_x^\alpha\partial_\xi^\beta\rho(X+\theta''Y)\big|\leq CM^{-2+2n}\langle\xi+\theta''\langle\xi\rangle_\gamma\eta\rangle_\gamma^{-|\beta|}\leq C'(C_0\rho(X)^{-1})^{n-1}\langle\xi\rangle_\gamma^{-|\beta|}$$

since $\sqrt{2}\langle\xi+\theta\langle\xi\rangle_\gamma\eta\rangle_\gamma\geq\langle\xi\rangle_\gamma/2$ for $|\eta|\leq 1/2$ and $|\theta|<1$. Take $s=\rho(X)^{1/2}$ in (4.10) to get

$$\Big|\sum_{j=0}^{2n-1}\frac{1}{j!}d^j a(X;Y)\rho(X)^{j/2}\Big|\leq C\Big(\sum_{j=0}^{2n-1}\frac{1}{j!}d^j\rho(X;Y)\rho(X)^{j/2}\Big)^n+C\rho(X)^n$$

which is bounded by $C\rho(X)^n$ because $|d\rho(X;Y)|\leq C''\rho(X)^{1/2}$ in view of (4.8) and

$$|d^j\rho(X;Y)|\leq CM^{-2+j}\leq C(C_0\rho^{-1}(X))^{j/2-1}$$

for $j\geq 3$. This gives

$$\Big|\sum_{j=1}^{2n-1}\frac{1}{j!}d^j a(X;Y)\frac{\rho(X)^{j/2}}{\rho(X)^n}\Big|\leq C_1.$$

Replacing (y,η) by $s(y,\eta)$, $|(y,\eta)|=1/2$, $0<|s|<1$ one obtains

$$\Big|\sum_{j=1}^{2n-1}\frac{s^j}{j!}d^j a(X;Y)\frac{\rho(X)^{j/2}}{\rho(X)^n}\Big|\leq C_1.$$

Since two norms $\sup_{|s|\leq 1}|p(s)|$ and $\max\{|c_j|\}$ on the vector space consisting of all polynomials $p(s)=\sum_{j=0}^{2n-1}c_js^j$ are equivalent one obtains $|d^ja(X;Y)|\leq B'\rho(X)^{n-j/2}$. Since $|(y,\eta)|=1/2$ is arbitrary one concludes (4.9). □

Corollary 4.1. *On has* $\big|\partial_x^\alpha\partial_\xi^\beta\rho(x,\xi)\big|\precsim\rho(x,\xi)^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$.

Lemma 4.3. *Let* $s\in\mathbb{R}$. *Then* $\big|\partial_x^\alpha\partial_\xi^\beta\rho^s\big|\precsim\rho^{s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$.

Proof. Since

$$\partial_x^\alpha \partial_\xi^\beta \rho^s = \sum C_{\alpha^{(j)}\beta^{(j)}} \rho^s \Big(\frac{\partial_x^{\alpha^{(1)}} \partial_\xi^{\beta^{(1)}} \rho}{\rho} \Big) \cdots \Big(\frac{\partial_x^{\alpha^{(k)}} \partial_\xi^{\beta^{(k)}} \rho}{\rho} \Big)$$

the assertion follows from Corollary 4.1. □

Lemma 4.4. *Let $a_j(x,\xi) = a_j(y(x), \eta(\xi), \epsilon(\xi))$. Then*

$$\big|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)\big| \precsim \rho(x,\xi)^{j-|\alpha+\beta|/2} \langle \xi \rangle_\gamma^{-|\beta|}, \quad j = 1, 2, 3.$$

Proof. The assertion follows from Lemmas 3.3 and 4.2. □

For the extension $\psi(x,\xi) = \psi(y(x), \eta(\xi), \epsilon(\xi))$ of $\psi(X,\epsilon)$ we have

Lemma 4.5. *One has $\big|\partial_x^\alpha \partial_\xi^\beta \psi(x,\xi)\big| \le C_{\alpha\beta} \rho(x,\xi)^{1-|\alpha+\beta|/2} \langle \xi \rangle_\gamma^{-|\beta|}$.*

Proof. Since Lemma 4.2 is not available for $\psi(X,\epsilon)$ because it is not defined for $\epsilon = 0$ then we show the assertion directly. Let $\nu_1(x,\xi)$, $a_1(x,\xi)$ and $\bar\Delta(t,x,\xi)$ be extensions of $\nu_1(X,\epsilon)$, $a_1(X,\epsilon)$ and $\bar\Delta(t,X,\epsilon)$ and hence one has $\bar\Delta(\nu_1(x,\xi),x,\xi) = 0$. Note that $|\partial_t \bar\Delta(\nu_1,x,\xi)| \ge 4\,c_2^2\,\rho^2(x,\xi)$ if $\nu_1(x,\xi) + a_1(x,\xi) < 2c_1\rho(x,\xi)$ thanks to Lemma 3.5. Starting with

$$\partial_t \bar\Delta(\nu_1,x,\xi) \partial_x^\alpha \partial_\xi^\beta \nu_1 + \partial_x^\alpha \partial_\xi^\beta \bar\Delta(\nu_1,x,\xi) = 0 \quad (|\alpha+\beta| = 1)$$

a repetition of the same argument proving the estimates for λ_2 in Lemma 6.3 below together with Lemma 4.4 one obtains

$$\big|\partial_x^\alpha \partial_\xi^\beta \nu_1\big| \precsim \rho^{1-|\alpha+\beta|/2} \langle \xi \rangle_\gamma^{-|\beta|}, \quad \nu_1 + a_1 < 2c_1\rho. \tag{4.11}$$

Here we have used $|\nu_1| \precsim \rho$ which also follows from Lemma 4.4. Using (4.11) and Lemmas 4.3 and 4.4 the assertion follows easily. □

4.3 Remarks on the condition (3.10)

In this subsection we work near $(0,\bar\xi)$ and (x,ξ) varies in a neighborhood of $(0,\bar\xi)$. First note that p has no triple characteristic root in $t > 0$ because $t + \alpha(x,\xi) > 0$ for $t > 0$. Define

$$\bar\psi(x,\xi) = -(\nu_1(x,\xi,0) + a_1(x,\xi,0))/2$$

then it is clear from the proof of Proposition 3.1 that

$$C\,\Delta(t,x,\xi,0) \ge \min\big\{t^2, (t - \bar\psi(x,\xi))^2\big\}(t + \alpha(x,\xi)) \quad (t \ge 0). \tag{4.12}$$

Assume that p has a double characteristic root at (t,x,ξ) with $t > 0$. Denoting by $\mu(t,x,\xi)$ the other characteristic root of p, which is simple and hence smooth in (t,x,ξ) near the reference point, one can write

$$\begin{aligned} p(t,x,\tau,\xi) &= \tau^3 - a(t,x,\xi)|\xi|^2\tau - b(t,x,\xi)|\xi|^3 \\ &= (\tau - \mu(t,x,\xi))\big(\tau^2 + c_1(t,x,\xi)\tau + c_2(t,x,\xi)\big). \end{aligned}$$

Note that

$$\Delta(t,x,\xi,0)|\xi|^6 = \big(4\,a^3 - 27\,b^2\big)|\xi|^6 = (2c_1^2 + c_2)\big(c_1^2 - 4c_2\big)$$

where $\Delta_2 = (c_1^2 - 4c_2)/4$ is the discriminant of $\tau^2 + c_1\tau + c_2$. Since $\mu^2 + c_1\mu + c_2 \neq 0$ hence $2c_1^2 + c_2 \neq 0$ it follows from (4.12) that

$$\{(t,x,\xi) \mid \Delta_2 = 0, t > 0\} \subset \{(t,x,\xi) \mid t = \bar\psi(x,\xi) > 0\}.$$

Note that $\bar\psi(x,\xi) > 0$ implies that $\nu_1(x,\xi,0) + a_1(x,\xi,0) < 0 \leq 2c_1\alpha(x,\xi)$ hence $\bar\psi(x,\xi)$ is smooth there and

$$\big|\nu_1(x,\xi,0) - \nu_j(x,\xi,0)\big| \geq c_2\alpha(x,\xi), \quad j = 2, 3 \tag{4.13}$$

by Lemma 3.5. Now we can prove

Lemma 4.6. *Near $\bar X = (0,\bar\xi)$ the doubly characteristic set of p with $t > 0$ is contained in $\{(t,x,\xi) \mid t = \bar\psi(x,\xi) > 0\}$ and $t - \bar\psi(x,\xi)$ is a time function for p.*

It remains to show that $t - \bar\psi$ is a time function (see e.g. [24]) for p. Let $q = \tau^2 + c_1\tau + c_2$ then $F_p = c\,F_q$ at a double characteristic with $t > 0$ with some $c \neq 0$ then it is enough to prove that $t - \bar\psi$ is a time function for q. Write

$$q = \big(\tau + c_1/2\big)^2 - \Delta_2$$

and recall [24, Lemma 2.1.3] that $t - \bar\psi$ is a time function for q if and only if

$$\{\tau + c_1/2, t - \bar\psi\} > 0, \quad \{\Delta_2, t - \bar\psi\}^2 \leq 4\,c\{\tau + c_1/2, t - \bar\psi\}^2\Delta_2 \tag{4.14}$$

with some $0 < c < 1$. Since $\Delta_2 \geq 0$ one obtains $|\{\Delta_2, t - \bar\psi\}| = |\{\Delta_2, \bar\psi\}| \leq C\sqrt{\Delta_2}\,|\nabla\bar\psi|$. Taking (4.13) into account, a repetition of the proof of Lemma 4.5 shows $|\nabla\bar\psi| \leq C'\sqrt{\alpha}$ and hence $|\{\Delta_2, t - \bar\psi\}|^2 \leq C\alpha\Delta_2$. On the other hand one has $\{\tau + c_1/2, t - \bar\psi\} = 1 - \{c_1, \bar\psi\}/2 \geq 1 - C|\nabla\bar\psi| \geq 1 - C''\sqrt{\alpha}$ then (4.14) holds because α can be assumed to be small there.

4.4 Lower bound of perturbed discriminant

Recall that $\alpha(x,\xi)$, $a(t,x,\xi)$, $b(t,x,\xi)$, $e(t,x,\xi)$, $\Delta(t,x,\xi)$, … are extensions of $\alpha(X)$, $a(t,X,\epsilon)$, $b(t,X)$, $e(t,X,\epsilon)$, $\Delta(t,X,\epsilon)$, … defined in Section 4.2 so that

$$p = \tau^3 - a(t,x,\xi)|\xi|^2\tau + b(t,x,\xi)|\xi|^3, \quad a = e(t,x,\xi)\big(t + \alpha(x,\xi)\big)$$

is now defined in $\mathbb{R}^d \times \mathbb{R}^d$ and coincides with the original p in a conic neighborhood W_M of $(0,\bar\xi)$. We add a term $2Me(t,x,\xi)\langle\xi\rangle_\gamma^{-1}$ to p and consider

$$\tau^3 - e(t + \alpha + 2M\langle\xi\rangle_\gamma^{-1})|\xi|^2\tau - b\,|\xi|^3.$$

Denoting

$$a_M(t,x,\xi) = e(t,x,\xi)(t + \alpha(x,\xi) + 2M\langle\xi\rangle_\gamma^{-1}) \tag{4.15}$$

consider the discriminant

$$
\tag{4.16}
\begin{aligned}
\Delta_M(t,x,\xi) &= 4\,e^3\big(t+\alpha+2M\langle\xi\rangle_\gamma^{-1}\big)^3 - 27\,b^2\\
&= 4\,e^3\big(t+\alpha+M\langle\xi\rangle_\gamma^{-1}\big)^3 - 27\,b^2 + \Delta_r(t,x,\xi)
\end{aligned}
$$

where, recalling $\alpha+M\langle\xi\rangle_\gamma^{-1}=\rho$, we have

$$
\begin{aligned}
\Delta_r &= 4e^3\big(3(t+\rho)^2M\langle\xi\rangle_\gamma^{-1}+3(t+\rho)M^2\langle\xi\rangle_\gamma^{-2}+M^3\langle\xi\rangle_\gamma^{-3}\big)\\
&= 12e^3\big(c_1(x,\xi)t^2+c_2(x,\xi)t+c_3(x,\xi)\big) \ge 12e^3M(t+\rho)^2\langle\xi\rangle_\gamma^{-1}
\end{aligned}
$$

where it is clear that $c_j(x,\xi)$ verifies $\big|\partial_x^\alpha\partial_\xi^\beta c_j\big| \precsim \rho^{j-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. Thanks to Proposition 3.1 one has

$$
\bar{\Delta}(t,x,\xi) \ge \upsilon\min\{t^2,(t-\psi)^2\}(t+\rho).
$$

Since $\Delta(t,x,\xi)=\tilde{e}\,\bar{\Delta}$ then

$$
\tag{4.17}
\begin{aligned}
\Delta(t,x,\xi) = \tilde{e}\,\bar{\Delta} &\ge \tilde{e}\,\upsilon\min\{t^2,(t-\psi)^2\}(t+\rho)\\
&\ge \big(\tilde{e}/e\big)\upsilon\min\{t^2,(t-\psi)^2\}e(t+\rho).
\end{aligned}
$$

Therefore choosing a constant $\bar{\nu}>0$ such that $12\,e^2 \ge \big(\tilde{e}/e\big)\upsilon\bar{\nu}$ one obtains from (4.16), (4.17) that

$$
\tag{4.18}
\begin{aligned}
\Delta_M &\ge \big(\tilde{e}/e\big)\upsilon\min\big\{t^2,(t-\psi)^2\big\}e(t+\rho) + 12\,e^3(t+\rho)^2M\langle\xi\rangle_\gamma^{-1}\\
&\ge \big(\tilde{e}/e\big)\upsilon\big(\min\big\{t^2,(t-\psi)^2\big\}+\bar{\nu}(t+\rho)M\langle\xi\rangle_\gamma^{-1}\big)e(t+\rho)\\
&\ge \big(\tilde{e}/e\big)\upsilon\min\big\{t^2,(t-\psi)^2+\bar{\nu}M\rho\langle\xi\rangle_\gamma^{-1}\big\}e(t+\rho) \qquad (t\ge 0).
\end{aligned}
$$

Proposition 4.1. *One can write*

$$
\Delta_M = e\big(t^3+a_1(x,\xi)t^2+a_2(x,\xi)t+a_3(x,\xi)\big)
$$

where $0<e\in S(1,G)$ *uniformly in* t *and* a_j *satisfies*

$$
\tag{4.19}
\big|\partial_x^\alpha\partial_\xi^\beta a_j\big| \precsim \rho^{j-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.
$$

Moreover there exist $\bar{\nu}>0$ *and* $c>0$ *such that*

$$
\tag{4.20}
\frac{\Delta_M}{a_M} \ge \frac{\tilde{e}}{2e}\upsilon\,\min\big\{t^2,(t-\psi)^2+\bar{\nu}M\rho\langle\xi\rangle_\gamma^{-1}\big\},\quad \frac{\Delta_M}{a_M}\ge c\,M\langle\xi\rangle_\gamma^{-1}\,a_M
$$

for $0\le t\le T$ *where* ψ *and* ρ *satisfy*

$$
\tag{4.21}
\big|\partial_x^\alpha\partial_\xi^\beta\psi\big|,\qquad \big|\partial_x^\alpha\partial_\xi^\beta\rho\big| \precsim \rho^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.
$$

Proof. Choosing $\epsilon=\sqrt{2}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ in (4.4) and applying Lemma 3.1 one can write Δ_M as a third order polynomial in t, up to non-zero factor and can estimate the coefficients thanks to Lemmas 3.3 and 4.2 in terms of $\alpha+2M\langle\xi\rangle_\gamma^{-1}$. Noting $\rho(x,\xi)\le\alpha(x,\xi)+2M\langle\xi\rangle_\gamma^{-1}\le 2\rho(x,\xi)$ we have (4.19). The assertion (4.20) follows from (4.18) for $a_M\le 2\,e(t+\rho)$. The estimates (4.21) are nothing but Corollary 4.1 and Lemma 4.5 with the choice $\epsilon=M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. □

We estimate the ratio of $\partial_t b$ to $\sqrt{a_M}$ for later use.

Lemma 4.7. *We have*

$$\big|\partial_t b\big| \le (1+CM^{-2})\big(2\sqrt{2/3}\big)|e(0,0,\bar{\xi})|\sqrt{a_M} \qquad (0\le t\le M^{-2}).$$

Proof. Write $b=\beta_0(x,\xi)+t\beta_1(x,\xi)+t^2\beta_3(t,x,\xi)$. From $27b^2\le 4a^3$ for $0\le t\le T$ it is clear that $|\beta_0|\le (2/3\sqrt{3})e^{3/2}\alpha^{3/2}$. We first check that

$$|\beta_1|\le (1+CM^{-2})(2/\sqrt{3})e^{3/2}\sqrt{\alpha}. \tag{4.22}$$

If $\alpha(x,\xi)=0$ then $\beta_1(x,\xi)=0$ by $27b^2\le 4a^3$ hence (4.22) is clear. When $\alpha(x,\xi)>0$ take $t=3\alpha$ it follows from $27b^2\le 4a^3$ that

$$\begin{aligned}3\alpha|\beta_1|\le 2(4^{3/2}e^{3/2}/3\sqrt{3})\alpha^{3/2}+|\beta_0|+C\alpha^2 &\le (6/\sqrt{3})e^{3/2}\alpha^{3/2}+C\alpha^2\\ &\le (6/\sqrt{3})(1+CM^{-2})e^{3/2}\alpha^{3/2}\end{aligned}$$

because $\alpha\le CM^{-4}$ which proves (4.22). Since $|\partial_t b|\le|\beta_1|+Ct$ we see that

$$\begin{aligned}|\partial_t b| &\le (1+CM^{-2})(2/\sqrt{3})e^{3/2}\sqrt{\alpha}+CM^{-2}\sqrt{t}\\ &\le (1+CM^{-2})(2/\sqrt{3})e^{3/2}\big(\sqrt{\alpha}+\sqrt{t}\big)\end{aligned}$$

from which the proof is immediate. □

Remark 4.1. Here we make a remark on $e(0,0,\bar{\xi})=\partial_t a(0,0,\bar{\xi})$. In view of (3.1) it is clear that $e(0,0,\bar{\xi})$ is the nonzero positive real eigenvalue of $F_p(0,0,0,\bar{\xi})$. Since $\tilde{e}/e=4\,e^2(0,0,\bar{\xi})\big(1+O(M^{-2})\big)$ the coefficienet of the right-hand side of (4.20) is, essentially, constant times the square of the nonzero positive real eigenvalue of the Hamilton map.

In what follows we denote $\bar{e}=e(0,0,\bar{\xi})$.

5 Metric g and estimates of ω and ϕ

Introduce the metric

$$g=g_{(x,\xi)}(dx,d\xi)=M^{-1}\big(\langle\xi\rangle_\gamma|dx|^2+\langle\xi\rangle_\gamma^{-1}|d\xi|^2\big)$$

which is a basic metric with which we work in this paper. Note that

$$S(M^s,G)\subset S(M^s,g)$$

because $M^{s+2|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|}\le M^sM^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ in view of $\langle\xi\rangle_\gamma\ge\gamma\ge M^5$. The metric g is slowly varying and σ temperate (see [6, Chapter 18.5], in what follows we omit "σ" because we use only the Weyl calculus in this paper) *uniformly* in $\gamma\ge M^5\ge 1$ which will be checked in Section 7.

Lemma 5.1. *For* $|\alpha+\beta|\geq 1$ *one has*

$$\left|\partial_x^\alpha\partial_\xi^\beta\psi\right|\precsim M^{1/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}.$$

Proof. It is enough to remark

$$\left|\partial_x^\alpha\partial_\xi^\beta\psi\right|\precsim\rho^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}\precsim\rho^{1/2}\rho^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|}$$
$$\precsim\rho^{1/2}(M^{-1}\langle\xi\rangle_\gamma)^{(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta}=M^{1/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$$

which proves the assertion. □

Corollary 5.1. *For* $|\alpha+\beta|\geq 1$ *one has*

$$\partial_x^\alpha\partial_\xi^\beta\psi\in S(M^{-(|\alpha+\beta|-1)/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},g).$$

5.1 Estimate ω by metric g

Taking Proposition 4.1 into account we introduce a preliminary weight

$$\omega(t,x,\xi)=\sqrt{(t-\psi(x,\xi))^2+\bar{\nu}M\rho\langle\xi\rangle_\gamma^{-1}}.$$

Since the exact value of $\bar{\nu}>0$ is irrelevant in the following arguments so we assume $\bar{\nu}=1$ from now on. In what follows we work with symbols depending on t. We assume that t varies in some fixed interval $[0,T]$ and it is assumed that all constants are independent of $t\in[0,T]$ and γ, M if otherwise stated. Now $A\precsim B$ implies that A is bounded by constant, independent of t, M and γ, times B.

Lemma 5.2. *Let* $s\in\mathbb{R}$. *For* $|\alpha+\beta|\geq 1$ *we have*

$$\left|\partial_x^\alpha\partial_\xi^\beta\omega^s\right|\precsim\omega^s(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}.$$

Proof. Recall $\omega^2=(t-\psi)^2+M\rho\langle\xi\rangle_\gamma^{-1}$. Note that for $|\alpha+\beta|\geq 2$

$$\left|\partial_x^\alpha\partial_\xi^\beta(t-\psi)^2\right|\precsim\omega|\partial_x^\alpha\partial_\xi^\beta\psi|+\sum|\partial_x^{\alpha'}\partial_\xi^{\beta'}\psi||\partial^{\alpha''}\partial_\xi^{\beta''}\psi|$$
$$\precsim\omega\rho^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}+\rho^{2-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$$
$$\precsim\omega^2\big\{\omega^{-1}\rho^{1/2}\rho^{-(|\alpha+\beta|-1)/2}+\omega^{-2}\rho\rho^{-(|\alpha+\beta|-2)/2}\big\}\langle\xi\rangle_\gamma^{-|\beta|}$$
$$\precsim\omega^2\big\{\omega^{-1}\rho^{1/2}(M^{-1}\langle\xi\rangle_\gamma)^{(|\alpha+\beta|-1)/2}+\omega^{-2}\rho(M^{-1}\langle\xi\rangle_\gamma)^{(|\alpha+\beta|-2)/2}\big\}\langle\xi\rangle_\gamma^{-|\beta|}$$
$$\precsim\omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$$

since $\omega\geq\sqrt{M}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. When $|\alpha+\beta|=1$ there is no second term and hence

$$\left|\partial_x^\alpha\partial_\xi^\beta(t-\psi)^2\right|\precsim\omega\rho^{1/2-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|}$$
$$\precsim\omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}.$$

Next it is easy to see that for $|\alpha+\beta|\geq 1$

$$\begin{aligned}\big|\partial_x^\alpha\partial_\xi^\beta(M\rho\langle\xi\rangle_\gamma^{-1})\big| &\precsim M\rho\langle\xi\rangle_\gamma^{-1}\rho^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim M\rho^{1/2}\langle\xi\rangle_\gamma^{-1}\rho^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \omega^2\big(M\omega^{-2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1}\big)\big(M^{-1}\langle\xi\rangle_\gamma\big)^{(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\end{aligned}$$

because $\omega\geq\sqrt{M}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\geq M\langle\xi\rangle_\gamma^{-1}$. Therefore one concludes that

$$\big|\partial_x^\alpha\partial_\xi^\beta\omega^2\big|\precsim\omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$$

which proves the assertion for $s=2$. For general s noting

$$\begin{aligned}\big|\partial_x^\alpha\partial_\xi^\beta(\omega^2)^{s/2}\big| &\precsim \sum\big|(\omega^2)^{s/2-l}\big(\partial_x^{\alpha^1}\partial_\xi^{\beta^1}\omega^2\big)\cdots\big(\partial_x^{\alpha^l}\partial_\xi^{\beta^l}\omega^2\big)\big|\\ &\precsim\sum\omega^s\big(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\big)^lM^{-(|\alpha+\beta|-l)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\\ &\precsim\omega^s\big(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\big)M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\end{aligned}$$

for $\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\leq M^{-1/2}\leq 1$ the proof is immediate. □

Corollary 5.2. *We have $\omega^s\in S(\omega^s,g)$ for $s\in\mathbb{R}$.*

Corollary 5.3. *For $|\alpha+\beta|\geq 1$ one has*

$$\partial_x^\alpha\partial_\xi^\beta\omega^s\in S\big(M^{-(|\alpha+\beta|-1)/2}\omega^s\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2+(|\alpha|-\beta|)/2},g\big).$$

5.2 Estimate ϕ by metric g

Introduce a wight which plays a crucial role in deriving energy estimates

$$\phi(t,x,\xi)=\omega(t,x,\xi)+t-\psi(x,\xi).$$

Start with remarking

Lemma 5.3. *There is $C>0$ such that $\phi(t,x,\xi)\geq M\langle\xi\rangle_\gamma^{-1}/C$.*

Proof. When $t-\psi(x,\xi)\geq 0$ one has $\phi\geq\omega$ hence

$$\phi\geq\omega\geq M^{1/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\geq M\langle\xi\rangle_\gamma^{-1}$$

is obvious for $\rho\geq M\langle\xi\rangle_\gamma^{-1}$. Assume $t-\psi(x,\xi)<0$ then $0\leq t<\psi(x,\xi)\leq\delta\rho(x,\xi)$ with some $\delta>0$ by Lemma 4.5. Noticing that $|t-\psi(x,\xi)|=\psi(x,\xi)-t\leq\delta\rho(x,\xi)$ we have

$$\begin{aligned}\omega^2(t,x,\xi)=(t-\psi(x,\xi))^2+M\rho(x,\xi)\langle\xi\rangle_\gamma^{-1} &\leq\delta^2\rho^2+M\rho\langle\xi\rangle_\gamma^{-1}\\ &\leq\delta^2\rho^2+\rho^2=(\delta^2+1)\rho^2.\end{aligned}$$

Now remarking that

$$\phi(t,x,\xi) \geq \frac{M\rho\langle\xi\rangle_\gamma^{-1}}{\omega+|t-\psi|} \geq \frac{M\rho\langle\xi\rangle_\gamma^{-1}}{2\,\omega} \tag{5.1}$$

the proof is immediate. □

Next show

Lemma 5.4. *We have* $\left|\partial_x^\alpha\partial_\xi^\beta\phi\right| \precsim \phi M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$.

Proof. Let $|\alpha+\beta|=1$ and write

$$\partial_x^\alpha\partial_\xi^\beta\phi = \frac{-\partial_x^\alpha\partial_\xi^\beta\psi}{\omega}\phi + \frac{\partial_x^\alpha\partial_\xi^\beta(M\rho\langle\xi\rangle_\gamma^{-1})}{2\omega} = \phi_{\alpha\beta}\phi+\psi_{\alpha\beta}. \tag{5.2}$$

From Corollaries 5.2 and 4.1 it follows that

$$\left|\partial_x^\mu\partial_\xi^\nu\big(\psi_{\alpha\beta}\big)\right| \precsim \omega^{-1}M\rho\langle\xi\rangle_\gamma^{-1}M^{-|\alpha+\beta+\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}.$$

Noting (5.1) one obtains

$$\left|\partial_x^\mu\partial_\xi^\nu\big(\psi_{\alpha\beta}\big)\right| \precsim \phi M^{-|\alpha+\beta+\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}.$$

On the other hand thanks to Corollaries 5.1 and 5.2 one sees

$$|\partial_x^\mu\partial_\xi^\nu\phi_{\alpha\beta}| \precsim M^{-|\alpha+\beta+\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}.$$

Hence using (5.2) the assertion is proved by induction on $|\alpha+\beta|$. □

We refine this lemma.

Lemma 5.5. *Let* $|\alpha+\beta|\geq 1$ *then*

$$\partial_x^\alpha\partial_\xi^\beta\phi \in S(\phi\, M^{-(|\alpha+\beta|-1)/2}\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g).$$

Proof. From Corollary 5.1 one has $\partial_x^\alpha\partial_\xi^\beta\psi \in S(\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g)$ for $|\alpha+\beta|=1$ hence $\phi_{\alpha\beta}\in S(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g)$ for $|\alpha+\beta|=1$ by Corollary 5.2. From Corollary 5.3 it follows that

$$\left|\partial_x^\mu\partial_\xi^\nu\big(\psi_{\alpha\beta}\big)\right| \precsim \omega^{-1}\rho^{1/2}M\langle\xi\rangle_\gamma^{-1-|\beta|}M^{-|\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2}$$

for $|\alpha+\beta|=1$ because $\partial_x^\alpha\partial_\xi^\beta(M\rho\langle\xi\rangle_\gamma^{-1}) \in S(M\rho^{1/2}\langle\xi\rangle_\gamma^{-1-|\beta|}, g)$. Thanks to Lemma 5.3 one sees $M\langle\xi\rangle_\gamma^{-1}\leq C\phi(t,x,\xi)$ and hence

$$\psi_{\alpha\beta}\in S(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\phi, g), \quad |\alpha+\beta|=1.$$

Since $\phi\in S(\phi,g)$ by Lemma 5.4 we conclude the assertion from (5.2). □

6 Bézout matrix as symmetrizer

Add $-2M\mathrm{op}(e(t,x,\xi)\langle\xi\rangle_\gamma^{-1})[D]^2D_t$ to the principal part and subtract the same one from the lower order part so that the operator is left to be invariant;

$$\hat{P} = D_t^3 - a_M(t,x,D)[D]^2D_t - b(t,x,D)\,[D]^3 + b_1(t,x,D)D_t^2$$
$$\big(b_2(t,x,D) + d_M(t,x,D)\big)[D]D_t + b_3(t,x,D)[D]^2$$

where $b_j(t,x,\xi) \in S(1,G)$ and $d_M(t,x,\xi) = 2M(e\langle\xi\rangle_\gamma^{-1})\#[\xi] \in S(M,G)$. Here note that

$$d_M(t,x,\xi) - 2Me(t,x,\xi) \in S(M^{-1},g) \tag{6.1}$$

which follows from (4.6). With $U = {}^t(D_t^2u, [D]D_tu, [D]^2u)$ the equation $\hat{P}u = f$ is transformed to

$$D_tU = A(t,x,D)[D]U + B(t,x,D)U + F \tag{6.2}$$

where $F = {}^t(f,0,0)$ and

$$A(t,x,\xi) = \begin{bmatrix} 0 & a_M & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B(t,x,\xi) = \begin{bmatrix} b_1 & b_2 + d_M & b_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let S be the Bézout matrix of p and $\partial p/\partial\tau$, that is

$$S(t,x,\xi) = \begin{bmatrix} 3 & 0 & -a_M \\ 0 & 2a_M & 3b \\ -a_M & 3b & a_M^2 \end{bmatrix}$$

then S is nonnegative definite and symmetrizes A, that is SA is symmetric.

6.1 Eigenvalues of Beźout matrix

Consider the principal symbol $\tau^3 - a_M(t,x,\xi)[\xi]^2\tau - b(t,x,\xi)[\xi]^3$ of $\hat{P}$. Denote

$$\sigma(t,x,\xi) = t + \alpha(x,\xi) + 2M\langle\xi\rangle_\gamma^{-1} = t + \rho(x,\xi) + M\langle\xi\rangle_\gamma^{-1}$$

hence $a_M(t,x,\xi) = e(t,x,\xi)\sigma(t,x,\xi)$ and $(1-CM^{-2})\bar{e}\,\sigma \le a_M \le (1+CM^{-2})\bar{e}\,\sigma$. In what follows we assume that t varies in the interval

$$0 \le t \le M^{-4}.$$

Since $\rho \in S(M^{-4},G)$ it is clear that $\sigma(t,x,\xi) \in S(M^{-4},G)$.

Lemma 6.1. *We have* $\big|\partial_x^\alpha\partial_\xi^\beta\sigma\big| \precsim \sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. *In particular* $\sigma \in S(\sigma,g)$.

Proof. It is clear from (4.8) that $\big|\partial_x^\alpha\partial_\xi^\beta\sigma\big| \precsim \sqrt{\sigma}\,\langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta| = 1$. For $|\alpha+\beta| \ge 2$ it follows from $\rho \in S(M^{-4},G)$ that

$$\big|\partial_x^\alpha\partial_\xi^\beta\sigma\big| \precsim M^{2|\alpha+\beta|-4}\langle\xi\rangle_\gamma^{-|\beta|} \precsim \sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$$

since $C\sigma^{-1} \ge M^4$. The second assertion is clear from $\sigma^{-1} \le M^{-1}\langle\xi\rangle_\gamma$. □

Corollary 6.1. *Let $s \in \mathbb{R}$. Then $\left|\partial_x^\alpha \partial_\xi^\beta \sigma^s\right| \precsim \sigma^{s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. In particular $\sigma^s \in S(\sigma^s, g)$.*

Definition 6.1. To simplify notations we denote by $\mathcal{C}(\sigma^s)$ the set of symbols $r(t, x, \xi)$ satisfying

$$\left|\partial_x^\alpha \partial_\xi^\beta r\right| \precsim \sigma^{s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.$$

It is clear that $\mathcal{C}(\sigma^s) \subset S(\sigma^s, g)$ because $\sigma^{-|\alpha+\beta|/2} \leq M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{|\alpha+\beta|/2}$. It is also clear that if $p \in \mathcal{C}(\sigma^s)$ with $s > 0$ then $(1+p)^{-1} - 1 \in \mathcal{C}(\sigma^s)$.

Lemma 6.2. *One has*

$$a_M^s \in \mathcal{C}(\sigma^s), \ \ (s \in \mathbb{R}), \quad b \in \mathcal{C}(\sigma^{3/2}), \quad \partial_t a_M \in \mathcal{C}(1), \quad \partial_t b \in \mathcal{C}(\sqrt{\sigma}).$$

Proof. The first assertion is clear from Corollary 6.1 because $a_M = e\sigma$ and $e \in S(1, G)$, $1/C \leq e \leq C$. To show the second assertion, recalling $b(t, x, \xi)$ is the extension of $b(t, X)$, write

$$\begin{aligned} b(t,x,\xi) &= b(0, y(x), \eta(\xi)) + \partial_t b(0, y(x), \eta(\xi))t \\ &\quad + \int_0^1 (1-\theta)\partial_t^2 b(\theta t, y(x), \eta(\xi)) d\theta \cdot t^2. \end{aligned} \tag{6.3}$$

Since $\partial_x^\alpha \partial_\xi^\beta b(0, 0, \bar{\xi}) = 0$ for $|\alpha+\beta| \leq 2$ and $\partial_t b(0, 0, \bar{\xi}) = 0$ then thanks to Lemma 4.1 one has $b(0, y(x), \eta(\xi)) \in S(M^{-6}, G)$ and $\partial_t b(0, y(x), \eta(\xi)) \in S(M^{-2}, G)$. Since $0 \leq t \leq M^{-4}$ we conclude that $b(t, x, \xi) \in S(M^{-6}, G)$. Since $|b| \leq C\sigma^{3/2}$ and $\sigma \in S(M^{-4}, G)$ a repetition of the same arguments proving Lemma 4.2 shows the second assertion. The third assertion is clear because $\partial_t a_M = e + (\partial_t e)\sigma$. As for the last assertion, recall Lemma 4.7 that $|\partial_t b| \leq C a_M^{1/2} \leq C'\sigma^{1/2}$. Noting $\partial_t b \in S(M^{-2}, G)$ which results from (6.3) one sees $|\langle\xi\rangle_\gamma^{|\beta|} \partial_x^\alpha \partial_\xi^\beta \partial_t b| \precsim M^{2|\alpha+\beta|-2} \precsim \sigma^{1/2-|\alpha+\beta|/2}$ for $|\alpha+\beta| \geq 1$ hence the assertion. □

Let

$$0 \leq \lambda_1(t, x, \xi) \leq \lambda_2(t, x, \xi) \leq \lambda_3(t, x, \xi)$$

be the eigenvalues of $S(t, x, \xi)$. Recall [26, Proposition 2.1]

Proposition 6.1. *There exist M_0 and $K > 0$ such that*

$$\begin{gathered} \Delta_M/(6a_M + 2a_M^2 + 2a_M^3) \leq \lambda_1 \leq \left(2/3 + K a_M\right) a_M^2, \\ (2 - K a_M)\, a_M \leq \lambda_2 \leq (2 + K a_M)\, a_M, \\ 3 \leq \lambda_3 \leq 3 + K a_M^2 \end{gathered}$$

provided that $M \geq M_0$.

Proof. Since $a_M = e\,\sigma$ and $\sigma \in S(M^{-4}, G)$ then for any $\bar{\epsilon} > 0$ there is M_0 such that $e\, M_0^{-4} \leq \bar{\epsilon}$. Then the assertion follows from [26, Proposition 2.1]. □

Corollary 6.2. *The eigenvalues $\lambda_i(t, x, \xi)$ are smooth in $(0, M^{-4}] \times \mathbb{R}^d \times \mathbb{R}^d$.*

6.2 Estimates of eigenvalues

First we prove

Lemma 6.3. *One has* $\lambda_j \in \mathcal{C}(\sigma^{3-j})$ *for* $j = 1, 2, 3$.

Denote $q(\lambda) = \det(\lambda I - S)$ so that

$$q(\lambda) = \lambda^3 - (3 + 2a_M + a_M^2)\lambda^2 + (6a_M + 2a_M^2 + 2a_M^3 - 9b^2)\lambda - \Delta_M. \tag{6.4}$$

Note that

$$\partial_\lambda q(\lambda_i)\partial_x^\alpha \partial_\xi^\beta \lambda_i + \partial_x^\alpha \partial_\xi^\beta q(\lambda_i) = 0, \quad |\alpha + \beta| = 1. \tag{6.5}$$

Let us write $\partial_x^\alpha \partial_\xi^\beta = \partial_{x,\xi}^{\alpha,\beta}$ for simplicity. We show by induction on $|\alpha + \beta|$ that

$$\begin{aligned} \partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha,\beta}\lambda_i = &\sum_{2|\mu+\nu|+s\geq 2} C_{\mu,\nu,\gamma^{(j)},\delta^{(j)},s}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^s q(\lambda_i) \\ &\times\big(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i\big) \end{aligned} \tag{6.6}$$

where $\mu + \sum\gamma^{(i)} = \alpha$, $\nu + \sum\delta^{(j)} = \beta$ and $|\gamma^{(i)} + \delta^{(j)}| \geq 1$. The assertion holds for $|\alpha + \beta| = 1$ by (6.5). Suppose that (6.6) holds for $|\alpha + \beta| = m$. With $|e + f| = 1$ operating $\partial_{x,\xi}^{e,f}$ to (6.6) the resulting left-hand side is

$$\begin{aligned} &\partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha+e,\beta+f}\lambda_i + \partial_\lambda^2 q(\lambda_i)(\partial_{x,\xi}^{\alpha,\beta}\lambda_i)(\partial_{x,\xi}^{e,f}\lambda_i) + \partial_{x,\xi}^{e,f}\partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha,\beta}\lambda_i \\ &= \partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha+e,\beta+f}\lambda_i \\ &- \sum_{2|\mu+\nu|+s\geq 2} C_{\mu,\nu,\gamma^{(j)},\delta^{(j)},s}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^s q(\lambda_i)\big(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i\big). \end{aligned}$$

On the other hand, the resulting right-hand side is

$$\begin{aligned} &\sum C_{\ldots}\partial_{x,\xi}^{\mu+e,\nu+f}\partial_\lambda^s q(\lambda_i)\big(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i\big) \\ &+\sum C_{\ldots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s+1} q(\lambda_i)\big(\partial_{x,\xi}^{e,f}\lambda_i\big)\big(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i\big) \\ &+\sum_{j=1}^{s}\sum C_{\ldots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^s q(\lambda_i)\big(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(j)}+e,\delta^{(j)}+f}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i\big) \end{aligned}$$

which can be written as

$$\sum_{2|\mu+\nu|+s\geq 2} C_{\mu,\nu,\gamma^{(j)},\delta^{(j)},s}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^s q(\lambda_i)\big(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i\big)$$

where $\mu + \sum\gamma^{(i)} = \alpha + e$, $\nu + \sum\delta^{(i)} = \beta + f$ and $|\gamma^{(j)} + \delta^{(j)}| \geq 1$. Therefore we conclude (6.6). In order to estimate $\partial_{x,\xi}^{\alpha,\beta}\lambda_i$ one needs to estimate $\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^s q(\lambda_i)$.

Lemma 6.4. *For any $s \in \mathbb{N}$ and α, β it holds that*

$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s q(\lambda_j)| \precsim \sigma^{4-j-(3-j)s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}, \ \ j=1,2$$
$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s q(\lambda_3)| \precsim \sigma^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.$$

Proof. From Proposition 6.1 and (6.4) one sees that

$$|q(\lambda_i)| \precsim |\lambda_i|^2 + |a_M||\lambda_i| + |a_M|^3,$$
$$|\partial_{x,\xi}^{\alpha,\beta} q(\lambda_i)| \precsim \big(|\partial_{x,\xi}^{\alpha,\beta} a_M| + |\partial_{x,\xi}^{\alpha,\beta} b^2|\big)|\lambda_i| + |\partial_{x,\xi}^{\alpha,\beta} a_M^3| + |\partial_{x,\xi}^{\alpha,\beta} b^2| \quad (|\alpha+\beta|\geq 1)$$

because $|\Delta_M| \precsim a_M^3$ and $|b| \precsim a_M^{3/2}$. Therefore thanks to Proposition 6.1 and Lemma 6.2 one obtains the assertions for the case $s=0$. Since

$$|\partial_\lambda q(\lambda_i)| \precsim |\lambda_i| + |a_M|, \qquad |\partial_\lambda^s q(\lambda_i)| \precsim 1, \ \ s\geq 2,$$
$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda q(\lambda_i)| \precsim |\partial_{x,\xi}^{\alpha,\beta} a_M||\lambda_i| + |\partial_{x,\xi}^{\alpha,\beta} a_M| + |\partial_{x,\xi}^{\alpha,\beta} b^2| \quad (|\alpha+\beta|\geq 1),$$
$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^2 q(\lambda_i)| \precsim |\partial_{x,\xi}^{\alpha,\beta} a_M|, \quad \partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s q(\lambda_i) = 0, \ \ s\geq 3 \quad (|\alpha+\beta|\geq 1)$$

the assertions for the case $s\geq 1$ are clear from Proposition 6.1 and Lemma 6.2 again. □

Proof of Lemma 6.3: Since $\partial_\lambda q(\lambda_i) = \prod_{k\neq i}(\lambda_i - \lambda_k)$ it follows from Proposition 6.1 that

$$6a_M(1 - Ca_M) \leq \big|\partial_\lambda q(\lambda_i)\big| \leq 6a_M(1+Ca_M), \ i=1,2, \ \ \partial_\lambda q(\lambda_3) \simeq 1. \tag{6.7}$$

Then for $|\alpha+\beta| = 1$ one has

$$\big|\partial_{x,\xi}^{\alpha,\beta}\lambda_j\big| \precsim \big|\partial_{x,\xi}^{\alpha,\beta} q(\lambda_j)/\partial_\lambda q(\lambda_j)\big| \precsim \sigma^{3-j-1/2}\langle\xi\rangle_\gamma^{-|\beta|}, \ \ j=1,2,3$$

by Lemma 6.4 with $s=0$. Assume that

$$\big|\partial_{x,\xi}^{\alpha,\beta}\lambda_j\big| \precsim \sigma^{3-j-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}, \quad j=1,2,3$$

holds for $|\alpha+\beta|\leq m$. Lemma 6.4 and (6.6) show that

$$\big|\partial_\lambda q(\lambda_1)\partial_{x,\xi}^{\alpha,\beta}\lambda_1\big| \precsim \sum \sigma^{3-2s-|\mu+\nu|/2}\sigma^{2-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{2-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|}$$
$$\precsim \sum \sigma^{3-|\mu+\nu|/2}\sigma^{-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|} \precsim \sigma^{3-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.$$

This together with (6.7) proves the estimate for λ_1. The same arguments show the assertion for λ_2. The estimate for λ_3 is clear from (6.6) because of (6.7). Thus we have the assertion for $|\alpha+\beta| = m+1$ and the proof is completed by induction on $|\alpha+\beta|$. □

Turn to estimate $\partial_t\lambda_i$.

Lemma 6.5. *One has $\partial_t\lambda_1 \in \mathcal{C}(\sigma)$, $\partial_t\lambda_2 \in \mathcal{C}(1)$ and $\partial_t\lambda_3 \in \mathcal{C}(1)$.*

Proof. First examine that $\partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha,\beta}\partial_t\lambda_i$ can be written as

$$
\begin{aligned}
(6.8)\quad &\sum_{|\alpha'+\beta'|<|\alpha+\beta|} C_{\dots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s+1}q(\lambda_i)\big(\partial_{x,\xi}^{\alpha',\beta'}\partial_t\lambda_i\big)\big(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_i\big)\\
&\qquad + \sum C_{\dots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s}\partial_t q(\lambda_i)\big(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_i\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_i\big)
\end{aligned}
$$

where $\alpha'+\mu+\sum\gamma^{(i)}=\alpha$, $\beta'+\nu+\sum\delta^{(i)}=\beta$ and $|\gamma^{(i)}+\delta^{(i)}|\geq 1$. Indeed (6.8) is clear when $|\alpha+\beta|=0$ from

$$
(6.9)\qquad \partial_\lambda q(\lambda_i)\partial_t\lambda_i+\partial_t q(\lambda_i)=0.
$$

Differentiating (6.9) by $\partial_{x,\xi}^{e,f}$ and repeating the same arguments proving (6.6) one obtains (6.8) by induction. To prove Lemma 6.5 first check that

$$
(6.10)\qquad |\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s\partial_t q(\lambda_j)|\precsim\sigma^{3-j-(3-j)s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|},\quad j=1,2,3.
$$

In fact from

$$
(6.11)\quad \partial_t q(\lambda)=-\partial_t(2a_M+a_M^2)\lambda^2+\partial_t(6a_M+2a_M^2+2a_M^3-9b^2)\lambda-\partial_t\Delta_M
$$

it follows that $|\partial_t q(\lambda_i)|\precsim\lambda_i+\sigma^2$ and $|\partial_{x,\xi}^{\alpha,\beta}\partial_t q(\lambda_i)|\precsim(\lambda_i+\sigma^2)\sigma^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta|\geq 1$ in view of Lemma 6.2 and hence the assertion for $s=0$. Since $|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s\partial_t q(\lambda_i)|\precsim\sigma^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$ for $s\geq 1$ the assertion can be proved. We now show Lemma 6.5 for λ_1 by induction on $|\alpha+\beta|$. Assume

$$
(6.12)\qquad \big|\partial_{x,\xi}^{\alpha,\beta}\partial_t\lambda_1\big|\precsim\sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.
$$

It is clear from (6.9), (6.7) and (6.10) that (6.12) holds for $|\alpha+\beta|=0$. Assume that (6.12) holds for $|\alpha+\beta|\leq m$. For $|\alpha+\beta|=m+1$, thanks to the inductive assumption, Lemma 6.4 and Lemma 6.3 it follows that

$$
\begin{aligned}
&\sum_{|\alpha'+\beta'|<|\alpha+\beta|}\big|\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s+1}q(\lambda_1)\big(\partial_{x,\xi}^{\alpha',\beta'}\partial_t\lambda_1\big)\big(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_1\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_1\big)\big|\\
&\precsim\sum\sigma^{3-2(s+1)-|\mu+\nu|/2}\sigma^{1-|\alpha'+\beta'|/2}\sigma^{2-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{2-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|}\\
&\qquad\precsim\sigma^{2-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.
\end{aligned}
$$

On the other hand one sees

$$
\begin{aligned}
&\sum\big|\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s}\partial_t q(\lambda_1)\big(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_1\big)\cdots\big(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_1\big)\big|\\
&\preceq\sum\sigma^{2-2s-|\mu+\nu|/2}\sigma^{2-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{2-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|}\precsim\sigma^{2-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}
\end{aligned}
$$

in view of (6.10) and Lemma 6.3. This proves that (6.12) holds for $|\alpha+\beta|=m+1$ and hence for all α, β. As for λ_2, λ_3 the proof is similar. □

6.3 Eigenvectors of Bézout matrix

We sometimes denote by $\mathcal{C}(\sigma^s)$ a function belonging to $\mathcal{C}(\sigma^s)$. If we write n_{ij} for the (i,j)-cofactor of $\lambda_k I - S$ then ${}^t(n_{j1}, n_{j2}, n_{j3})$ is, if non-trivial, an eigenvector of S corresponding to λ_k. We take $k = 1$, $j = 3$ and hence

$$\begin{bmatrix} a_M(2\,a_M - \lambda_1) \\ 3\,b(\lambda_1 - 3) \\ (\lambda_1 - 3)(\lambda_1 - 2\,a_M) \end{bmatrix} = \begin{bmatrix} \ell_{11} \\ \ell_{21} \\ \ell_{31} \end{bmatrix}$$

is an eigenvector of S corresponding to λ_1 and therefore

$$\mathbf{t}_1 = \begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \end{bmatrix} = \frac{1}{d_1} \begin{bmatrix} \ell_{11} \\ \ell_{21} \\ \ell_{31} \end{bmatrix}, \quad d_1 = \sqrt{\ell_{11}^2 + \ell_{21}^2 + \ell_{31}^2}$$

is a unit eigenvector of S corresponding to λ_1. Thanks to Proposition 6.1 and recalling $b \in \mathcal{C}(\sigma^{3/2})$ it is clear that

$$d_1 = \sqrt{36\,a_M^2 + \mathcal{C}(\sigma^3)} = 6a_M\sqrt{1 + \mathcal{C}(\sigma)} = 6a_M(1 + \mathcal{C}(\sigma)).$$

Therefore since $\ell_{11} = \mathcal{C}(\sigma^2)$, $\ell_{21} = \mathcal{C}(\sigma^{3/2})$ and $\ell_{31} = 6\,a + \mathcal{C}(\sigma^2)$ we have

$$\mathbf{t}_1 = \begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \end{bmatrix} = \begin{bmatrix} a_M/3 + \mathcal{C}(\sigma^2) \\ -3b/(2a_M) + \mathcal{C}(\sigma) \\ 1 + \mathcal{C}(\sigma) \end{bmatrix}.$$

Similarly choosing $k = 2, j = 2$ and $k = 3, j = 1$

$$\begin{bmatrix} -3a_M b \\ (\lambda_2 - 3)(\lambda_2 - a_M^2) - a_M^2 \\ 3b(\lambda_2 - 3) \end{bmatrix} = \begin{bmatrix} \ell_{12} \\ \ell_{22} \\ \ell_{32} \end{bmatrix}, \quad \begin{bmatrix} (\lambda_3 - 2a_M)(\lambda_3 - a_M^2) - 9b^2 \\ -3a_M b \\ -a_M(\lambda_3 - 2a_M) \end{bmatrix} = \begin{bmatrix} \ell_{13} \\ \ell_{23} \\ \ell_{33} \end{bmatrix}$$

are eigenvectors of S corresponding to λ_2 and λ_3 respectively and

$$\mathbf{t}_j = \begin{bmatrix} t_{1j} \\ t_{2j} \\ t_{3j} \end{bmatrix} = \frac{1}{d_j} \begin{bmatrix} \ell_{1j} \\ \ell_{2j} \\ \ell_{3j} \end{bmatrix}, \quad d_j = \sqrt{\ell_{1j}^2 + \ell_{2j}^2 + \ell_{3j}^2}$$

are unit eigenvectors of S corresponding to λ_j, $j = 2, 3$. Thanks to Proposition 6.1 it is easy to check

$$d_2 = 3\lambda_2(1 + \mathcal{C}(\sigma)), \qquad d_3 = \lambda_3^2(1 + \mathcal{C}(\sigma)).$$

Then repeating the same arguments one concludes

$$\begin{bmatrix} t_{12} \\ t_{22} \\ t_{32} \end{bmatrix} = \begin{bmatrix} \mathcal{C}(\sigma^{3/2}) \\ -1 + \mathcal{C}(\sigma) \\ -3b/\lambda_2 + \mathcal{C}(\sigma) \end{bmatrix}, \qquad \begin{bmatrix} t_{13} \\ t_{23} \\ t_{33} \end{bmatrix} = \begin{bmatrix} 1 + \mathcal{C}(\sigma) \\ \mathcal{C}(\sigma^{5/2}) \\ -a_M/\lambda_3 + \mathcal{C}(\sigma^2) \end{bmatrix}.$$

Now $T = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) = (t_{ij})$ is an orthogonal matrix which diagonalizes S;

$$\Lambda = T^{-1}ST = {}^tTST = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Note that Λ symmetrizes $A^T = T^{-1}AT$;

$${}^t(\Lambda A^T) = {}^t({}^tTSAT) = {}^tT\,{}^t(SA)T = {}^tTSAT = \Lambda A^T.$$

Summarize what we have proved above in

Lemma 6.6. *Let T be defined as above. Then there is M_0 such that T has the form*

$$T = \begin{bmatrix} a_M/3 + \mathcal{C}(\sigma^2) & \mathcal{C}(\sigma^{3/2}) & 1 + \mathcal{C}(\sigma) \\ -3b/(2a_M) + \mathcal{C}(\sigma) & -1 + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{5/2}) \\ 1 + \mathcal{C}(\sigma) & -3b/\lambda_2 + \mathcal{C}(\sigma) & -a_M/\lambda_3 + \mathcal{C}(\sigma^2) \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{3/2}) & 1 + \mathcal{C}(\sigma) \\ \mathcal{C}(\sigma^{1/2}) & -1 + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{5/2}) \\ 1 + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{1/2}) & \mathcal{C}(\sigma) \end{bmatrix}$$

for $M \geq M_0$. In particular $T,\ T^{-1} \in S(1, g)$.

Lemma 6.7. *We have*

$$\partial_t T = \begin{bmatrix} \partial_t(a_M/3) + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{1/2}) & \mathcal{C}(1) \\ -\partial_t(3b/2a_M) + \mathcal{C}(1) & \mathcal{C}(1) & \mathcal{C}(\sigma^{3/2}) \\ \mathcal{C}(1) & -\partial_t(3b/\lambda_2) + \mathcal{C}(1) & -\partial_t(a_M/\lambda_3) + \mathcal{C}(\sigma) \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{C}(1) & \mathcal{C}(\sigma^{1/2}) & \mathcal{C}(1) \\ \mathcal{C}(\sigma^{-1/2}) & \mathcal{C}(1) & \mathcal{C}(\sigma^{3/2}) \\ \mathcal{C}(1) & \mathcal{C}(\sigma^{-1/2}) & \mathcal{C}(1) \end{bmatrix}.$$

Proof. Note that every entry of T is a function in a_M, b and λ_j. Then the assertion is clear from Lemmas 6.2 and 6.5. □

From Lemma 6.6 it follows that

$$\langle \xi \rangle_\gamma^{|\beta|} \partial_x^\alpha \partial_\xi^\beta T = \begin{bmatrix} \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) \\ \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(\sigma^2) \\ \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) \end{bmatrix}, \quad |\alpha + \beta| = 1. \tag{6.13}$$

Lemma 6.8. *There is M_0 such that $A^T = T^{-1}AT$ has the form*

$$A^T = \begin{bmatrix} \mathcal{C}(\sqrt{\sigma}) & -1 + \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) \\ \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) & -1 + \mathcal{C}(\sigma) \\ \mathcal{C}(\sigma^{3/2}) & \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{5/2}) \end{bmatrix}, \quad M \geq M_0.$$

Proof. Writing $A^T = (\tilde{a}_{ij})$ it is clear that

$$\tilde{a}_{ij} = t_{1i}\, a_M\, t_{2j} + t_{1i}\, b\, t_{3j} + t_{2i} t_{1j} + t_{3i} t_{2j}.$$

Then the assertion follows from Lemma 6.6. □

Corollary 6.3. *Let $A^T = (\tilde{a}_{ij})$. Then*

$$\tilde{a}_{31} = \lambda_1\, \mathcal{C}(\sqrt{\sigma}), \quad \tilde{a}_{32} = \lambda_2\, \mathcal{C}(1), \quad \tilde{a}_{21} = \lambda_1\, \mathcal{C}(\sigma^{-1}).$$

Proof. Lemma 6.8 gives

$$\Lambda A^T = \begin{bmatrix} \lambda_1 \mathcal{C}(\sqrt{\sigma}) & \lambda_1(-1+\mathcal{C}(\sigma)) & \lambda_1 \mathcal{C}(\sqrt{\sigma}) \\ \lambda_2 \tilde{a}_{21} & \lambda_2 \tilde{a}_{22} & \lambda_2(-1+\mathcal{C}(\sigma)) \\ \lambda_3 \tilde{a}_{31} & \lambda_3 \tilde{a}_{32} & \lambda_3 \tilde{a}_{33} \end{bmatrix}.$$

Since ΛA^T is symmetric it follows immediately

$$\tilde{a}_{31} = \frac{\lambda_1 \mathcal{C}(\sqrt{\sigma})}{\lambda_3}, \quad \tilde{a}_{32} = \frac{\lambda_2(-1+\mathcal{C}(\sigma))}{\lambda_3}, \quad \tilde{a}_{21} = \frac{\lambda_1(-1+\mathcal{C}(\sqrt{\sigma}))}{\lambda_2}.$$

This proves the assertion because $1/\lambda_3 \in \mathcal{C}(1)$ and $1/\lambda_2 \in \mathcal{C}(\sigma^{-1})$. □

From Corollary 6.3 one can improve Lemma 6.8 such that $\tilde{a}_{31} = \mathcal{C}(\sigma^{5/2})$ for $\lambda_1 \in \mathcal{C}(\sigma^2)$.

Corollary 6.4. *We have*

$$\langle \xi \rangle_\gamma^{|\beta|} \partial_x^\alpha \partial_\xi^\beta A^T = \begin{bmatrix} \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(1) \\ \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) \\ \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(\sigma^2) \end{bmatrix}, \quad |\alpha + \beta| = 1.$$

Proof. The proof is clear since $\langle \xi \rangle_\gamma^{|\beta|} \partial_x^\alpha \partial_\xi^\beta (-1 + \mathcal{C}(\sigma)) = \mathcal{C}(\sqrt{\sigma})$. □

Before closing the section we consider $T^{-1}(\partial_t T)$. Note that

$$(\partial_t T^{-1}) T = (\partial_t ({}^t T)) T = (\langle \partial_t \mathbf{t}_i, \mathbf{t}_j \rangle)$$

and $\langle \partial_t \mathbf{t}_i, \mathbf{t}_j \rangle = -\langle \mathbf{t}_i, \partial_t \mathbf{t}_j \rangle = -\langle \partial_t \mathbf{t}_j, \mathbf{t}_i \rangle$ so that $(\partial_t T^{-1}) T$ is antisymmetric. From Lemmas 6.6 and 6.7 one has

$$T^{-1}(\partial_t T) = \begin{bmatrix} 0 & -\partial_t(3b/2a_M) + \mathcal{C}(1) & \mathcal{C}(1) \\ \partial_t(3b/2a_M) + \mathcal{C}(1) & 0 & \mathcal{C}(\sqrt{\sigma}) \\ \partial_t(a_M/3) + \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) & 0 \end{bmatrix}. \tag{6.14}$$

For later use we estimate $(2,1)$-th and $(3,1)$-th entries of $T^{-1}(\partial_t T)$. Recalling $a_M = e(t + \alpha + 2M\langle \xi \rangle_\gamma^{-1})$ and $0 \le t \le M^{-4}$ it is clear

$$\partial_t a_M - \bar{e} \in S(M^{-2}, g). \tag{6.15}$$

Taking $|b^2/a_M^3| \le 4/27$ into account, thanks to Lemma 4.7 it follows that

$$\begin{aligned} \big|\sqrt{a_M}\,\partial_t(3b/2a_M)\big| &\le 3\big(|\partial_t b/\sqrt{a_M}| + |b/a_M^{3/2}||\partial_t a_M|\big)/2 \\ &\le (1 + CM^{-2})\big((1+3\sqrt{2})/\sqrt{3}\big)\,\bar{e}. \end{aligned} \tag{6.16}$$

7 ϕ and λ_j are admissible weights for g

Write $z = (x, \xi)$ and $w = (y, \eta)$. It is clear that

$$g^{\sigma}_z(dx, d\xi) = M\big(\langle\xi\rangle_\gamma |dx|^2 + \langle\xi\rangle_\gamma^{-1}|d\xi|^2\big) = M^2 g_z(dx, d\xi).$$

Note that $|\xi - \eta| \le c\,\langle\xi\rangle_\gamma$ with $0 < c < 1$ implies

$$(1-c)\langle\xi\rangle_\gamma/\sqrt{2} \le \langle\eta\rangle_\gamma \le \sqrt{2}\,(1+c)\langle\xi\rangle_\gamma.$$

If $g_z(w) < c$ then $|\xi - \eta|^2 < c\,M\langle\xi\rangle_\gamma = c\,M\langle\xi\rangle_\gamma^{-1}\langle\xi\rangle_\gamma^2 \le c\,\langle\xi\rangle_\gamma^2$ then

$$g_z(dx, d\xi)/C \le g_w(dx, d\xi) \le C g_z(dx, d\xi)$$

with C independent of $\gamma \ge M^5 \ge 1$ that is g_z is slowly varying uniformly in $\gamma \ge M^5 \ge 1$. Similarly noting that $|\xi - \eta| \ge (\gamma + |\xi|)/2 \ge \langle\xi\rangle_\gamma/2$ if $\langle\eta\rangle_\gamma \le \langle\xi\rangle_\gamma/2\sqrt{2}$ and $|\xi - \eta| \ge (\gamma + |\eta|)/2 \ge \langle\eta\rangle_\gamma/2$ if $\langle\eta\rangle_\gamma \ge 2\sqrt{2}\langle\xi\rangle_\gamma$ it is clear that

$$\frac{\langle\xi\rangle_\gamma}{\langle\eta\rangle_\gamma} + \frac{\langle\eta\rangle_\gamma}{\langle\xi\rangle_\gamma} \le C\big(1 + \langle\eta\rangle_\gamma^{-1}|\xi - \eta|^2\big) \le C\big(1 + g^{\sigma}_w(z - w)\big) \tag{7.1}$$

that is g is temperate uniformly in $\gamma \ge 0$ and $M \ge 1$ (see [6, Chapter 18.5]). Therefore g is an *admissible metric*. It is clear from (7.1) that

$$g^{\sigma}_z(z - w) \le C\big(1 + g^{\sigma}_w(z - w)\big)^2. \tag{7.2}$$

7.1 ρ and σ are admissible weights for g

We adapt the same convention as in Sections 5, 6 even to weights for g so that we omit to say uniformly in $t \in [0, M^{-4}]$.

Lemma 7.1. *ρ is an admissible weight for g.*

Proof. First study $\rho^{1/2}$. Assume

$$g_z(w) = M^{-1}\langle\xi\rangle_\gamma\big(|y|^2 + \langle\xi\rangle_\gamma^{-2}|\eta|^2\big) < c\ (< 1/2)$$

so that $M^{-1}\langle\xi\rangle_\gamma^{-1}|\eta|^2 < c$ hence $|\eta| < c\langle\xi\rangle_\gamma$ for $M\langle\xi\rangle_\gamma^{-1} \le 1$. Thus $\langle\xi + s\eta\rangle_\gamma^{-1} \le C\langle\xi\rangle_\gamma^{-1}$ ($|s| < 1$) and Lemma 4.3 shows

$$|\rho^{1/2}(z + w) - \rho^{1/2}(z)| \le C\big(|y| + \langle\xi + s\eta\rangle_\gamma^{-1}|\eta|\big) \le CM^{1/2}\langle\xi\rangle_\gamma^{-1/2} g_z^{1/2}(w).$$

Since $\rho(z) \ge M\langle\xi\rangle_\gamma^{-1}$ this yields

$$|\rho^{1/2}(z + w) - \rho^{1/2}(z)| \le C\rho^{1/2}(z) g_z^{1/2}(w). \tag{7.3}$$

Choosing c such that $C\,c < 1/2$ one has $\big|\rho(z + w)/\rho(z) - 1\big| < 1/2$ hence

$$\rho^{1/2}(z + w)/2 \le \rho^{1/2}(z) \le 3\,\rho^{1/2}(z + w)/2$$

that is $\rho^{1/2}$ is g continuous hence so is ρ. Note that

$$M\langle\xi\rangle_\gamma^{-1} \le \rho(z) \le CM^{-4} \le C. \tag{7.4}$$

If $|\eta| \ge c\,\langle\xi\rangle_\gamma/2$ then $g_z^\sigma(w) \ge Mc^2\langle\xi\rangle_\gamma/4$ and $g_z^\sigma(w) \ge Mc|\eta|/2$ thus

$$\rho(z+w) \le C \le C\langle\xi\rangle_\gamma \rho(z) \le C'\rho(z)(1+g_z^\sigma(w)).$$

If $|\eta| \le c\langle\xi\rangle_\gamma$ then (7.3) gives

$$\rho^{1/2}(z+w) \le C\rho^{1/2}(z)(1+g_z(w))^{1/2} \le C\rho^{1/2}(z)(1+g_z^\sigma(w))^{1/2} \tag{7.5}$$

so that

$$\sigma(t,z+w) \le C\,\sigma(t,z)(1+g_z^\sigma(w)) \tag{7.6}$$

hence ρ is g temperate in view of (7.2). Thus ρ is an admissible weight. □

Lemma 7.2. *σ is an admissible weight for g and $\sigma \in S(\sigma, g)$.*

Proof. Since $\sigma(t,z) = t + \rho(z) + M\langle\xi\rangle_\gamma^{-1}$ and $\rho(z) + M\langle\xi\rangle_\gamma^{-1}$ is admissible for g by Lemma 7.1 it is clear that σ is admissible for g. The second assertion is clear from

$$\begin{aligned}\left|\partial_x^\alpha \partial_\xi^\beta \sigma\right| &\precsim \sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|} \precsim \sigma(M^{-1}\langle\xi\rangle_\gamma)^{|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \sigma\, M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}.\end{aligned}$$

for $\sigma \ge M\langle\xi\rangle_\gamma^{-1}$. □

7.2 ω and ϕ are admissible weights for g

We start with showing

Lemma 7.3. *ω and ϕ are g continuous.*

Proof. Denote $f = t - \psi$ and $h = M^{1/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ so that $\omega^2 = f^2 + h^2$. Note that

$$\begin{aligned}|\omega(z+w) - \omega(z)| &= \frac{|\omega^2(z+w) - \omega^2(z)|}{|\omega(z+w) + \omega(z)|}\\ &\le 2|f(z+w) - f(z)| + 2|h(z+w) - h(z)|\end{aligned} \tag{7.7}$$

because

$$\frac{|f(z+w) + f(z)|}{|\omega(z+w) + \omega(z)|} \le 2, \quad \frac{|h(z+w) + h(z)|}{|\omega(z+w) + \omega(z)|} \le 2.$$

Assume $g_z(w) < c$ $(\le 1/2)$ which implies $|\eta| < \sqrt{c}\langle\xi\rangle_\gamma$ for $M\langle\xi\rangle_\gamma^{-1} \le 1$ hence

$$\langle\xi + s\eta\rangle_\gamma/C \le \langle\xi\rangle_\gamma \le C\langle\xi + s\eta\rangle_\gamma \tag{7.8}$$

where C is independent of $|s| \le 1$. It is assumed that constants C may change from line to line but independent of $\gamma \ge M^5 \ge 1$. Noting $|f(z+w) - f(z)| = |\psi(z+w) - \psi(z)|$ it follows from Lemma 5.1 that

$$
\begin{aligned}
|f(z+w) - f(z)| &\le C\rho^{1/2}(z+sw)\big(|y| + \langle \xi + s\eta\rangle_\gamma^{-1}|\eta|^2\big) \\
\le C\rho^{1/2}(z+sw)\big(|y| + \langle\xi\rangle_\gamma^{-1}|\eta|\big) &\le CM^{1/2}\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}g_z^{1/2}(w)
\end{aligned}
\tag{7.9}
$$

since ρ is g continuous. Noting that $\omega(z) \ge M^{1/2}\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}$ it results

$$
|f(z+w) - f(z)| \le C\omega(z)g_z^{1/2}(w). \tag{7.10}
$$

Similar arguments shows that $|h(z+w) - h(z)| \le CM^{1/2}\langle\xi\rangle_\gamma^{-1}g_z^{1/2}(w)$. Taking $\omega(z) \ge M^{1/2}\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2} \ge M\langle\xi\rangle_\gamma^{-1}$ into account we have

$$
|h(z+w) - h(z)| \le CM^{-1/2}\omega(z)g_z^{1/2}(w).
$$

Therefore from (7.7) one has $|\omega(z+w) - \omega(z)| \le C\omega(z)g_z^{1/2}(w)$. Choosing c such that $C\,c < 1/2$ we conclude that ω is g continuous.

Next consider ϕ. Since $\phi = \omega + f$ one can write

$$
\begin{aligned}
&\phi(z+w) - \phi(z) \\
&= \frac{(f(z+w) - f(z))(\phi(z+w) + \phi(z)) + h^2(z+w) - h^2(z)}{\omega(z+w) + \omega(z)}.
\end{aligned}
\tag{7.11}
$$

Since ω is g continuous, decreasing $c > 0$ if necessary, one has

$$
\omega(z+w)/C \le \omega(z) \le C\,\omega(z+w)
$$

which together with (7.10) gives

$$
|f(z+w) - f(z)|/(\omega(z+w) + \omega(z)) \le Cg_z^{1/2}(w).
$$

Recalling $h^2(z) = M\rho(z)\langle\xi\rangle_\gamma^{-1}$ and repeating similar arguments as above one sees

$$
\begin{aligned}
|h^2(z+w) - h^2(z)| &\le CM\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-3/2}g_z^{1/2}(w) \\
&\le CM^{1/2}\rho(z)\langle\xi\rangle_\gamma^{-1}g_z^{1/2}(w)
\end{aligned}
\tag{7.12}
$$

for $\rho^{1/2}(z) \ge M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. Taking (5.1) into account it follows from (7.12) that

$$
|h^2(z+w) - h^2(z)|/(\omega(z+w) + \omega(z)) \le C\phi(z)g_z^{1/2}(w).
$$

Combining these estimates we obtain from (7.11) that

$$
\Big|\frac{\phi(z+w)}{\phi(z)} - 1\Big| \le C\Big|\frac{\phi(z+w)}{\phi(z)} + 1\Big|g_z^{1/2}(w) + Cg_z^{1/2}(w)
$$

which proves $\phi(z)/C \le \phi(z+w) \le C\,\phi(z)$ choosing $c > 0$ small. Then we conclude that ϕ is g continuous. □

Lemma 7.4. *ω and ϕ are admissible weights for g and $\omega \in S(\omega, g)$, $\phi \in S(\phi, g)$.*

Proof. Note that

$$\langle\xi\rangle_\gamma^{-1} \le M\langle\xi\rangle_\gamma^{-1} \le \sqrt{M}\sqrt{\rho}\langle\xi\rangle_\gamma^{-1/2} \le \omega \le CM^{-4} \le C. \tag{7.13}$$

Assume $|\eta| \ge c\,\langle\xi\rangle_\gamma$ hence $g_z^\sigma(w) \ge Mc^2\langle\xi\rangle_\gamma \ge c^2\langle\xi\rangle_\gamma$. Therefore

$$\omega(z+w) \le C \le C\langle\xi\rangle_\gamma\omega(z) \le C'\omega(z)(1+g_z^\sigma(w)). \tag{7.14}$$

Assume $|\eta| \le c\,\langle\xi\rangle_\gamma$ and note that (7.5) holds provided $|\eta| \le c\langle\xi\rangle_\gamma$. Then checking the proof of Lemma 7.3 we see that $|f(z+w)-f(z)| \le C\omega(z)(1+g_z^\sigma(w))$ and $|h(z+w)-h(z)| \le C\omega(z)(1+g_z^\sigma(w))^{1/2}$. Then (7.14) follows from (7.7) which proves that ω is g temperate hence admissible for g.

Turn to ϕ. From (5.1) and (7.4), (7.13) it follows that

$$\langle\xi\rangle_\gamma^{-2}/C \le M^6\langle\xi\rangle_\gamma^{-2}/C \le \phi(z) = \omega(z) + f(z) \le CM^{-4} \le C.$$

If $|\eta| \ge \langle\xi\rangle_\gamma/2$ then $g_z^\sigma(w) \ge M\langle\xi\rangle_\gamma/4 \ge \langle\xi\rangle_\gamma/4$ hence

$$\phi(z+w) \le C \le C^2\langle\xi\rangle_\gamma^2\phi(z) \le C\phi(z)(1+g_z^\sigma(w))^2.$$

Assume $|\eta| \le \langle\xi\rangle_\gamma/2$ so that (7.8) holds. From (7.5) and (7.9) it results that

$$|f(z+w) - f(z)| \le C\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}(1+g_z^\sigma(w)).$$

Recalling (7.5) and $M^2 g_z(w) = g_z^\sigma(w)$ the same arguments obtaining (7.12) shows that

$$|h^2(z+w) - h^2(z)| \le C\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-3/2}(1+g_z^\sigma(w)).$$

Taking these into account (7.11) yeilds

$$\begin{aligned} |\phi(z+w)-\phi(z)| \le C\Big(&\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}\big(\phi(z+w)+\phi(z)\big) \\ &+\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-3/2}}{\omega(z+w)+\omega(z)}\Big)(1+g_z^\sigma(w)). \end{aligned} \tag{7.15}$$

Applying Lemma 5.3 to (7.15) to obtain

$$\begin{aligned} |\phi(z+w)-\phi(z)| &\le C\Big(\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}\big(\phi(z+w)+\phi(z)\big) \\ &\qquad+\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}\phi(z)\Big)(1+g_z^\sigma(w)) \\ &= C\big(\phi(z+w)+2\phi(z)\big)\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(1+g_z^\sigma(w)). \end{aligned}$$

If $\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}(1+g_z^\sigma(w))/(\omega(z+w)+\omega(z))<\varepsilon$ then it follows

$$\Big|\frac{\phi(z+w)}{\phi(z)}-1\Big|\leq\varepsilon\Big(\frac{\phi(z+w)}{\phi(z)}+2\Big)$$

from which we have $\phi(z+w)/C\leq\phi(z)\leq C\,\phi(z+w)$. If

$$\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(1+g_z^\sigma(w))\geq\varepsilon$$

we have

$$\varepsilon^{-2}(1+g_z^\sigma(w))^2\geq\frac{2\langle\xi\rangle_\gamma}{\rho(z)}\omega(z+w)\omega(z)\geq\phi(z+w)\frac{1}{2\,\phi(z)}$$

by (5.1) and an obvious inequality $\phi(z+w)\leq 2\,\omega(z+w)$. Thus we conclude that ϕ is g temperate hence ϕ is an admissible weight for g. □

7.3 λ_j are admissible weights for g

Lemma 7.5. *Assume that $\lambda\in\mathcal{C}(\sigma^2)$ and $\lambda\geq cM\sigma\langle\xi\rangle_\gamma^{-1}$ with some $c>0$. Then λ is an admissible weight for g.*

Proof. Consider $\sqrt{\lambda}$. Assume $g_z(w)<c$ and hence $\langle\xi+s\eta\rangle_\gamma\approx\langle\xi\rangle_\gamma$. Since $\sqrt{\lambda}\in\mathcal{C}(\sigma)$ it follows that

$$\begin{aligned}\big|\sqrt{\lambda(z+w)}-\sqrt{\lambda(z)}\big|&\leq C\sqrt{\sigma(z+sw)}\big(|y|+\langle\xi+s\eta\rangle_\gamma^{-1}|\eta|\big)\\&\leq C\sqrt{\sigma(z+sw)}\langle\xi\rangle_\gamma^{-1/2}g_z^{1/2}(w)\end{aligned}\tag{7.16}$$

which is bounded by $C'\sqrt{\sigma(z)}\langle\xi\rangle_\gamma^{-1/2}g_z^{1/2}(w)$ since σ is g continuous. By assumption $\lambda(z)\geq cM\sigma(z)\langle\xi\rangle_\gamma^{-1}$ one has

$$\big|\sqrt{\lambda(z+w)}-\sqrt{\lambda(z)}\big|\leq C''M^{-1/2}\sqrt{\lambda(z)}\,g_z^{1/2}(w)\leq C''\sqrt{\lambda(z)}\,g_z^{1/2}(w).$$

Choosing $c>0$ such that $C''\sqrt{c}<1$ shows that $\sqrt{\lambda(z)}$ is g continuous and so is $\lambda(z)$. From $cM^2\langle\xi\rangle_\gamma^{-2}\leq cM\sigma\langle\xi\rangle_\gamma^{-1}\leq\lambda\leq C'\sigma^2\leq C'M^{-4}$ one sees

$$c_1M\langle\xi\rangle_\gamma^{-1}\leq c_1M^{1/2}\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}\leq\sqrt{\lambda(z)}\leq C.$$

If $|\eta|\geq\langle\xi\rangle_\gamma/2$ hence $g_z^\sigma(w)\geq M\langle\xi\rangle_\gamma/4$ then

$$\sqrt{\lambda(z+w)}\leq C\leq C(c_1M)^{-1}\langle\xi\rangle_\gamma\sqrt{\lambda(z)}\leq C'\sqrt{\lambda(z)}\,g_z^\sigma(w).$$

If $|\eta|\leq\langle\xi\rangle_\gamma/2$ it follows from (7.16) and (7.6)

$$\begin{aligned}\big|\sqrt{\lambda(z+w)}-\sqrt{\lambda(z)}\big|&\leq C\sqrt{\sigma(z)}\langle\xi\rangle_\gamma^{-1/2}(1+g_z^\sigma(w))\\&\leq C'\sqrt{\lambda(z)}(1+g_z^\sigma(w))\end{aligned}$$

which proves that $\sqrt{\lambda}$ is g temperate and hence so is λ. □

Lemma 7.6. *Assume that $\lambda \in \mathcal{C}(\sigma)$ and $\lambda \geq cM\langle\xi\rangle_\gamma^{-1}$ with some $c > 0$. Then λ is an admissible weight for g. If $\lambda \in \mathcal{C}(1)$ and $\lambda \geq c$ with some $c > 0$ then λ is an admissible weight for g.*

Proof. It is enough to repeat the proof of Lemma 7.5. □

Lemma 7.7. *Assume that $\lambda \in \mathcal{C}(\sigma^2)$ and $\lambda \geq cM\sigma\langle\xi\rangle_\gamma^{-1}$ with some $c > 0$. Then*

$$\partial_x^\alpha \partial_\xi^\beta \lambda \in S(\sqrt{\sigma}\sqrt{\lambda}\langle\xi\rangle_\gamma^{-|\beta|}, g), \quad |\alpha+\beta| = 1.$$

In particular $\lambda \in S(\lambda, g)$.

Proof. From $\lambda \in \mathcal{C}(\sigma^2)$ we have $|\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta\lambda| \leq C\,\sigma$ for $|\alpha+\beta| = 2$. Since $\lambda \geq 0$, thanks to the Glaeser's inequality one has

$$\big|\partial_x^\alpha\partial_\xi^\beta\lambda\big| \leq C'\,\sqrt{\sigma}\,\sqrt{\lambda}\,\langle\xi\rangle_\gamma^{-|\beta|}, \quad |\alpha+\beta| = 1.$$

For $|\alpha'+\beta'| \geq 1$ note that

$$\begin{aligned}\big|\partial_x^{\alpha'}\partial_{\xi'}^{\beta'}\big(\partial_x^\alpha\partial_\xi^\beta\lambda\big)\big| &\precsim \sigma^{3/2-|\alpha'+\beta'|/2}\langle\xi\rangle_\gamma^{-|\beta|-|\beta'|} \precsim \sigma^{1-(|\alpha'+\beta'|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|}\langle\xi\rangle_\gamma^{-|\beta'|}\\ &\precsim \sigma(M^{-1}\langle\xi\rangle_\gamma)^{(|\alpha'+\beta'|-1)/2}\langle\xi\rangle_\gamma^{-|\beta'|}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \sigma M^{-|\alpha'+\beta'|/2}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \sqrt{\sigma}M^{-|\alpha'+\beta'|/2}\sqrt{\lambda}\,\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}\langle\xi\rangle_\gamma^{-|\beta|}\end{aligned}$$

because $\lambda \geq cM\sigma\langle\xi\rangle_\gamma^{-1}$ which proves the first assertion. Noting

$$\sqrt{\sigma}\langle\xi\rangle_\gamma^{-|\beta|} = \sqrt{\sigma}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \leq CM^{-1/2}\sqrt{\lambda}\,\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$$

it is clear that $\lambda \in S(\lambda, g)$. □

Lemma 7.8. *Assume that $\lambda \in \mathcal{C}(\sigma)$ and $\lambda \geq cM\langle\xi\rangle_\gamma^{-1}$ with some $c > 0$. Then $\lambda \in S(\lambda, g)$. If $\lambda \in \mathcal{C}(1)$ and $\lambda \geq c$ with some $c > 0$. Then $\lambda \in S(\lambda, g)$.*

Proof. It suffices to repeat the proof of Lemma 7.7. □

Corollary 7.1. *For $s \in \mathbb{R}$ we have $\lambda_j^s \in S(\lambda_j^s, g)$, $j = 1, 2, 3$.*

Define

$$\kappa = \frac{1}{t} + \frac{1}{\omega} = \frac{t+\omega}{t\,\omega}, \qquad (t > 0).$$

Lemma 7.9. *κ is an admissible weight for g and $\kappa^s \in S(\kappa^s, g)$ for $s \in \mathbb{R}$.*

Proof. Since ω^{-1} is g continuous and g temperate it is clear that $\kappa = t^{-1} + \omega^{-1}$ is g continuous and g temperate. Noting that $\omega^{-1} \in S(\omega^{-1}, g)$ and $\omega^{-1} \leq \kappa$ it is also clear that

$$\begin{aligned}\big|\partial_x^\alpha\partial_\xi^\beta\kappa\big| = \big|\partial_x^\alpha\partial_\xi^\beta\omega^{-1}\big| &\precsim M^{-|\alpha+\beta|/2}\omega^{-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\\ &\precsim M^{-|\alpha+\beta|/2}\kappa\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \quad |\alpha+\beta| \geq 1\end{aligned}$$

which proves $\kappa \in S(\kappa, g)$. □

Lemma 7.10. *One has*

$$\partial_x^\alpha \partial_\xi^\beta \kappa^s \in S\big(M^{-(|\alpha+\beta|-1)/2}\kappa^s\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2+(|\alpha|-\beta|)/2}, g\big), \;\; |\alpha+\beta|\geq 1.$$

Proof. Since $\partial_x^\alpha \partial_\xi^\beta \kappa^s = \kappa^{s-1}\partial_x^\alpha \partial_\xi^\beta \kappa$ it is enough to show the case $s=1$. The proof of the case $s=1$ follows easily from Corollary 5.3. □

Lemma 7.11. *There is $C>0$ such that*

$$\frac{1}{\kappa\lambda_1}\leq \frac{3}{\bar{e}^2\,\upsilon}(1+CM^{-4})\kappa, \qquad \frac{1}{\sigma^2\kappa}\leq \kappa.$$

Proof. In view of Propositions 4.1 and 6.1 one sees

$$\lambda_1 \geq \frac{\bar{e}^2}{3}\upsilon(1-CM^{-4})\min\big\{t^2,\omega^2\big\}.$$

Denote $c=3/(\bar{e}^2\,\upsilon(1-CM^{-4}))=(3/\bar{e}^2\,\upsilon)(1+CM^{-4})$. If $\omega^2\geq t^2$ and hence $\lambda_1\geq t^2/c$ then $1/\lambda_1\leq c/t^2$ which shows that

$$\frac{1}{\kappa\lambda_1}\leq \frac{c}{\kappa t^2}=\frac{c\,t\,\omega}{(t+\omega)t^2}=\frac{c\,\omega}{(t+\omega)t}\leq \frac{c(t+\omega)}{t\omega}=c\kappa.$$

If $t^2\geq\omega^2$ and hence $\lambda_1\geq\omega^2/c$ then $1/\lambda_1\leq c/\omega^2$ and hence

$$\frac{1}{\kappa\lambda_1}\leq \frac{c}{\kappa\omega^2}=\frac{c\,t\,\omega}{(t+\omega)\omega^2}=\frac{c\,t}{(t+\omega)\omega}\leq \frac{c(t+\omega)}{t\,\omega}=c\kappa$$

then the first assertion. To show the second assertion it suffices to note $\sigma\geq t$ and then $\sigma^2(t+\omega)^2\geq t^2(t+\omega)^2\geq t^2\omega^2$. □

8 Lower bounds of $\mathrm{op}(\lambda_i)$

8.1 Some preliminary lemmas

Introduce a metric independent of M

$$\bar{g}=\langle\xi\rangle_\gamma|dx|^2+\langle\xi\rangle_\gamma^{-1}|d\xi|^2$$

so that $g=M^{-1}\,\bar{g}$. We start with

Lemma 8.1. *Let m be an admissible weight for g and $p\in S(m,g)$ satisfy $p\geq c\,m$ with some constant $c>0$. Then $p^{-1}\in S(m^{-1},g)$ and there exist $k,\tilde{k}\in S(M^{-1},g)$ such that*

$$p\#p^{-1}\#(1+k)=1,\; (1+k)\#p\#p^{-1}=1,\; p^{-1}\#(1+k)\#p=1,$$
$$p^{-1}\#p\#(1+\tilde{k})=1,\; (1+\tilde{k})\#p^{-1}\#p=1,\; p\#(1+\tilde{k})\#p^{-1}=1.$$

Proof. It is clear that $p^{-1} \in S(m^{-1}, g)$. Write $p\#p^{-1} = 1 - r$ where $r \in S(M^{-1}, g)$. Fix any M. Since

$$|r|_{S(1,\bar{g})}^{(l)} = \sup_{|\alpha+\beta|\le l,(x,\xi)\in\mathbb{R}^{2d}} \left|\langle\xi\rangle_\gamma^{(|\beta|-|\alpha|)/2}\partial_x^\alpha\partial_\xi^\beta r\right| \le C_l M^{-1}$$

from the Calderón-Vaillancourt theorem we have $\|\mathrm{op}(r)\| \le CM^{-1}$. Therefore for large M there exists the inverse $(1-\mathrm{op}(r))^{-1}$ which is given by $1+\sum_{\ell=1}^\infty r^{\#\ell} \in S(1,\bar{g})$. (see [1]). Denote $k = \sum_{\ell=1}^\infty r^{\#\ell} \in S(1,\bar{g})$ and prove $k \in S(M^{-1}, g)$. It is easy to see from the proof (see, e.g. [16], [18]) that there is M_0 such that for any $l \in \mathbb{N}$ one can find $C_l > 0$ such that

$$|k|_{S(1,\bar{g})}^{(l)} \le C_l$$

holds uniformly in $M \ge M_0$. Note that k satisfies $(1-r)\#(1+k) = 1$, that is

$$k = r + r\#k. \tag{8.1}$$

Since $r \in S(M^{-1}, g)$ it follows from (8.1) that $\left|k\right|_{S(1,\bar{g})}^{(l)} \le C_l M^{-1}$ uniformly in $M \ge M_0$. Assume that

$$\sup\left|\langle\xi\rangle_\gamma^{(|\beta|-|\alpha|)/2}\partial_x^\alpha\partial_\xi^\beta k\right| \le C_{\alpha,\beta,\nu}M^{-1-l/2}, \quad |\alpha+\beta| \ge l \tag{8.2}$$

for $0 \le l \le \nu$. Let $|\alpha+\beta| \ge \nu+1$ and note that

$$\partial_x^\alpha\partial_\xi^\beta k = \partial_x^\alpha\partial_\xi^\beta r + \sum C_{\dots}\big(\partial_x^{\alpha''}\partial_\xi^{\beta''}r\big)\#\big(\partial_x^{\alpha'}\partial_\xi^{\beta'}k\big)$$

where $\alpha'+\alpha'' = \alpha$ and $\beta'+\beta'' = \beta$. From the assumption (8.2) we have $\partial_x^{\alpha'}\partial_\xi^{\beta'}k \in S(M^{-1-|\alpha'+\beta'|/2}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}, \bar{g})$ if $|\alpha'+\beta'| \le \nu$ and $\partial_x^{\alpha'}\partial_\xi^{\beta'}k \in S(M^{-1-\nu/2}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}, \bar{g})$ if $|\alpha'+\beta'| \ge \nu+1$. Since $r \in S(M^{-1}, g)$ one has

$$\big(\partial_x^{\alpha''}\partial_\xi^{\beta''}r\big)\#\big(\partial_x^{\alpha'}\partial_\xi^{\beta'}k\big) \in S(M^{-1-(\nu+2)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \bar{g})$$

which implies that (8.2) holds for $0 \le l \le \nu+1$ and hence for all ν by induction on ν. This proves that $k \in S(M^{-1}, g)$. The proof of the assertions for $\tilde{k}$ is similar. □

Here recall [24, Lemmas 3.1.6, 3.1.7].

Lemma 8.2. *Let $q \in S(1, g)$ satisfy $q \ge c$ with a constant c independent of M. Then there is $C > 0$ such that*

$$\big(\mathrm{op}(q)u, u\big) \ge (c - CM^{-1/2})\|u\|^2.$$

Proof. One can assume $c = 0$. We see $q(x,\xi)+M^{-1/2}$ is an admissible weight for $\bar{g}$ and $(q+M^{-1/2})^{1/2} \in S((q+M^{-1/2})^{1/2}, \bar{g})$. Moreover $\partial_x^\alpha\partial_\xi^\beta(q+M^{-1/2})^{1/2} \in S(M^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \bar{g})$ for $|\alpha+\beta| = 1$. Therefore

$$q + M^{-1/2} = (q+M^{-1/2})^{1/2}\#(q+M^{-1/2})^{1/2} + r, \quad r \in S(M^{-1}, \bar{g})$$

which proves the assertion. □

Lemma 8.3. *Let $q \in S(1,g)$ then there is $C>0$ such that*

$$\|\mathrm{op}(q)u\| \le \big(\sup|q| + CM^{-1/2}\big)\|u\|.$$

Lemma 8.4. *Let $m>0$ be an admissible weight for g and $m \in S(m,g)$. If $q \in S(m,g)$ then there is $C>0$ such that*

$$\big|(\mathrm{op}(q)u,u)\big| \le \big(\sup\big(|q|/m\big) + CM^{-1/2}\big)\|\mathrm{op}(\sqrt{m}\,)u\|^2.$$

Proof. First note that $m^{\pm 1/2}$ are admissible weights and $m^{\pm 1/2} \in S(m^{\pm 1/2},g)$. Write

$$\tilde{q} = (1+k)\#m^{-1/2}\#q\#m^{-1/2}\#(1+\tilde{k}) \in S(1,g)$$

where $m^{1/2}\#(1+k)\#m^{-1/2} = 1$ and $m^{-1/2}\#(1+\tilde{k})\#m^{1/2} = 1$ such that

$$m^{1/2}\#\tilde{q}\#m^{1/2} = q.$$

Since k, $\tilde{k} \in S(M^{-1},g)$ one can write $\tilde{q} = qm^{-1} + r$ with $r \in S(M^{-1},g)$. Thanks to Lemma 8.3 we have $\|\mathrm{op}(qm^{-1})v\| \le (\sup\big(|q|/m\big) + CM^{-1/2})\|v\|$ hence

$$\big|(\mathrm{op}(q)u,u)\big| \le \big|(\mathrm{op}(qm^{-1})\mathrm{op}(m^{1/2})u, \mathrm{op}(m^{1/2})u)| + CM^{-1}\|\mathrm{op}(m^{1/2})u\|^2$$

proves the assertion. □

Lemma 8.5. *Let $m_i > 0$ be two admissible weights for g and assume that $m_i \in S(m_i,g)$ and $m_2 \le C\, m_1$ with $C>0$. Then there is $C'>0$ such that*

$$\big\|\mathrm{op}(m_2)u\big\| \le C'\big\|\mathrm{op}(m_1)u\big\|.$$

Proof. Write $\tilde{m}_2 = m_2\#m_1^{-1}\#(1+k)$ such that $m_2 = \tilde{m}_2\#m_1$ with $k \in S(M^{-1},g)$. Since $\tilde{m}_2 \in S(1,g)$ one has

$$\|\mathrm{op}(m_2)u\| = \|\mathrm{op}(\tilde{m}_2)\mathrm{op}(m_1)u\| \le C'\|\mathrm{op}(m_1)u\|$$

which proves the assertion. □

8.2 Lower bounds of $\mathrm{op}(\lambda_j)$

Lemma 8.6. *There exist $C>0$ and M_0 such that*

$$\mathsf{Re}\,\big(\mathrm{op}(\lambda_j\#\kappa)u,u\big) \ge (1-CM^{-2})\big\|\mathrm{op}(\kappa^{1/2}\lambda_j^{1/2})u\big\|^2, \quad M \ge M_0.$$

Proof. Since $\kappa \in S(\kappa,g)$ and $\lambda_j \in S(\lambda_j,g)$ one can write

$$\lambda_j\#\kappa = \kappa\lambda_j + r_{j1} + r_{j2}$$

where r_{j1} is pure imaginary and $r_{j2} \in S(M^{-2}\kappa\lambda_j,g)$. Thanks to Lemma 8.4 it follows that

$$\mathsf{Re}\,(\mathrm{op}(\lambda_j\#\kappa)u,u) \ge (\mathrm{op}(\kappa\lambda_j)u,u) - CM^{-2}\|\mathrm{op}({\lambda_j}^{1/2}\kappa^{1/2})u\|^2.$$

Consider $(\mathrm{op}(\kappa\lambda_j)u, u)$. Since $\lambda_j^{1/2}\kappa^{1/2} \in S(\lambda_j^{1/2}\kappa^{1/2}, g)$ then

$$(\lambda_j^{1/2}\kappa^{1/2})\#(\lambda_j^{1/2}\kappa^{1/2}) = \lambda_j\kappa + \tilde{r}_j$$

with $\tilde{r}_j \in S(M^{-2}\lambda_j\kappa, g)$. Applying Lemma 8.4 to $\mathrm{op}(\tilde{r}_j)$ one obtains

$$\big(\mathrm{op}(\lambda_j\kappa)u, u\big) \geq (1 - CM^{-2})\|\mathrm{op}(\lambda_j^{1/2}\kappa^{1/2})u\|^2$$

which proves the assertion. □

Lemma 8.7. *There exist $c >$ and M_0 such that*

$$\mathsf{Re}\,\big(\mathrm{op}(\lambda_1)u, u\big) \geq c\,\big\|\mathrm{op}(\lambda_1^{1/2})u\big\|^2 + cM^2\big\|\langle D\rangle_\gamma^{-1}u\big\|^2, \quad M \geq M_0.$$

Proof. From Propositions 4.1 and 6.1 it follows that $\lambda_1 \geq c'\,M\sigma\langle\xi\rangle_\gamma^{-1}$ with some $c' > 0$. Denote

$$\tilde{\lambda}_1 = \lambda_1/2 - c\,M\sigma\langle\xi\rangle_\gamma^{-1}$$

where $c > 0$ is chosen so that $\tilde{\lambda}_1 \geq c_1 M\sigma\langle\xi\rangle_\gamma^{-1}$ with $c_1 > 0$. Note that $\tilde{\lambda}_1 \in \mathcal{C}(\sigma^2)$ since $M\sigma\langle\xi\rangle_\gamma^{-1} \in \mathcal{C}(\sigma^2)$. Thanks to Lemmas 7.5 and 7.7 it follows that $\tilde{\lambda}_1 \in S(\tilde{\lambda}_1, g)$ and $\tilde{\lambda}_1$ is an admissible weight for g. Thus a repetition of the above arguments shows

$$\big(\mathrm{op}(\tilde{\lambda}_1)u, u\big) \geq (1 - CM^{-2})\|\mathrm{op}(\tilde{\lambda}_1^{1/2})u\|^2$$

where the right-hand side is nonnegative if $M \geq \sqrt{C} = M_0$. Since

$$(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2})\#(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}) = \sigma\,\langle\xi\rangle_\gamma^{-1} + r$$

with $r \in S(M^{-2}\sigma\,\langle\xi\rangle_\gamma^{-1}, g)$ and then

$$\big(\mathrm{op}(\sigma\,\langle\xi\rangle_\gamma^{-1})u, u\big) \geq (1 - CM^{-2})\|\mathrm{op}(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2})u\|^2.$$

Recalling $\mathrm{op}(\lambda_1/2) = \mathrm{op}(\tilde{\lambda}_1) + c\,M\mathrm{op}(\sigma\,\langle\xi\rangle_\gamma^{-1})$ it follows that

$$(\mathrm{op}(\lambda_1/2)u, u) \geq c\,M(1 - CM^{-2})\|\mathrm{op}(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2})u\|^2 \tag{8.3}$$

for $M \geq M_0$. Since $M^2\langle\xi\rangle_\gamma^{-2} \leq M\sigma\langle\xi\rangle_\gamma^{-1}$ it follows from Lemma 8.5 that

$$M^2\|\langle D\rangle_\gamma^{-1}u\|^2 \leq CM\|\mathrm{op}(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2})u\|^2. \tag{8.4}$$

Finally writing $\lambda_1 = \lambda_1^{1/2}\#\lambda_1^{1/2} + r$ with $r \in S(M^{-1}\lambda_1, g)$ one obtains

$$\big(\mathrm{op}(\lambda_1/2)u, u\big) \geq (1/2 - CM^{-1})\big\|\mathrm{op}(\lambda_1^{1/2})u\big\|^2$$

which together with (8.3) and (8.4) proves the assertion. □

Lemma 8.8. *There exist $c > 0$ and M_0 such that*

$$\mathsf{Re}\,\big(\mathrm{op}(\lambda_2)u, u\big) \geq c\,\big\|\mathrm{op}(\lambda_2^{1/2})u\big\|^2 + c\,M\big\|\langle D\rangle_\gamma^{-1/2}u\big\|^2, \quad M \geq M_0.$$

Proof. A repetition of the same arguments shows that

$$(\mathrm{op}(\lambda_2)u,u)\geq(1-CM^{-2})\|\mathrm{op}(\lambda_2^{1/2})u\|^2.$$

Note that one can find $C>0$, M_0 such that

$$\|\mathrm{op}(\sigma^{1/2})u\|/C\leq\|\mathrm{op}(\lambda_2^{1/2})u\|^2\leq C\|\mathrm{op}(\sigma^{1/2})u\|$$

for $M\geq M_0$. Noting $\sigma\geq M\langle\xi\rangle_\gamma^{-1}$ we conclude the assertion. □

Lemma 8.9. *There exist $c>0$ and M_0 such that*

$$\big(\mathrm{op}(\lambda_3)u,u\big)\geq c\,\big\|u\big\|^2,\quad M\geq M_0.$$

Summarize what we have proved in

Proposition 8.1. *There exist $c>0$, $C>0$ and M_0 such that*

$$\begin{aligned}\mathsf{Re}\,(\mathrm{op}(\Lambda\#\kappa)W,W)&\geq(1-CM^{-2})\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2,\\ \mathsf{Re}\,(\mathrm{op}(\Lambda)W,W)&\geq c\sum_{j=1}^3\big(\|\mathrm{op}(\Lambda^{1/2})W\|^2+\|\mathrm{op}(\mathcal{D})W\|^2\big)\end{aligned}$$

for $M\geq M_0$ where

$$\mathcal{D}=\begin{bmatrix}M\langle\xi\rangle_\gamma^{-1}&0&0\\0&M^{1/2}\langle\xi\rangle_\gamma^{-1/2}&0\\0&0&1\end{bmatrix}.$$

9 System with diagonal symmetrizer

Diagonalizing the Bézout matrix introduced in Section 6 we reduce the system (6.2) to a system with a diagonal symmetrizer.

Lemma 9.1. *Let $p\in\mathcal{C}(\sigma^k)$ then $\partial_x^\alpha\partial_\xi^\beta p\in S(\sigma^{k-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|},g)$.*

Proof. The proof is clear from

$$\begin{aligned}\big|\partial_x^{\alpha'}\partial_\xi^{\beta'}(\partial_x^\alpha\partial_\xi^\beta p)\big|&\precsim\sigma^{k-|\alpha'+\beta'+\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta'+\beta|}\\ \precsim\sigma^{k-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}&\sigma^{-|\alpha'+\beta'|/2}\langle\xi\rangle_\gamma^{-|\alpha'+\beta'|/2}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}\\ &\precsim\sigma^{k-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}M^{-|\alpha'+\beta'|/2}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}\end{aligned}$$

since $\sigma\geq\rho\geq M\langle\xi\rangle_\gamma^{-1}$. □

Lemma 9.2. *Let $p\in\mathcal{C}(\sigma^k)$ and $q\in\mathcal{C}(\sigma^\ell)$. Then*

$$p\#p-p^2\in S(\sigma^{2k-2}\langle\xi\rangle_\gamma^{-2},g),\quad p\#q-pq\in S(\sigma^{k+\ell-1}\langle\xi\rangle_\gamma^{-1},g).$$

Proof. The assertions follows from Lemma 9.1 and the Weyl calculus of pseudodifferential operators. □

Since $a \in \mathcal{C}(\sigma)$, $b \in \mathcal{C}(\sigma^{3/2})$ one sees $A\#[\xi] = A(t,x,\xi)[\xi] + R$ with $R \in S(M^{-2}, g)$ for $\partial_\xi^\beta[\xi] \in S(1,g)$ by (4.6) one can replace $A(t,x,D)\#[D]$ by $\mathrm{op}(A[\xi])$ in (6.2), moving R to B. Denote

$$L = D_t - \mathrm{op}(\tilde{A}) - \mathrm{op}(B), \quad \tilde{A} = \begin{bmatrix} 0 & a & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [\xi].$$

Consider $T^{-1}\#T = I - R$ where $R \in S(M^{-1}, g)$. Thanks to Lemma 8.1 there is $K \in S(M^{-1}, g)$ such that $(I-R)\#(I+K) = I = (I+K)\#(I-R)$ and hence

$$T^{-1}\#T\#(I+K) = I, \quad (I+K)\#T^{-1}\#T = I, \quad T\#(I+K)\#T^{-1} = I.$$

Therefore one can write

$$L\,\mathrm{op}(T) = \mathrm{op}(T)\,\tilde{L} \tag{9.1}$$

where

$$\tilde{L} = D_t - \mathrm{op}((I+K)\#T^{-1}\#(\tilde{A}+B)\#T) + \mathrm{op}((I+K)\#T^{-1}\#(D_tT)).$$

Lemma 9.3. *One has $K \in S(M^{-1}\langle\xi\rangle_\gamma^{-1}, g)$.*

Proof. Write $T = (t_{ij})$ then $T^{-1}\#T = (\sum_{k=1}^3 t_{ki}\#t_{kj})$ and denote

$$\sum_{k=1}^3 t_{ki}\#t_{kj} = \delta_{ij} + r_{ij}.$$

Taking Lemma 6.6 into account, we see $r_{ii} \in S(\sigma^{-1}\langle\xi\rangle_\gamma^{-2}, g) \subset S(M^{-1}\langle\xi\rangle_\gamma^{-1}, g)$ and $r_{ij} \in S(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1}, g) \subset S(M^{-2}\langle\xi\rangle_\gamma^{-1}, g)$ for $i \neq j$ thanks to Lemma 9.2 hence $R \in S(M^{-1}\langle\xi\rangle_\gamma^{-1}, g)$. Since $K \in S(M^{-1}, g)$ satisfies $K = R + R\#K$ we conclude the assertion. □

Therefore $K\#T^{-1}\#(\tilde{A}+B)\#T \in S(M^{-1}, g)$ is clear. Hence

$$\tilde{L} = D_t - \mathrm{op}(T^{-1}\#(\tilde{A}+B)\#T - T^{-1}\#(D_tT)) + \mathrm{op}(S(M^{-1}, g)).$$

To simplify notations sometimes we abbreviate $S(m, g)$ to $S(m)$ where m is admissible for g. In view of Lemmas 6.6 and 6.7 it follows from Lemma 9.2 that

$$\begin{aligned} T^{-1}\#(\partial_t T) &= T^{-1}\partial_t T \\ &+ \begin{bmatrix} S(\sigma^{-1}\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(\langle\xi\rangle_\gamma^{-1}) \\ S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1}\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) \\ S(\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(\langle\xi\rangle_\gamma^{-1}) \end{bmatrix} \end{aligned} \tag{9.2}$$

hence $T^{-1}\#(\partial_t T) = T^{-1}\partial_t T + S(M^{-1}, g)$ because $\sigma \geq M\langle\xi\rangle_\gamma^{-1}$.

Turn to study $T^{-1}\#\tilde{A}\#T$. Noting that $\partial_x^\alpha\partial_\xi^\beta a \in S(\sigma^{1/2}\langle\xi\rangle_\gamma^{-|\beta|}, g)$, $\partial_x^\alpha\partial_\xi^\beta b \in S(\sigma\langle\xi\rangle_\gamma^{-|\beta|}, g)$ for $|\alpha+\beta| = 1$ and $\partial_\xi^\beta[\xi] \in S(1, g)$, $|\beta| = 1$ we have

$$T^{-1}\#\tilde{A} = T^{-1}\tilde{A} + R, \quad R = \begin{bmatrix} S(1) & S(M^{-2}) & S(M^{-6}) \\ S(M^{-2}) & S(1) & S(M^{-8}) \\ S(M^{-8}) & S(M^{-2}) & S(M^{-6}) \end{bmatrix}.$$

Therefore $T^{-1}\#\tilde{A}\#T = (T^{-1}\tilde{A})\#T + R_1$ with

$$R_1 = R\#T = \begin{bmatrix} S(M^{-4}) & S(M^{-2}) & S(1) \\ S(M^{-2}) & S(1) & S(M^{-2}) \\ S(M^{-4}) & S(M^{-2}) & S(M^{-8}) \end{bmatrix}.$$

Note that

$$T^{-1}\tilde{A} = \begin{bmatrix} C(\sigma^{1/2}) & 1 + C(\sigma) & C(\sigma^{5/2}) \\ -1 + C(\sigma) & C(\sigma^{1/2}) & C(\sigma^3) \\ C(\sigma^{5/2}) & C(\sigma) & C(\sigma^{3/2}) \end{bmatrix} [\xi]$$

and hence

$$\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta\big(T^{-1}\tilde{A}\big) = \begin{bmatrix} S(1) & S(1) & S(M^{-8}) \\ S(1) & S(1) & S(M^{-10}) \\ S(M^{-8}) & S(M^{-2}) & S(M^{-4}) \end{bmatrix}$$

for $|\alpha+\beta| = 1$. Then thanks to (6.13) one sees

$$(T^{-1}\tilde{A})\#T = T^{-1}\tilde{A}T + R_2, \quad R_2 = \begin{bmatrix} S(1) & S(M^{-2}) & S(M^{-2}) \\ S(1) & S(M^{-2}) & S(M^{-2}) \\ S(M^{-2}) & S(M^{-4}) & S(M^{-6}) \end{bmatrix}.$$

Thus we obtain $T^{-1}\#\tilde{A}\#T = T^{-1}\tilde{A}T + R_1 + R_2$ where

$$R_1 + R_2 = \begin{bmatrix} S(1) & S(M^{-2}) & S(M^{-2}) \\ S(1) & S(M^{-2}) & S(M^{-2}) \\ S(M^{-2}) & S(M^{-4}) & S(M^{-6}) \end{bmatrix}.$$

Recall

$$B = \begin{bmatrix} b_1 & b_2 + d_M & b_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and consider $T^{-1}\#B\#T$. Since $d_M \in S(M, g)$ one sees by Lemma 6.6 that

$$T^{-1}\#B = \begin{bmatrix} S(\sigma) & S(M\sigma) & S(\sigma) \\ S(\sigma^{3/2}) & S(M\sigma^{3/2}) & S(\sigma^{3/2}) \\ b_1 + S(\sigma) & b_2 + d_M + S(M\sigma) & b_3 + S(\sigma) \end{bmatrix}.$$

Noting that $\sigma \le CM^{-4}$ we conclude that $T^{-1}\#B\#T$ is written

$$\begin{bmatrix} S(\sigma) & S(M\sigma) & S(\sigma) \\ S(\sigma^{3/2}) & S(M\sigma^{3/2}) & S(\sigma^{3/2}) \\ b_3 + S(M\sigma^{1/2}) & -b_2 - d_M + S(\sigma^{1/2}) & b_1 + S(\sigma) \end{bmatrix}. \tag{9.3}$$

Thus using $\sigma \le CM^{-4}$ again

$$T^{-1}\#B\#T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_3 & -2Me + S(1) & S(1) \end{bmatrix} + S(M^{-1}, g)$$

where we have used (6.1). We summarize what we have proved in

Proposition 9.1. *One can write* $L \cdot \mathrm{op}(T) = \mathrm{op}(T) \cdot \tilde{L}$ *where*

$$\tilde{L} = D_t - \mathrm{op}(\mathcal{A} + \mathcal{B}_1 - T^{-1}D_tT), \qquad \mathcal{A} = (T^{-1}AT)[\xi]\,,$$

$$\mathcal{B}_1 = \begin{bmatrix} S(1) & S(1) & S(1) \\ S(1) & S(1) & S(1) \\ b_3 + S(M^{-1}) & -2Me + S(1) & S(1) \end{bmatrix}.$$

Note that from Lemma 4.1 it follows that

$$b_3(t,x,\xi) - \bar{b}_3 \in S(M^{-2}, g), \quad \bar{b}_3 = b_3(0,0,\bar{\xi}). \tag{9.4}$$

10 Weighted energy estimates

10.1 Energy form

Let $w = t\phi(t,x,\xi)$ and consider the energy with the scalar weight $\mathrm{op}(w^{-n})$;

$$\mathcal{E}(V) = e^{-\theta t}\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V\big)$$

where $\theta > 0$ is a large positive parameter and n is fixed such that

$$n > \upsilon^{-1/2}\Big(\frac{|3\,\bar{b}_3 + i\bar{e}|}{\bar{e}} + 6 + \sqrt{2}\Big) + C^* + 2 \tag{10.1}$$

where $\upsilon^{-1} = \big(2(18\sqrt{2}+1)\big)$ and C^* is given by (3.13) and $\bar{e}$ is the nonzero positive real eigenvalue of $F_p(0,0,0,\bar{\xi})$ (cf. [24, (7.2.3)]).

Note that $\partial_t\phi = \omega^{-1}\phi$ and hence

$$\partial_t w^{-n} = -n\Big(\frac{1}{t} + \frac{1}{\omega}\Big)w^{-n} = -n\kappa\, w^{-n}.$$

Recall that V satisfies

$$\partial_t V = \mathrm{op}(i\mathcal{A} + i\mathcal{B})V + F, \quad \mathcal{B} = \mathcal{B}_1 - T^{-1}D_tT. \tag{10.2}$$

Noting that Λ is real and diagonal hence $\mathrm{op}(\Lambda)^* = \mathrm{op}(\Lambda)$ one has

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E} &= -\theta e^{-\theta t}\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V\big)\\
&-2n\mathsf{Re}\, e^{-\theta t}\big(\mathrm{op}(\Lambda)\mathrm{op}(\kappa w^{-n})V, \mathrm{op}(w^{-n})V\big)\\
&+e^{-\theta t}\big(\mathrm{op}(\partial_t\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V\big)\\
+2\mathsf{Re}\, e^{-\theta t}&\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})(\mathrm{op}(i\mathcal{A}+i\mathcal{B})V+F), \mathrm{op}(w^{-n})V\big).
\end{aligned}
\tag{10.3}
$$

Consider $\mathrm{op}(w^{-n})\mathrm{op}(\Lambda)\mathrm{op}(\kappa w^{-n}) = \mathrm{op}(w^{-n}\#\Lambda\#(\kappa w^{-n}))$. Since κ and ϕ^{-n} are admissible weights for g one can write

$$
\kappa\#\phi^{-n} = \kappa\phi^{-n} - r, \quad r \in S(M^{-1}\kappa\phi^{-n}, g).
$$

Let $\tilde{r} = r\#\phi^{n}\#(1+k) \in S(M^{-1}\kappa, g)$ such that $r = \tilde{r}\#\phi^{-n}$ and hence $\kappa\phi^{-n} = (\kappa+\tilde{r})\#\phi^{-n}$ thus

$$
\kappa w^{-n} = (\kappa+\tilde{r})\#w^{-n}.
$$

Therefore we have

$$
\begin{aligned}
\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(\kappa w^{-n})V, \mathrm{op}(w^{-n})V\big) &\geq \mathsf{Re}\,\big(\mathrm{op}(\Lambda\#\kappa)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V\big)\\
&-\big|\big(\mathrm{op}(\Lambda\#\tilde{r})\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V\big)\big|.
\end{aligned}
$$

Since $\lambda_j\#\tilde{r} \in S(M^{-1}\kappa\lambda_j, g)$ thanks to Lemma 8.4 the second term on the right-hand side is bounded by

$$
CM^{-1}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})V\|.
$$

Applying Proposition 8.1, one can conclude, denoting $W_j = \mathrm{op}(w^{-n})V_j$, that

$$
\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(\kappa w^{-n})V, \mathrm{op}(w^{-n})V\big) \geq (1-CM^{-1})\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2.
$$

Applying Proposition 8.1 again one obtains

$$
\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V\big) \geq c\big(\|\mathrm{op}(\Lambda^{1/2})W\|^2 + \|\mathrm{op}(\mathcal{D})W\|^2\big)
$$

for $M \geq M_0$.

Definition 10.1. To simplify notations we denote

$$
\begin{aligned}
\mathcal{E}_1(V) &= \|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})V\|^2 = t^{-2n}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})V\|^2,\\
\mathcal{E}_2(V) &= \|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(w^{-n})V\|^2 + \|\mathrm{op}(\mathcal{D})\mathrm{op}(w^{-n})V\|^2\\
&= t^{-2n}\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})V\|^2 + t^{-2n}\|\mathrm{op}(\mathcal{D})\mathrm{op}(\phi^{-n})V\|^2.
\end{aligned}
$$

Now we summarize

Lemma 10.1. *One can find $C>0$, $c>0$ and M_0 such that*

$$
\begin{aligned}
n\,\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(\kappa w^{-n})V, \mathrm{op}(w^{-n})V\big) &+ \theta\,\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V\big)\\
&\geq n(1-CM^{-1})\mathcal{E}_1(V) + c\,\theta\mathcal{E}_2(V), \quad M \geq M_0.
\end{aligned}
$$

10.2 Term $(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{B})V, \mathrm{op}(w^{-n})V)$

First recall that $\lambda_i \in S(\lambda_i, g)$ and $\lambda_1 \le C\sigma\lambda_2 \le C\sigma^2\lambda_3$ with some $C > 0$. Let $b \in S(\sigma^{-1/2}, g)$ and consider $(\mathrm{op}(\lambda_j)\mathrm{op}(b)W_i, W_j)$ for $i \ge j$. Write

$$\begin{aligned} r &= (1+k)\#(\kappa^{-1/2}\lambda_j^{-1/2})\#(\lambda_j\#b)\#\lambda_i^{-1/2}\#(1+\tilde{k}) \\ &\in S(\kappa^{-1/2}\sigma^{-1/2}\lambda_j^{1/2}\lambda_i^{-1/2}, g) \subset S(\sigma^{(i-j)/2}, g) \quad (i \ge j) \end{aligned}$$

for $\sigma\kappa \ge 1$, such that $(\kappa^{1/2}\lambda_j^{1/2})\#r\#\lambda_i^{1/2} = \lambda_j\#b$. Then we have

$$|(\mathrm{op}(\lambda_j)\mathrm{op}(b)W_i, W_j)| \le M^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 + CM^2\|\mathrm{op}(\Lambda^{1/2})W\|^2$$

for $i \ge j$. Turn to study $(\mathrm{op}(\lambda_i)\mathrm{op}(b)W_j, W_i)$ for $1 \le j < i$. Let $b \in S(M^l, g)$ and denote

$$r = (1+k)\#(\kappa^{-1/2}\lambda_2^{-1/2})\#(\lambda_3\#b)\#\lambda_3^{-1/2}\#(1+\tilde{k})$$

such that $(\kappa^{1/2}\lambda_2^{1/2})\#r\#\lambda_3^{1/2} = \lambda_3\#b$. Since $r \in S(\kappa^{-1/2}\lambda_3^{1/2}\lambda_2^{-1/2}, g)$ hence $r \in S(\sigma^{1/2}\lambda_2^{-1/2}, g) \subset S(1, g)$ in view of Lemma 7.11 which proves

$$\begin{aligned} \big|(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_2, W_3)\big| &= \big|(\mathrm{op}(r)\mathrm{op}(\kappa^{1/2}\lambda_2^{1/2})W_2, \mathrm{op}(\lambda_3)W_3)\big| \\ &\le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 + CM^{2+2l}\|\mathrm{op}(\Lambda^{1/2})W\|^2. \end{aligned}$$

We next check $(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_1, W_3)$ for $b \in S(1, g)$. Write

$$r = (1+k)\#(\kappa^{-1/2}\lambda_1^{-1/2})\#(\lambda_3\#b)\#(\kappa^{-1/2}\lambda_3^{-1/2})\#(1+\tilde{k})$$

such that $(\kappa^{1/2}\lambda_1^{1/2})\#r\#(\kappa^{1/2}\lambda_3^{1/2}) = \lambda_3\#b$. Since $k,\ \tilde{k} \in S(M^{-1}, g)$ it is easy to see that

$$r = b\#(\lambda_3^{1/2}\lambda_1^{-1/2}\kappa^{-1}) + \tilde{r}$$

with $\tilde{r} \in S(M^{-1/2}, g)$. By Proposition 6.1 and Lemma 7.11 one sees that

$$\big|\lambda_3^{1/2}\lambda_1^{-1/2}\kappa^{-1}\big| \le 3/(\bar{e}\,\upsilon^{1/2}) + CM^{-4}$$

hence $\|\mathrm{op}(\lambda_3^{1/2}\lambda_1^{-1/2}\kappa^{-1})u\| \le 3/(\bar{e}\,\upsilon^{1/2})\,(1 + C'M^{-1/2})\|u\|$. Therefore

$$\begin{aligned} |(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_1, W_3)| &= |(\mathrm{op}(r_{31})\mathrm{op}(\kappa^{1/2}\lambda_1^{1/2})W_1, \mathrm{op}(\kappa^{1/2}\lambda_1^{1/2})W_3)| \\ &\le \big(3/(\bar{e}\,\upsilon^{1/2})\|\mathrm{op}(b)\| + CM^{-1/2}\big)\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2. \end{aligned}$$

Now consider $(\mathrm{op}(\lambda_2)\mathrm{op}(b)W_1, W_2)$ for $b \in S(\sigma^{-1/2}, g) = S(\lambda_2^{-1/2}, g)$. Denote

$$r = (1+k)\#(\kappa^{-1/2}\lambda_2^{-1/2})\#(\lambda_2\#b)\#(\lambda_1^{-1/2}\kappa^{-1/2})\#(1+\tilde{k})$$

such that $(\kappa^{1/2}\lambda_2^{1/2})\#r\#(\lambda_1^{1/2}\kappa^{1/2}) = \lambda_2\#b$. Since one can write

$$r = (\lambda_2^{1/2}b)\#(\kappa^{-1}\lambda_1^{-1/2}) + \tilde{r}$$

with $\tilde{r} \in S(M^{-1}, g)$ because $\kappa^{-1}\lambda_1^{-1/2} \in S(1, g)$. Thus repeating the same arguments as above one conclude

$$\big|(\mathrm{op}(\lambda_2)\mathrm{op}(b)W_1, W_2)\big| \le \big(\sqrt{3}/(\bar{e}\,\upsilon^{1/2})\|\mathrm{op}(\lambda_2^{1/2}b)\| + CM^{-1/2}\big)\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2.$$

We summarize the above estimates in

Lemma 10.2. *We have*

$$\big|(\mathrm{op}(\lambda_j)\mathrm{op}(b)W_i, W_j)\big| \le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 + CM^2\|\mathrm{op}(\Lambda^{1/2})W\|^2$$

for $b \in S(\sigma^{-1/2}, g)$ *and* $i \ge j$ *and*

$$|(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_2, W_3)| \le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 + CM^{2+2l}\|\mathrm{op}(\Lambda^{1/2})W\|^2$$

for $b \in S(M^l, g)$ *and*

$$|(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_1, W_3)| \le \big(3/(\bar{e}\,\upsilon^{1/2})\|\mathrm{op}(b)\| + CM^{-1/2}\big)\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2$$

for $b \in S(1, g)$ *and*

$$\big|(\mathrm{op}(\lambda_2)\mathrm{op}(b)W_1, W_2)\big| \le \big(\sqrt{3}/(\bar{e}\,\upsilon^{1/2})\|\mathrm{op}(\lambda_2^{1/2}b)\| + CM^{-1/2}\big)\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2$$

for $b \in S(\sigma^{-1/2}, g)$.

In particular, this lemma implies

Corollary 10.1. *Let* $B = (b_{ij}) \in S(1, g)$. *Then*

$$\big|(\mathrm{op}(\Lambda)\mathrm{op}(B)W, W)\big| \le (3/(\bar{e}\,\upsilon^{1/2})\|\mathrm{op}(b_{31})\| + CM^{-1/2})\mathcal{E}_1(V) + C\mathcal{E}_2(V).$$

From Proposition 9.1 it results $\phi^{-n}\#\mathcal{B}_1 - \mathcal{B}_1\#\phi^{-n} \in S(M^{-1}\phi^{-n}, g)$ then one concludes by Corollary 10.1 that

$$\big|(\mathrm{op}(\Lambda)[\mathrm{op}(w^{-n}), \mathrm{op}(\mathcal{B}_1)]V, W)\big| \le CM^{-1}\mathcal{E}_1(V) + C\mathcal{E}_2(V) \tag{10.4}$$

where $W = \mathrm{op}(w^{-n})V$ again. Write $T^{-1}\partial_t T = (\tilde{t}_{ij})$ and recall (6.14) and note that $\tilde{t}_{12} = -\tilde{t}_{21} \in \mathcal{C}(\sigma^{-1/2})$ and $\tilde{t}_{31} \in S(1, g)$. Then thanks to Lemma 5.5 one has

$$\begin{aligned}
\lambda_2\#(\phi^{-n}\#\tilde{t}_{21} - \tilde{t}_{21}\#\phi^{-n})\#\phi^n &\in S(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1}, g)\\
&\subset S(M^{-1}\sqrt{\kappa\lambda_1}\,\sqrt{\kappa\lambda_2}, g),\\
\lambda_3\#(\phi^{-n}\#\tilde{t}_{31} - \tilde{t}_{31}\#\phi^{-n})\#\phi^n &\in S(\sigma^{-1/2}\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1}, g)\\
&\subset S(M^{-1}\sqrt{\kappa\lambda_1}\,\sqrt{\kappa\lambda_3}, g)
\end{aligned}$$

because $C\lambda_1 \ge M\sigma\langle\xi\rangle_\gamma^{-1}$, $C\lambda_2 \ge \sigma \ge M\langle\xi\rangle_\gamma^{-1}$ and $\omega^{-1} \le \kappa$. Therefore repeating similar arguments one concludes

$$\big|(\mathrm{op}(\Lambda)[\mathrm{op}(w^{-n}), \mathrm{op}(T^{-1}\partial_t T)]V, W)\big| \le CM^{-1}\mathcal{E}_1(V). \tag{10.5}$$

Recalling $\mathcal{B} = \mathcal{B}_1 - T^{-1}D_tT$ it follows from (10.4) and (10.5) that

$$\big|(\mathrm{op}(\Lambda)[\mathrm{op}(w^{-n}), \mathrm{op}(\mathcal{B})]V, W)\big| \le CM^{-1}\mathcal{E}_1(V) + C\mathcal{E}_2(V). \tag{10.6}$$

With $\mathcal{B} = (q_{ij})$ we see that

$$q_{21} = i\partial_t(3b/2a_M) + S(1),\ q_{31} = b_3 + i\partial_t a_M/3 + S(M^{-1})$$

and $q_{32} = -2Me + S(1)$, $q_{ij} \in S(\sigma^{-1/2}, g)$ for $j \ge i$. Applying Lemma 10.2 we have from (6.15), (6.16), (9.4) and Proposition 6.1 that

$$\begin{aligned} &\big|(\mathrm{op}(\Lambda)\mathrm{op}(\mathcal{B})\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V)\big| \\ &\le \big(v^{-1/2}(|3\bar{b}_3 + i\bar{e}|/\bar{e} + 6 + \sqrt{2}) + CM^{-1/2}\big)\mathcal{E}_1(V) + CM^4\mathcal{E}_2(V). \end{aligned} \tag{10.7}$$

Combining the estimates (10.7) and (10.6) we obtain

Lemma 10.3. *We have*

$$\begin{aligned} &\big|(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{B})V, \mathrm{op}(w^{-n})V)\big| \\ &\le \big(v^{-1/2}(|3\bar{b}_3 + i\bar{e}|/\bar{e} + 6 + \sqrt{2}) + CM^{-1/2}\big)\mathcal{E}_1(V) + CM^4\mathcal{E}_2(V). \end{aligned}$$

10.3 Term $(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(i\mathcal{A})V, \mathrm{op}(w^{-n})V)$

Study $g^{-n}\#\mathcal{A} - \mathcal{A}\#g^{-n}$. Recall Lemma 6.8

$$\mathcal{A} = [\xi]\begin{bmatrix} \mathcal{C}(\sqrt{\sigma}) & -1+\mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) \\ \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) & -1+\mathcal{C}(\sigma) \\ \mathcal{C}(\sigma^{3/2}) & \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{5/2}) \end{bmatrix}. \tag{10.8}$$

Let $r \in \mathcal{C}(\sigma^s)$ then thanks to Lemma 5.5 it follows that

$$\phi^{-n}\#([\xi]r) - ([\xi]r)\#\phi^{-n} \in S(\phi^{-n}\sigma^{s-1/2}\omega^{-1}\rho^{1/2}, g).$$

Denoting $\phi^{-n}\#\mathcal{A} - \mathcal{A}\#\phi^{-n} = (r_{ij})$, in view of Lemma 5.5 it follows that

$$r_{ij} \in S(\phi^{-n}\omega^{-1}\rho^{1/2}, g) \subset S(M^{-2}\kappa\phi^{-n}, g)$$

for $i \le j$ because $\omega^{-1} \le \kappa$. Writing $\tilde{r}_{ij} = r_{ij}\#\phi^n\#(1+k) \in S(M^{-2}\kappa, g)$ such that $r_{ij} = \tilde{r}_{ij}\#\phi^{-n}$ one obtains

$$\begin{aligned} \big|(\mathrm{op}(\lambda_i)\mathrm{op}(r_{ij})V_j, W_i)\big| &= \big|(\mathrm{op}(\lambda_i\#\tilde{r}_{ij})W_j, W_i)\big| \\ &\le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 \end{aligned}$$

since $\lambda_i\#\tilde{r}_{ij} \in S(M^{-2}\kappa\lambda_i, g)$. It rests to estimate $(\mathrm{op}(\lambda_i)\mathrm{op}(r_{ij})V_j, W_i)$ for $i > j$. From Corollary one sees $\tilde{a}_{21} = \lambda_1\mathcal{C}(\sigma^{-1})$ hence thanks to Lemmas 5.5 and 7.7

$$\begin{aligned} r_{21} = \phi^{-n}\#(\tilde{a}_{21}[\xi]) - \phi^{-n}\tilde{a}_{21}[\xi] &\in S(\sigma^{-1/2}\lambda_1^{1/2}\omega^{-1}\rho^{1/2}\phi^{-n}, g) \\ &\subset S(\lambda_1^{1/2}\kappa\phi^{-n}, g) \end{aligned}$$

because $\omega^{-1} \le \kappa$ hence $r_{21} = \tilde{r}_{21}\#\phi^{-n}$ with $\tilde{r}_{21} \in S(\kappa\lambda_1^{1/2}, g)$. Then noting $\lambda_2^{1/2} \le CM^{-2}$ we have

$$\begin{aligned}\big|(\mathrm{op}(\lambda_2)\mathrm{op}(r_{21})V_1, W_2)\big| &= \big|(\mathrm{op}(\lambda_2\#\tilde{r}_{21})W_1, W_2)\big| \\ &\le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2.\end{aligned}$$

Similarly from $\tilde{a}_{31} = \lambda_1\,\mathcal{C}(\sigma^{1/2})$, $\tilde{a}_{32} = \lambda_2\,\mathcal{C}(1)$ and Lemma 7.7 it follows that

$$\begin{aligned}r_{31} &\in S(\sigma\lambda_1^{1/2}\omega^{-1}\rho^{1/2}\phi^{-n}, g) \subset S(M^{-6}\lambda_1^{1/2}\kappa\phi^{-n}, g), \\ r_{32} &\in S(\lambda_2^{1/2}\omega^{-1}\rho^{1/2}\phi^{-n}, g) \subset S(M^{-2}\lambda_2^{1/2}\kappa\phi^{-n}, g).\end{aligned}$$

Here we have used

$$\partial_x^\alpha\partial_\xi^\beta\lambda_2 \in S(\lambda_2^{1/2}\langle\xi\rangle_\gamma^{-|\beta|}, g), \qquad |\alpha+\beta| = 1 \tag{10.9}$$

which follows from $\lambda_2 \in \mathcal{C}(\sigma)$ easily. Then one obtains

$$\begin{aligned}\big|(\mathrm{op}(\lambda_3)\mathrm{op}(r_{31})V_1, W_3)\big| &\le CM^{-6}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2, \\ \big|(\mathrm{op}(\lambda_3)\mathrm{op}(r_{32})V_2, W_3)\big| &\le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2.\end{aligned}$$

Therefore $(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{A})V, \mathrm{op}(w^{-n})V) - (\mathrm{op}(\Lambda)\mathrm{op}(\mathcal{A})W, W)$ is bounded by constant times $M^{-2}\mathcal{E}_1(V)$.

Next study $\Lambda\#\mathcal{A} - \Lambda\mathcal{A} = (q_{ij})$. From Lemmas 6.8 and 7.7 it follows that

$$\begin{aligned}\lambda_1\#(\tilde{a}_{1j}[\xi]) - \lambda_1\tilde{a}_{1j}[\xi] &\in S(\sigma^{1/2}\lambda_1^{1/2}, g) \subset S(M^{-2}\lambda_1\kappa, g), \\ \lambda_2\#(\tilde{a}_{2j}[\xi]) - \lambda_2\tilde{a}_{2j}[\xi] &\in S(\lambda_2^{1/2}, g) \subset S(\lambda_2\kappa^{1/2}, g)\end{aligned}$$

because $\lambda_1^{1/2}\kappa \ge 1$ and $C\lambda_2\kappa \ge 1$. Then

$$\big|(\mathrm{op}(q_{1j})W_j, W_1)\big| \le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 + C\|\mathrm{op}(\Lambda^{1/2})W\|^2$$

for $j = 1, 2, 3$ and

$$\big|(\mathrm{op}(q_{2j})W_j, W_2)\big| \le CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 + CM^2\|\mathrm{op}(\Lambda^{1/2})W\|^2$$

for $j = 2, 3$. Repeating similar arguments, applying Lemmas 6.8 and 7.7, one has

$$\begin{aligned}\lambda_2\#(\tilde{a}_{21}[\xi]) - \lambda_2\tilde{a}_{21}[\xi] &\in S(\sigma^{-1/2}\lambda_2^{1/2}\lambda_1^{1/2}, g) \subset S(\kappa^{1/2}\lambda_2^{1/2}\lambda_1^{1/2}, g), \\ \lambda_3\#(\tilde{a}_{31}[\xi]) - \lambda_3\tilde{a}_{31}[\xi] &\in S(\sigma^{1/2}\lambda_1^{1/2}, g), \\ \lambda_3\#(\tilde{a}_{32}[\xi]) - \lambda_3\tilde{a}_{32}[\xi] &\in S(\sigma^{-1/2}\lambda_2^{1/2}, g) \subset S(\kappa^{1/2}\lambda_2^{1/2}, g)\end{aligned}$$

since $\kappa\sigma \ge 1$. Therefore we have

$$\begin{aligned}&|(\mathrm{op}(q_{21})W_2, W_1)| + |(\mathrm{op}(q_{31})W_3, W_1)| + |(\mathrm{op}(q_{32})W_3, W_2)| \\ &\qquad \le CM^{-2}\|\mathrm{op}(\Lambda^{1/2}\kappa^{1/2})W\|^2 + CM^2\|\mathrm{op}(\Lambda^{1/2})W\|^2.\end{aligned}$$

Thus we conclude that

$$
\begin{aligned}
(10.10) \qquad & \big|(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{A})V, \mathrm{op}(w^{-n})V) - (\mathrm{op}(\Lambda\mathcal{A})W, W)\big| \\
& \qquad \leq CM^{-2}\mathcal{E}_1(V) + CM^2\mathcal{E}_2(V).
\end{aligned}
$$

Since $\Lambda^* = \Lambda$ a repetition of the same arguments proves that

$$
\big|(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})\mathrm{op}(\mathcal{A})V) - (\mathrm{op}(\mathcal{A}\Lambda^*)W, W)\big|
$$

is also bounded by the right-hand side of (10.10). Recalling that $\Lambda\mathcal{A} = \mathcal{A}^*\Lambda$ and $\Lambda^* = \Lambda$ we have

Lemma 10.4. *One can find $C > 0$ such that*

$$
\big|\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(i\mathcal{A})V, \mathrm{op}(w^{-n})V\big)\big| \leq CM^{-2}\mathcal{E}_1(V) + CM^2\mathcal{E}_2(V).
$$

10.4 Term $(\mathrm{op}(\partial_t\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V)$

Start with

Lemma 10.5. *We have $\partial_t\lambda_j \in S(\kappa\lambda_j, g)$, $j = 1, 2$.*

Proof. Note that Lemma 3.6 with $\epsilon = \sqrt{2}M\langle\xi\rangle_\gamma^{-1}$ implies

$$
|\partial_t\Delta_M| \leq C^*\Big(\frac{1}{t} + \frac{1}{\omega}\Big)\Delta_M = C^*\kappa\Delta_M.
$$

Recalling $\partial_t\lambda_1 = -\partial_t q(\lambda_1)/\partial_\lambda q(\lambda_1)$ it follows from (6.11) and (6.7) that

$$
|\partial_t\lambda_1| \leq (1 + CM^{-2})\big(|\partial_t a_M/a_M|\lambda_1 + |\partial_t\Delta_M|/6a_M\big).
$$

Since $(1+CM^{-2})\lambda_1 \geq \Delta_M/6a_M$ by Proposition 6.1 and $1/a_M \leq \kappa/e$ by Lemma 7.11 one concludes $|\partial_t\lambda_1| \leq (1 + CM^{-2})(C^* + 1)\kappa\lambda_1$. Since $\partial_t\lambda_1 \in \mathcal{C}(\sigma)$ then

$$
|\partial_x^\alpha\partial_\xi^\beta\partial_t\lambda_1| \leq C\sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|} \leq C\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}.
$$

From Lemma 7.11 and $C\lambda_1 \geq M\sigma\langle\xi\rangle_\gamma^{-1}$ it follows that

$$
\kappa\lambda_1 \geq \kappa\sqrt{\lambda_1}M^{1/2}\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}/C \geq M^{1/2}\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}/C
$$

which proves $|\partial_x^\alpha\partial_\xi^\beta\partial_t\lambda_1| \leq CM^{-1/2}\kappa\lambda_1\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ for $|\alpha + \beta| = 1$. For $|\alpha + \beta| \geq 2$ it follows that

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta\partial_t\lambda_1| &\precsim \sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|} \precsim \sigma^{-(|\alpha+\beta|-2)/2}\langle\xi\rangle_\gamma^{-|\beta|} \\
&\precsim (M^{-1}\langle\xi\rangle_\gamma)^{(|\alpha+\beta|-2)/2}\langle\xi\rangle_\gamma^{-|\beta|} = M\langle\xi\rangle_\gamma^{-1}M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \\
&\leq \sigma^{-1}M\sigma\langle\xi\rangle_\gamma^{-1}M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \leq C\kappa\lambda_1M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}
\end{aligned}
$$

because $\kappa\sigma \geq 1$. Therefore we conclude $\partial_t\lambda_1 \in S(\kappa\lambda_1, g)$. On the other hand $\partial_t\lambda_j \in S(\kappa\lambda_j, g)$, $j = 2, 3$ is clear since $\partial_t\lambda_j \in \mathcal{C}(1) \subset S(1, g) \subset S(\kappa\lambda_2, g)$ for $C\lambda_2\kappa \geq 1$. This completes the proof. □

Note that from (6.10), (6.11), (6.7) and Proposition 6.1 we see that

$$\big|\partial_t\lambda_2\big| \le |\partial_t a_M|\lambda_2/a_M + Ca_M \le \big(1+CM^{-2}\big)\kappa\lambda_2$$

for $\kappa\sigma \ge 1$. Now applying Lemma 8.4 one obtains

$$\big|(\mathrm{op}(\partial_t\lambda_1)W_1, W_1)\big| \le (C^* + 1 + CM^{-1/2})\|\mathrm{op}(\kappa^{1/2}\lambda_1^{1/2})W_1\|^2,$$
$$\big|(\mathrm{op}(\partial_t\lambda_2)W_2, W_2)\big| \le (1 + CM^{-1/2})\|\mathrm{op}(\kappa^{1/2}\lambda_2^{1/2})W_2\|^2.$$

Since $\big|(\mathrm{op}(\partial_t\lambda_3)W_3, W_3)\big| \le C\|\mathrm{op}(\lambda_3)W_3\|^2$ is clear summarizing the above estimates we obtain

Lemma 10.6. *We have*

$$\big|(\mathrm{op}(\partial_t\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V)\big| \le (C^* + 2 + CM^{-1/2})\mathcal{E}_1(V) + C\mathcal{E}_2(V).$$

10.5 Conclusion

Consider the term $\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})F, \mathrm{op}(w^{-n})V\big)$ where $F = {}^t(F_1, F_2, F_3)$. Write

$$R = (1+K)\#(\kappa^{-1/2}\Lambda^{1/2})\#\Lambda\#(\kappa^{1/2}\Lambda^{-1/2})\#(1+\tilde{K})$$

such that $\Lambda = (\kappa^{1/2}\Lambda^{1/2})\#R\#(\kappa^{-1/2}\Lambda^{1/2})$. Since $R \in S(1, g)$ it follows that

$$\begin{aligned}
&\big|(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})F, \mathrm{op}(w^{-n})V)\big| \\
&= \big|(\mathrm{op}(R)\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})F, \mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})V)\big| \\
&\le CM^{-1}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})V\|^2 + CM\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})F\|^2.
\end{aligned}$$

Therefore we have

Lemma 10.7. *There exist $C > 0$, M_0 such that*

$$\begin{aligned}
\big|\mathsf{Re}\,\big(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})F, \mathrm{op}(w^{-n})V\big)\big| &\le CM^{-1}\mathcal{E}_1(V) \\
&+ CM\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})F\|^2, \quad M \ge M_0.
\end{aligned}$$

Because of the choice of n it follows from (10.3) and Lemmas 10.1, 10.3, 10.4, 10.6, 10.7 one can find $c_i > 0$ and M_0, γ_0, θ_0 such that

$$\frac{d}{dt}\mathcal{E} \le -c_1e^{-\theta t}\mathcal{E}_1 - c_2\theta e^{-\theta t}\mathcal{E}_2 + CMe^{-\theta t}\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(w^{-n})F\|^2$$

for $M \ge M_0$, $\gamma \ge \gamma_0$ and $\theta \ge \theta_0$. Recalling $w^{-n} = t^{-n}\phi^{-n}$ and integrating the above differential inequality in t we obtain

Proposition 10.1. *There exist $c_i > 0$ and M_0, γ_0, θ_0 such that*

$$c_1 t^{-2n} e^{-\theta t}\Big(\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})V(t)\|^2 + \|\mathrm{op}(\mathcal{D})\mathrm{op}(\phi^{-n})V(t)\|^2\Big)$$
$$+c_2\int_0^t e^{-\theta s}s^{-2n}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})V(s)\|^2 ds$$
$$+c_3\theta\int_0^t e^{-\theta s}s^{-2n}\Big(\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})V(s)\|^2 + \|\mathrm{op}(\mathcal{D})\mathrm{op}(\phi^{-n})V(s)\|^2\Big)ds$$
$$\leq CM\int_0^t e^{-\theta s}s^{-2n}\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})F(s)\|^2 ds$$

for V satisfying

$$\lim_{t\to+0} t^{-2n}\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})V(t), \mathrm{op}(\phi^{-n})V(t)\big) = 0$$

and for $0 \leq t \leq M^{-4}$, $M \geq M_0$, $\gamma \geq \gamma_0$ and $\theta \geq \theta_0$.

Fix M such that Proposition 10.1 holds. Since $\phi > 0$ is bounded and $\kappa \geq t^{-1}$ and $\langle\xi\rangle_\gamma^{-3/2+j/2} \leq C\lambda_j^{1/2}$ one sees that

$$t^{-1/2}\langle\xi\rangle_\gamma^{-3/2+j/2} \leq C\kappa^{1/2}\lambda_j^{1/2}\phi^{-n}, \quad 1 \leq j \leq 3.$$

Hence it follows from Lemma 8.5 that

$$t^{-1/2}\|\langle D\rangle_\gamma^{-1}V\| \leq t^{-1/2}\|\mathrm{op}(\mathcal{D})V\| \leq C\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2}\phi^{-n})V\|.$$

Writing $\kappa^{1/2}\lambda_j\phi^{-n} = r_j\#\phi^{-n}$ with $r_j \in S(\kappa^{1/2}\lambda_j^{1/2}, g)$ it is clear that

$$t^{-1/2}\|\langle D\rangle_\gamma^{-1}V\| \leq t^{-1/2}\|\mathrm{op}(\mathcal{D})V\| \leq C\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})V\|. \tag{10.11}$$

Similarly we see that

$$\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})F\| \leq C\|\mathrm{op}(\kappa^{-1/2}\phi^{-n}\Lambda^{1/2})F\|.$$

Thanks to Lemma 5.3 one has

$$\kappa^{-1/2}\phi^{-n}\lambda_j^{1/2} \in S(\sqrt{t}\,\langle\xi\rangle_\gamma^n, g)$$

hence applying Lemma 8.5 again

$$\begin{aligned}\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})F\| &\leq C\|\mathrm{op}(\kappa^{-1/2}\phi^{-n}\Lambda^{1/2})F\| \\ &\leq C\sqrt{t}\,\|\langle D\rangle_\gamma^n F\|.\end{aligned} \tag{10.12}$$

Remarking that $\big|(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})V(t), \mathrm{op}(\phi^{-n})V(t))\big| \leq C\|\langle D\rangle_\gamma^n V(t)\|^2$ one concludes that

Corollary 10.2. *We have*

$$t^{-2n}e^{-\theta t}\|\mathrm{op}(\mathcal{D})V(t)\|^2 + \int_0^t s^{-2n-1}e^{-\theta s}\|\mathrm{op}(\mathcal{D})V(s)\|^2 ds$$
$$\leq C\int_0^t s^{-2n+1}e^{-\theta s}\|\langle D\rangle_\gamma^n F(s)\|^2 ds$$

for V satisfying $\lim_{t\to+0} t^{-n}\|\langle D\rangle_\gamma^n V(t)\| = 0$.

11 Preliminary existence result

Let $s \in \mathbb{R}$ and try to obtain estimates for $\langle D\rangle_\gamma^s V$. In what follows we fix M and γ (actually it is enough to choose $\gamma = M^5$, see (4.1)) such that Proposition 10.1 holds, while θ remains to be free. From (10.2) one has

$$\partial_t(\langle D\rangle_\gamma^s V) = \big(\mathrm{op}(i\mathcal{A}+i\mathcal{B}) + i[\langle D\rangle_\gamma^s, \mathrm{op}(\mathcal{A}+\mathcal{B})]\langle D\rangle_\gamma^{-s}\big)\langle D\rangle_\gamma^s V + \langle D\rangle_\gamma^s F.$$

Lemma 11.1. *For any $s \in \mathbb{R}$ there is $C>0$ such that*

$$\big|([\langle D\rangle_\gamma^s, \mathrm{op}(\mathcal{A})]V, \mathrm{op}(\Lambda)\langle D\rangle_\gamma^s V)\big| \le C\mathcal{E}_2(\langle D\rangle_\gamma^s V).$$

Proof. Denoting $T^{-1}AT = (\tilde a_{ij})$ thanks to Corollary 6.3 and Lemma 7.7 and (10.9) we see that

$$\begin{aligned}
((\tilde a_{31}[\xi])\#\langle\xi\rangle_\gamma^s - \langle\xi\rangle_\gamma^s\#(\tilde a_{31}[\xi]))\#\langle\xi\rangle_\gamma^{-s} &\in S(\sigma\sqrt{\lambda_1}, g),\\
((\tilde a_{32}[\xi])\#\langle\xi\rangle_\gamma^s - \langle\xi\rangle_\gamma^s\#(\tilde a_{32}[\xi]))\#\langle\xi\rangle_\gamma^{-s} &\in S(\sqrt{\lambda_2}, g),\\
((\tilde a_{21}[\xi])\#\langle\xi\rangle_\gamma^s - \langle\xi\rangle_\gamma^s\#(\tilde a_{21}[\xi]))\#\langle\xi\rangle_\gamma^{-s} &\in S(\sigma^{-1/2}\sqrt{\lambda_1}, g)
\end{aligned}\tag{11.1}$$

where $S(\sigma\sqrt{\lambda_1}, g) \subset S(M^{-4}\sqrt{\lambda_1}, g)$ and $S(\sigma^{-1/2}\sqrt{\lambda_1}, g) = S(\lambda_2^{-1/2}\sqrt{\lambda_1}, g)$. From Lemma 6.8 it is easy to see $((\tilde a_{ij}[\xi])\#\langle\xi\rangle_\gamma^s - \langle\xi\rangle_\gamma^s\#(\tilde a_{ij}[\xi]))\#\langle\xi\rangle_\gamma^{-s} \in S(1,g)$ for $j \ge i$ then taking (11.1) into account the assertion is easily proved. □

Lemma 11.2. *For any $s \in \mathbb{R}$ and any $\epsilon>0$ there is $C>0$ such that*

$$\big|([\langle D\rangle_\gamma^s, \mathrm{op}(\mathcal{B})]V, \mathrm{op}(\Lambda)\langle D\rangle_\gamma^s V)\big| \le \epsilon\,\mathcal{E}_1(\langle D\rangle_\gamma^s V) + C\mathcal{E}_2(\langle D\rangle_\gamma^s V).$$

Proof. Write $\mathcal{B}_1 = (\tilde b_{ij})$. Since $\tilde b_{ij} \in S(1,g)$ by (9.3) it suffices to consider $\tilde b_{ij}$ with $i>j$. Taking b_1, $b_2 \in S(1,G)$ and $d_M \in S(M,G)$ (here recall that M being fixed) into account, it follows from (9.3) that

$$\begin{aligned}
\lambda_2\#(\langle\xi\rangle_\gamma^s\#\tilde b_{21} - \tilde b_{21}\#\langle\xi\rangle_\gamma^s)\#\langle\xi\rangle_\gamma^{-s} &\in S(\sigma^{5/2}\langle\xi\rangle_\gamma^{-1/2}, g) \subset S(\sigma^{3/2}\lambda_2^{1/2}\lambda_1^{1/2}, g),\\
\lambda_3\#(\langle\xi\rangle_\gamma^s\#\tilde b_{31} - \tilde b_{31}\#\langle\xi\rangle_\gamma^s)\#\langle\xi\rangle_\gamma^{-s} &\in S(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}, g) \subset S(\lambda_1^{1/2}\lambda_3^{1/2}, g),\\
\lambda_3\#(\langle\xi\rangle_\gamma^s\#\tilde b_{32} - \tilde b_{32}\#\langle\xi\rangle_\gamma^s)\#\langle\xi\rangle_\gamma^{-s} &\in S(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}, g) \subset S(\langle\xi\rangle_\gamma^{-1/2}\lambda_2^{1/2}\lambda_3^{1/2}, g)
\end{aligned}$$

since $\lambda_1 \ge M\sigma\langle\xi\rangle_\gamma^{-1}$. This proves

$$\big|([\langle D\rangle_\gamma^s, \mathrm{op}(\mathcal{B}_1)]V, \mathrm{op}(\Lambda)\langle D\rangle_\gamma^s V)\big| \le C\mathcal{E}_2(\langle D\rangle_\gamma^s V).\tag{11.2}$$

Next consider $T^{-1}\partial_t T = (\tilde t_{ij})$. Recalling $\tilde t_{21} \in \mathcal{C}(\sigma^{-1/2})$ and $\tilde t_{31} \in \mathcal{C}(1)$ we have

$$\begin{aligned}
\lambda_2\#(\langle\xi\rangle_\gamma^s\#\tilde t_{21} - \tilde t_{21}\#\langle\xi\rangle_\gamma^s)\#\langle\xi\rangle_\gamma^{-s} &\in S(\langle\xi\rangle_\gamma^{-1}, g) \subset S(M^{-1}\sqrt{\kappa\lambda_1}\,\sqrt{\lambda_2}, g),\\
\lambda_3\#(\langle\xi\rangle_\gamma^s\#\tilde t_{31} - \tilde t_{31}\#\langle\xi\rangle_\gamma^s)\#\langle\xi\rangle_\gamma^{-s} &\in S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}, g) \subset S(M^{-1}\sqrt{\kappa\lambda_1}\,\sqrt{\lambda_3}, g)
\end{aligned}$$

since $\sigma\kappa \ge 1$, $C\lambda_1 \ge M\sigma\langle\xi\rangle_\gamma^{-1}$ and $C\lambda_2 \ge \sigma \ge M\langle\xi\rangle_\gamma^{-1}$. Therefore we have

$$\begin{aligned}
\big|([\langle D\rangle_\gamma^s, \mathrm{op}(T^{-1}\partial_t T)]V, \mathrm{op}(\Lambda)\langle D\rangle_\gamma^s V)\big| &\le CM^{-1}\sqrt{\mathcal{E}_1(\langle D\rangle_\gamma^s V)}\sqrt{\mathcal{E}_2(\langle D\rangle_\gamma^s V)}\\
&\le \epsilon\,\mathcal{E}_1(\langle D\rangle_\gamma^s V) + C^2M^{-2}\epsilon^{-1}\mathcal{E}_2(\langle D\rangle_\gamma^s V)
\end{aligned}$$

which together with (11.2) proves the assertion. □

Choosing $\epsilon > 0$ smaller than c_2 in Proposition 10.1 and choosing θ large we conclude

Proposition 11.1. *Let $s \in \mathbb{R}$ be given. There exist $C > 0$, $\theta_0 > 0$ such that*

$$\begin{aligned}
t^{-2n}e^{-\theta t}\Big(\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s V(t)\|^2 + \|\mathrm{op}(\mathcal{D})\mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s V(t)\|^2\Big) \\
+\int_0^t e^{-\theta\tau}\tau^{-2n}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s V(\tau)\|^2 d\tau \\
+\theta\int_0^t e^{-\theta\tau}\tau^{-2n}\Big(\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s V(\tau)\|^2 \\
+\|\mathrm{op}(\mathcal{D})\mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s V(\tau)\|^2\Big)d\tau \\
\leq C\int_0^t e^{-\theta\tau}\tau^{-2n}\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s F(\tau)\|^2 d\tau
\end{aligned}$$

for $0 \leq t \leq M^{-4} = \delta$, $\theta_0 \leq \theta$ and V satisfying

$$\lim_{t\to+0} t^{-2n}\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s V(t), \mathrm{op}(\phi^{-n})\langle D\rangle_\gamma^s V(t)\big) = 0.$$

Lemma 11.3. *For any $s \in \mathbb{R}$ there exists $C_s > 0$ such that*

$$\begin{aligned}
t^{-2n}\|\mathrm{op}(\mathcal{D})\langle D\rangle_\gamma^s V(t)\|^2 + \int_0^t \tau^{-2n-1}\|\mathrm{op}(\mathcal{D})\langle D\rangle_\gamma^s V(\tau)\|^2 d\tau \\
\leq C_s\int_0^t \tau^{-2n+1}\|\langle D\rangle_\gamma^{n+s}\tilde{L}V(\tau)\|^2 d\tau
\end{aligned}$$

for $0 \leq t \leq \delta$ and for V satisfying $\lim_{t\to+0} t^{-n}\|\langle D\rangle_\gamma^{n+s}V(t)\| = 0$.

Recall (9.1) so that $\tilde{L} = \mathrm{op}(I+K)\mathrm{op}(T^{-1})\cdot L\cdot\mathrm{op}(T)$ with $T, T^{-1} \in S(1,g)$ then

$$\|\langle D\rangle_\gamma^{n+s}\tilde{L}V\| \leq C_s\|\langle D\rangle_\gamma^{n+s}L\cdot\mathrm{op}(T)V\|.$$

Since $\|\langle D\rangle_\gamma^{s-1}\mathrm{op}(T)V\| \leq C_s\|\langle D\rangle_\gamma^{s-1}V\| \leq C_s\|\mathrm{op}(\mathcal{D})\langle D\rangle_\gamma^s V\|$ it results from Lemma 11.3 that

$$\begin{aligned}
t^{-2n}\|\langle D\rangle_\gamma^{s-1}\mathrm{op}(T)V(t)\| + \int_0^s \tau^{-2n-1}\|\langle D\rangle_\gamma^{s-1}\mathrm{op}(T)V(\tau)\|^2 d\tau \\
\leq C_s\int_0^t \tau^{-2n+1}\|\langle D\rangle_\gamma^{n+s}L\cdot\mathrm{op}(T)V(\tau)\|^2 d\tau.
\end{aligned}$$

Replacing $\mathrm{op}(T)V$ by U one obtains

Lemma 11.4. *For any $s \in \mathbb{R}$ there exists $C_s > 0$ such that*

$$\begin{aligned}
t^{-2n}\|\langle D\rangle_\gamma^{s-1}U(t)\|^2 + \int_0^t \tau^{-2n-1}\|\langle D\rangle_\gamma^{s-1}U(\tau)\|^2 d\tau \\
\leq C_s\int_0^t \tau^{-2n+1}\|\langle D\rangle_\gamma^{n+s}LU(\tau)\|^2 d\tau, \quad 0 \leq t \leq \delta
\end{aligned}$$

for V satisfying $\lim_{t\to+0} t^{-n}\|\langle D\rangle_\gamma^{n+s}U(t)\| = 0$.

Return to $\hat{P}$. Since $U = {}^t(D_t^2 u, \langle D\rangle_\gamma D_t u, \langle D\rangle_\gamma^2 u)$ and $LU = {}^t(\hat{P}u, 0, 0)$ we have

$$
\begin{aligned}
&t^{-2n}\sum_{j=0}^{2}\|\langle D\rangle_\gamma^{s+1-j}D_t^j u(t)\|^2+\sum_{j=0}^{2}\int_0^t \tau^{-2n-1}\|\langle D\rangle_\gamma^{s+1-j}D_t^j u(\tau)\|^2 d\tau\\
&\qquad\leq C_s\int_0^t \tau^{-2n+1}\|\langle D\rangle_\gamma^{n+s}\hat{P}u(\tau)\|^2 d\tau,\qquad 0\leq t\leq\delta.
\end{aligned}
\tag{11.3}
$$

Now consider the adjoint operator $\hat{P}^*$ of $\hat{P}$. Noting $a_M \in \mathcal{C}(\sigma)$, $b \in \mathcal{C}(\sigma^{3/2})$ and (4.6) we see that

$$
\begin{gathered}
\hat{P}^* = D_t^3 - a_M(t,x,D)[D]^2 D_t - b(t,x,D)\,[D]^3\\
+b_1 D_t^2 + \big(\tilde{b}_2 + d_M\big)[D]D_t + \tilde{b}_3[D]^2 + \tilde{c}_1 D_t + \tilde{c}_2[D]
\end{gathered}
$$

with $\tilde{b}_j \in S(1,g)$ and $\tilde{c}_j \in S(M^3,g)$ hence $\tilde{c}_j[D]^{-1} \in S(M^{-2},g)$ where it is not difficult to check that

$$
\tilde{b}_3 - \big(b_3 + ie\big) \in S(M^{-3},g).
$$

Since the power n of the weight ϕ^{-n} depends only on a, b and b_3 (see (10.1)) then we can assume that one can choose the same n for $\hat{P}^*$ as for $\hat{P}$. Then employing the weighted energy

$$
\mathcal{E}^*(V) = e^{\theta t}\big(\mathrm{op}(\Lambda)\mathrm{op}(w^n)V, \mathrm{op}(w^n)V\big)
$$

and repeating the same arguments as before and making the integration

$$
-\int_t^\delta \frac{d}{dt}\mathcal{E}^*\,dt
$$

we have

Proposition 11.2. *There exist $c_i > 0$ and M_0, γ_0, θ_0 such that*

$$
\begin{aligned}
&c_1 t^{2n}e^{\theta t}\Big(\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^n)V(t)\|^2+\|\mathrm{op}(\mathcal{D})\mathrm{op}(\phi^n)V(t)\|^2\Big)\\
&\qquad+c_2\int_t^\delta e^{\theta\tau}\tau^{2n}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})\mathrm{op}(\phi^n)V(\tau)\|^2 d\tau\\
&+c_3\theta\int_t^\delta e^{-\theta\tau}\tau^{2n}\Big(\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^n)V(\tau)\|^2+\|\mathrm{op}(\mathcal{D})\mathrm{op}(\phi^n)V(\tau)\|^2\Big)d\tau\\
&\qquad\leq C\delta^{2n}e^{\theta\delta}(\mathrm{op}(\Lambda)\mathrm{op}(\phi^n)V(\delta),\mathrm{op}(\phi^n)V(\delta))\\
&+CM\int_t^\delta e^{\theta\tau}\tau^{2n}\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(\phi^n)F^*(\tau)\|^2 d\tau,\quad 0\leq t\leq\delta=M^{-4}
\end{aligned}
$$

for $M \geq M_0$, $\gamma \geq \gamma_0$ and $\theta \geq \theta_0$ where $F^* = \mathrm{op}(T)^t(\hat{P}^* f, 0, 0)$.

Fix M such that Proposition 11.2 holds. From Lemma 5.3 and $C\lambda_j \geq M^2\langle\xi\rangle_\gamma^{-1}$ we have

$$t^{-1/2}\langle\xi\rangle_\gamma^{-n-3/2+j/2} \in S(\kappa^{1/2}\lambda_j^{1/2}\phi^n, g), \quad \kappa^{1/2}\lambda_j^{1/2}\phi^n \in S(\sqrt{t}, g)$$

which shows that

$$\begin{aligned} t^{2n}e^{\theta t}\|\mathrm{op}(\mathcal{D})\langle D\rangle_\gamma^{-n}V(t)\|^2 + \int_t^\delta \tau^{2n-1}e^{\theta\tau}\|\mathrm{op}(\mathcal{D})\langle D\rangle_\gamma^{-n}V(\tau)\|^2 d\tau \\ \leq C\delta^{2n}e^{\theta\delta}\|V(\delta)\|^2 + C\int_t^\delta \tau^{2n+1}e^{\theta\tau}\|F^*(s)\|^2 d\tau, \quad 0 < t \leq \delta. \end{aligned}$$

Therefore repeating the same arguments as before we have

Lemma 11.5. *For any $s \in \mathbb{R}$ there is $C_s > 0$ such that*

$$\begin{aligned} t^{2n}\|\langle D\rangle_\gamma^{s-n-1}U(t)\|^2 + \int_t^\delta \tau^{2n-1}\|\langle D\rangle_\gamma^{s-n-1}U(\tau)\|^2 d\tau \\ \leq C_s\Big(\delta^{2n}\|\langle D\rangle_\gamma^s U(\delta)\|^2 + \int_t^\delta \tau^{2n+1}\|\langle D\rangle_\gamma^s F^*(\tau)\|^2 d\tau\Big), \quad 0 < t \leq \delta. \end{aligned}$$

Lemma 11.5 implies that

(11.4)
$$\begin{aligned} \sum_{j=0}^{2}\Big(t^{2n}\|\langle D\rangle_\gamma^{s+1-j}D_t^j u(t)\|^2 + \int_t^\delta \tau^{2n-1}\|\langle D\rangle_\gamma^{s+1-j}D_t^j u(\tau)\|^2 d\tau\Big) \\ \leq C_s\Big(\|\delta^{2n}\sum_{j=0}^{2}\|\langle D\rangle_\gamma^{n+s+2-j}D_t^j u(\delta)\|^2 + \int_t^\delta \tau^{2n+1}\|\langle D\rangle_\gamma^{n+s}\hat{P}^*u(\tau)\|^2 d\tau\Big) \end{aligned}$$

for $0 < t \leq \delta$. Replacing s by $-n-1-s$ then (11.4) gives

$$\int_0^\delta t^{2n-1}\|\langle D\rangle^{-n-s}u(t)\|^2 dt \leq C\int_0^\delta t^{2n+1}\|\langle D\rangle^{-1-s}\hat{P}^*u(t)\|^2 dt$$

for $u \in C_0^\infty((0,\delta)\times\mathbb{R}^d)$. This implies

$$\begin{aligned} \Big|\int_0^\delta (f,v)dt\Big| \leq \Big(\int_0^\delta t^{-2n+1}\|\langle D\rangle^{n+s}f\|^2 dt\Big)^{1/2}\Big(\int_0^\delta t^{2n-1}\|\langle D\rangle^{-n-s}v\|^2 dt\Big)^{1/2} \\ \leq C\Big(\int_0^\delta t^{-2n+1}\|\langle D\rangle^{n+s}f\|^2 dt\Big)^{1/2}\Big(\int_0^\delta t^{2n-1}\|\langle D\rangle^{-1-s}\hat{P}^*v\|^2 dt\Big)^{1/2} \end{aligned}$$

for all $v \in C_0^\infty((0,\delta)\times\mathbb{R}^d)$ and f such that $\int_0^\delta t^{-2n+1}\|\langle D\rangle^{n+s}f\|^2 dt < \infty$. Using the Hahn-Banach theorem to extend the anti-linear form in $\hat{P}^*v$;

$$\hat{P}^*v \mapsto \int_0^\delta (f,v)dt \tag{11.5}$$

we conclude that there is some u with $\int_0^\delta t^{-2n+1}\|\langle D\rangle^{1+s}u\|^2 dt < +\infty$ such that

$$\int_0^\delta (f,v)dt = \int_0^\delta (u,\hat{P}^* v)dt.$$

This implies that $\hat{P}u = f$. Since we may assume that $2n-1 \geq 0$ and hence $\langle D\rangle^{1+s}u \in L^2((0,\delta)\times\mathbb{R}^d)$ it follows from [6, Theorem B.2.9] that

$$\langle D\rangle^{1+s-j}D_t^j u \in L^2((0,\delta)\times\mathbb{R}^d), \quad j=0,1,2.$$

In view of the estimate (11.3) the following estimate holds for this u

$$\begin{aligned} t^{-2n}\sum_{j=0}^{2}\|\langle D\rangle^{1-j+s}D_t^j u(t)\|^2 + \sum_{j=0}^{2}\int_0^t \tau^{-2n-1}\|\langle D\rangle^{1+s-j}D_t^j u(\tau)\|^2 d\tau \\ \leq C\int_0^t \tau^{-2n+1}\|\langle D\rangle^{n+s}f(\tau)\|^2 d\tau, \quad 0\leq t\leq\delta. \end{aligned} \tag{11.6}$$

Theorem 11.1. *There exists $\delta > 0$ such that for any $s\in\mathbb{R}$ and any f with $t^{-n+1/2}\langle D\rangle^{n+s}f \in L^2((0,\delta)\times\mathbb{R}^d)$ there is a unique u with*

$$t^{-n-1/2}\langle D\rangle^{1+s-j}D_t^j u \in L^2((0,\delta)\times\mathbb{R}^d), \quad j=0,1,2$$

satisfying $\hat{P}u = f$ and (11.6).

Instead of (11.5) considering the anti-linear form in $\hat{P}v$;

$$\begin{aligned} \hat{P}v \mapsto \int_0^\delta (f,v)dt + \sum_{j=0}^{1}\big(w_{2-j}, D_t^j v(\delta,\cdot)\big) \\ +\big(w_0, (D_t^2 - \langle D\rangle^2 a(\delta,x,D))v(\delta,\cdot)\big) \end{aligned}$$

for $v\in C_0^\infty((0,\infty)\times\mathbb{R}^d)$ and repeating similar arguments adopting (11.3) we conclude

Theorem 11.2. *There exists $\delta > 0$ such that for any $s\in\mathbb{R}$ and any f with $t^{n+1/2}\langle D\rangle^{n+s}f \in L^2((0,\delta)\times\mathbb{R}^d)$ and any w_j with $\langle D\rangle_\gamma^{n+s+2-j}w_j \in L^2(\mathbb{R}^d)$, $j=0,1,2$, there is a unique u with*

$$t^{n-1/2}\langle D\rangle^{1+s-j}D_t^j u \in L^2((0,\delta)\times\mathbb{R}^d), \quad D_t^j u(\delta,\cdot) = w_j, \quad j=0,1,2 \tag{11.7}$$

satisfying $\hat{P}^* u = f$ and (11.4).

Indeed we first see that there is u with $t^{n-1/2}\langle D\rangle^{1+s}u \in L^2((0,\delta)\times\mathbb{R}^d)$ satisfying $D_t^j u(\delta) = w_j$, $j=0,1,2$ (e.g. [6, Chapter XXIII]). Since $\langle D\rangle^{n+s}f \in L^2((\varepsilon,\delta)\times\mathbb{R}^d)$ and $\langle D\rangle^{1+s}u \in L^2((\varepsilon,\delta)\times\mathbb{R}^d)$ for any $\varepsilon > 0$ it follows from [6, Theorem B.2.9] that $\langle D\rangle^{1+s-j}D_t^j u \in L^2((\varepsilon,\delta)\times\mathbb{R}^d)$, $0\leq j\leq 2$. Applying (11.4) with $t=\varepsilon$ we conclude (11.7), since $\varepsilon > 0$ is arbitrary.

Remark 11.1. It is clear from the proof that for any $n' \geq n$, Theorems 11.1 and 11.2 hold.

12 Propagation of micro support

In Section 11 we have proved an existence result of the Cauchy problem for $\hat{P}$, which coincides with the original P only in W_M. Following [21], [22], [9] (also [24]) we show that the micro support of $u(t,\cdot)$, obtained by Theorem 11.1, propagates with a finite speed *via* estimates of Sobolev norms of Φu, cut off by a suitable Φ. This fact enables us to solve the Cauchy problem for the original P through that of $\hat{P}$.

12.1 Estimate of cut off solution

Let $\chi(x)\in C_0^\infty(\mathbb{R}^d)$ be equal to 1 near $x=0$ and vanish in $|x|\geq 1$. Set

$$d_\epsilon(x,\xi;y,\eta)=\big\{\chi(x-y)|x-y|^2+|\xi\langle\xi\rangle_\gamma^{-1}-\eta\langle\eta\rangle_\gamma^{-1}|^2+\epsilon^2\big\}^{1/2},$$
$$f_\epsilon(t,x,\xi;y,\eta)=t-T+\nu d_\epsilon(x,\xi;y,\eta)$$

where $(y,\eta)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$ and ν is a positive small parameter and $T>0$. Note that

$$\big|\partial_x^\alpha\partial_\xi^\beta d_\epsilon\big|\leq C\langle\xi\rangle_\gamma^{-|\beta|},\quad |\alpha+\beta|=1 \tag{12.1}$$

where C is independent of $\epsilon>0$. Define Φ_ϵ by

$$\Phi_\epsilon(t,x,\xi)=\begin{cases}\exp\big(1/f_\epsilon(t,x,\xi)\big) & \text{if } f_\epsilon<0\\ 0 \quad \text{otherwise}\end{cases} \tag{12.2}$$

and set

$$\Phi_{\epsilon 1}=f_\epsilon^{-1}\Phi_\epsilon.$$

Note that Φ_ϵ, $\Phi_{\epsilon 1}\in S(1,g_0)$ for any fixed $\epsilon>0$ where $g_0=|dx|^2+\langle\xi\rangle_\gamma^{-2}|d\xi|^2$ and

$$\Phi_\epsilon-f_\epsilon\#\Phi_{\epsilon 1}\in S(\langle\xi\rangle_\gamma^{-1},g_0).$$

Since $\partial_t\Phi_\epsilon=-\Phi_{\epsilon 1}/f_\epsilon$ writing

$$\begin{aligned}\partial_t(\text{op}(\Phi_\epsilon)V)=-\text{op}(f_\epsilon^{-1}\Phi_{\epsilon 1})V+(\text{op}(i\mathcal{A}+i\mathcal{B}))\text{op}(\Phi_\epsilon)V\\ +[\text{op}(\Phi_\epsilon),\text{op}(i\mathcal{A}+i\mathcal{B})]V+\text{op}(\Phi_\epsilon)F\end{aligned} \tag{12.3}$$

we estimate $\mathcal{E}(\text{op}(\Phi_\epsilon)V)=e^{-\theta t}\big(\text{op}(\Lambda)\text{op}(w^{-n})\text{op}(\Phi_\epsilon)V,\text{op}(w^{-n})\text{op}(\Phi_\epsilon)V\big)$. Since $\Phi_\epsilon\#\mathcal{B}_1-\mathcal{B}_1\#\Phi_\epsilon\in S(c(M)\langle\xi\rangle_\gamma^{-1/2},g)$ by Proposition 9.1 it is not difficult to see from the proof of Corollary 10.1 that

$$\begin{aligned}\big|\big(\text{op}(\Lambda)\text{op}(\phi^{-n})[\text{op}(\Phi),\text{op}(\mathcal{B}_1)]V,\text{op}(\phi^{-n})\text{op}(\Phi)V\big)\big|\\ \leq c(M,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V)\end{aligned}$$

where, to simplify notations, we have set

$$\mathcal{E}_1(V)+\mathcal{E}_2(V)=t^{-2n}\mathcal{N}(V).$$

Denote $\Phi_\epsilon\#(T^{-1}\partial_t T)-(T^{-1}\partial_t T)\#\Phi_\epsilon=(\varphi_{ij})$ hence $\varphi_{21}\in S(\sigma^{-1}\langle\xi\rangle_\gamma^{-1},g)$ and $\varphi_{31}\in S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1},g)$ in view of (6.14). Then we have

$$\lambda_2\#\varphi_{21}\in S(\langle\xi\rangle_\gamma^{-1},g)\subset S(\langle\xi\rangle_\gamma^{-1/2}\sqrt{\kappa\lambda_1}\,\sqrt{\kappa\lambda_2},g),$$
$$\lambda_3\#\varphi_{31}\in S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1},g)\subset S(\langle\xi\rangle_\gamma^{-1/2}\sqrt{\kappa\lambda_1}\,\sqrt{\kappa\lambda_3},g)$$

because $C\lambda_1\geq M\sigma\langle\xi\rangle_\gamma^{-1}$, $C\lambda_2\geq\sigma$ and $\kappa\sigma\geq 1$. A repetition of similar arguments proving (10.5) shows that

$$\big|\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})[\mathrm{op}(\Phi_\epsilon),\mathrm{op}(T^{-1}\partial_t T)]V,\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V\big)\big|$$
$$\leq c(M,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V).$$

Note that $\Phi_\epsilon\#\mathcal{A}-\mathcal{A}\#\Phi_\epsilon$ can be written

$$\sum_{|\alpha+\beta|=1}\frac{(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}\Big(\partial_x^\alpha\partial_\xi^\beta\Phi_\epsilon\partial_x^\beta\partial_\xi^\alpha\mathcal{A}-\partial_x^\beta\partial_\xi^\alpha\Phi_\epsilon\partial_x^\alpha\partial_\xi^\beta\mathcal{A}\Big)+R_\epsilon=H_\epsilon+R_\epsilon$$

where it follows from (10.8) that

$$R_\epsilon=\begin{bmatrix}S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(M^2\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1})\\ S(\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(M^2\langle\xi\rangle_\gamma^{-1})\\ S(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1}) & S(\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{3/2}\langle\xi\rangle_\gamma^{-1})\end{bmatrix}$$
$$\in S(c(M)\langle\xi\rangle_\gamma^{-1/2},g)$$

for $\sigma\geq M\langle\xi\rangle_\gamma^{-1}$. It is not difficult to see from the proof of Corollary 10.1 that

$$\big|\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(R_\epsilon)V,\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V\big)\big|$$
$$\leq c(M,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V).$$

Study $(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(iH_\epsilon)V,\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V)$. Note that $H_\epsilon\in S(1,g)$ because $\partial_x^\alpha\partial_\xi^\beta\mathcal{A}\in S(\langle\xi\rangle_\gamma^{1-|\beta|},g)$ for $|\alpha+\beta|=1$. Write

$$\Phi_\epsilon=f_\epsilon\#\Phi_{\epsilon 1}+r_\epsilon,\quad r_\epsilon\in S(\langle\xi\rangle_\gamma^{-1},g_0)$$

and note $\phi^{-n}\#f_\epsilon-f_\epsilon\#\phi^{-n}\in S(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1}\phi^{-n},g)\subset S(M^{-1/2}\phi^{-n}\langle\xi\rangle_\gamma^{-1/2},g)$, then a repetition of similar arguments proves that the difference

$$\big|\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(iH_\epsilon)V,\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V\big)$$
$$-\big(\mathrm{op}(f)\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(iH_\epsilon)V,\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\big)\big|$$

is bounded by $c(M,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V)$. Since $\lambda_j\in S(\lambda_j,g)$ it follows that

$$f_\epsilon\#\lambda_j-\lambda_j\#f_\epsilon\in S(M^{-1/2}\lambda_j\langle\xi\rangle_\gamma^{-1/2},g)$$

then applying a similar arguments one can see that the difference

$$\big|(\mathrm{op}(f_\epsilon)\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(iH_\epsilon)V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\big)$$
$$-\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(f_\epsilon)\mathrm{op}(iH_\epsilon)V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\big)\big|$$

is bounded again by $c(M,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V)$. Here look at iH_ϵ more carefully. Note that

$$iH_\epsilon = \Big(\sum_{|\alpha+\beta|=1}\partial_x^\alpha\partial_\xi^\beta(\tilde{a}_{ij}[\xi])(\partial_x^\beta\partial_\xi^\alpha f_\epsilon)\frac{1}{f_\epsilon}\Phi_{1\epsilon}\Big) = \big(h_{ij}^\epsilon\big)\frac{1}{f_\epsilon}\Phi_{1\epsilon}$$

Taking $h_{ij}^\epsilon \in S(1,g)$ and $f_\epsilon^{-1}\Phi_{\epsilon 1}$, $\Phi_{\epsilon 1} \in S(1,g_0)$ into account one can write

$$f_\epsilon\#(iH_\epsilon) = \big(h_{ij}^\epsilon\big)\#\Phi_{\epsilon 1} + R_\epsilon$$

where $R_\epsilon \in S(M^{-1/2}\langle\xi\rangle_\gamma^{-1/2}, g)$ hence denoting $\tilde{H}_\epsilon = \big(h_{ij}^\epsilon\big)$ it results that

$$\big|\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(f_\epsilon)\mathrm{op}(iH_\epsilon)V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V)$$
$$-(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\tilde{H}_\epsilon)\mathrm{op}(\Phi_{\epsilon 1})V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\big)\big|$$

is bounded by $c(M,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V)$. From Lemma 6.8 we see that

$$h_{ij}^\epsilon \in \mathcal{C}(1),\;\; j \geq i, \quad h_{21}^\epsilon,\, h_{32}^\epsilon \in \mathcal{C}(\sigma^{1/2}), \quad h_{31}^\epsilon \in \mathcal{C}(\sigma)$$

then in view of Lemma 5.5

$$\lambda_i\#\big(\phi^{-n}\#h_{ij}^\epsilon - h_{ij}^\epsilon\#\phi^{-n}\big) \in S(\kappa\lambda_i\langle\xi\rangle_\gamma^{-1}\phi^{-n}, g),\;\; j \geq i,$$
$$\lambda_2\#\big(\phi^{-n}\#h_{21}^\epsilon - h_{21}^\epsilon\#\phi^{-n}\big) \in S(M^{-1/2}\kappa\lambda_2\lambda_1^{1/2}\langle\xi\rangle_\gamma^{-1/2}\phi^{-n}, g),$$
$$\lambda_3\#\big(\phi^{-n}\#h_{32}^\epsilon - h_{32}^\epsilon\#\phi^{-n}\big) \in S(\kappa\lambda_3\lambda_2^{1/2}\langle\xi\rangle_\gamma^{-1}, g),$$
$$\lambda_3\#\big(\phi^{-n}\#h_{31}^\epsilon - h_{31}^\epsilon\#\phi^{-n}\big) \in S(M^{-5/2}\kappa\lambda_3\lambda_1^{1/2}\langle\xi\rangle_\gamma^{-1/2}, g).$$

From this it follows that

$$\big|\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\tilde{H}_\epsilon)\mathrm{op}(\Phi_{\epsilon 1})V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V)$$
$$-(\mathrm{op}(\Lambda)\mathrm{op}(\tilde{H}_\epsilon)\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\big)\big|$$

is bounded by $c(M,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V)$.

Lemma 12.1. *One can write*

$$h_{ij}^\epsilon = \sum_{|\alpha+\beta|=1} k_{ij\alpha\beta}^\epsilon\#l_{ij\alpha\beta} + r_{ij}^\epsilon$$

where $k_{ij\alpha\beta}^\epsilon \in S(1,g_0)$ *such that* $|k_{ij\alpha\beta}^\epsilon| \leq C\nu$ *with some* $C>0$ *independent of* ν *and* ϵ *for any* $1 \leq i,j \leq 3$. *As for* $l_{ij\alpha\beta}$ *and* r_{ij}^ϵ *one has*

$$l_{ij\alpha\beta} \in S(1,g), \quad r_{ij}^\epsilon \in S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}, g), \quad (j \geq i),$$
$$l_{21\alpha\beta} \in S(\sigma^{-1/2}\sqrt{\lambda_1}, g), \quad r_{21}^\epsilon \in S(M^{-1/2}\sigma^{-1/2}\sqrt{\lambda_1}\langle\xi\rangle_\gamma^{-1/2}, g),$$
$$l_{31\alpha\beta} \in S(\sigma\sqrt{\lambda_1}, g), \quad r_{31}^\epsilon \in S(M^{-1/2}\sigma\sqrt{\lambda_1}\langle\xi\rangle_\gamma^{-1/2}, g),$$
$$l_{32\alpha\beta} \in S(\sqrt{\lambda_2}, g), \quad r_{32}^\epsilon \in S(M^{-1/2}\sqrt{\lambda_2}\langle\xi\rangle_\gamma^{-1/2}, g).$$

Proof. Set $k^{\epsilon}_{ij\alpha\beta}=\langle\xi\rangle_{\gamma}^{|\alpha|}\partial_x^{\beta}\partial_{\xi}^{\alpha}f_{\epsilon}$ and $l_{ij\alpha\beta}=\langle\xi\rangle_{\gamma}^{-|\alpha|}\partial_x^{\alpha}\partial_{\xi}^{\beta}(\tilde a_{ij}[\xi])$ then the assertion for $k^{\epsilon}_{ij\alpha\beta}$ is clear from (12.1). The assertions for $l_{ij\alpha\beta}$ follow from Lemma 6.8, Corollary 6.3 and Lemma 7.7. To prove the assertions for r^{ϵ}_{ij} note that $\partial_x^{\mu}\partial_{\xi}^{\nu}l_{ij\alpha\beta}\in S(\sigma^{-1/2}\langle\xi\rangle_{\gamma}^{-|\nu|},g)$, $|\mu+\nu|=1$ for $j\geq i$ and

$$\partial_x^{\mu}\partial_{\xi}^{\nu}l_{21\alpha\beta},\;\;\partial_x^{\mu}\partial_{\xi}^{\nu}l_{32\alpha\beta}\in S(\langle\xi\rangle_{\gamma}^{-|\nu|},g),\quad \partial_x^{\mu}\partial_{\xi}^{\nu}l_{31\alpha\beta}\in S(\sigma^{1/2}\langle\xi\rangle_{\gamma}^{-|\nu|},g)$$

for $|\mu+\nu|=1$ which follows from $\tilde a_{21},\ \tilde a_{32}\in\mathcal{C}(\sigma)$ and $\tilde a_{31}\in\mathcal{C}(\sigma^{5/2})$. Then remarking that $\sigma\geq M\langle\xi\rangle_{\gamma}^{-1}$ and $\lambda_1\geq M\sigma\langle\xi\rangle_{\gamma}^{-1}$ the assertions for r^{ϵ}_{ij} are checked immediately. □

With $R^{\epsilon}=(r^{\epsilon}_{ij})$ and $W=\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V$, recalling $\lambda_1\leq C\sigma\lambda_2\leq C\sigma^2\lambda_3$, it is easy to see that

$$\big|(\mathrm{op}(R^{\epsilon})W,\mathrm{op}(\Lambda)W)\big|\leq c(M,\epsilon)\|\mathrm{op}(\Lambda^{1/2})\langle D\rangle_{\gamma}^{-1/4}W\|^2.$$

Turn to $(\mathrm{op}(h^{\epsilon}_{ij})W_j,\mathrm{op}(\lambda_i)W_i)$. Write $\lambda_i=\lambda_i^{1/2}\#(1+k_i)\#\lambda_i^{1/2}$ with $k_i\in S(M^{-1},g)$ then thanks to Lemma 12.1 it follows that

$$\begin{aligned}&\big|(\mathrm{op}(\lambda_i)\mathrm{op}(h^{\epsilon}_{ij})W_j,\mathrm{op}(\lambda_i)W_i)\big|\\ &\leq(1+CM^{-1})\sum_{|\alpha+\beta|=1}\big|(\mathrm{op}(\lambda_i^{1/2})\mathrm{op}(l_{ij\alpha\beta})W_j,\mathrm{op}(k^{\epsilon}_{ij\alpha\beta})\mathrm{op}(\lambda_i^{1/2})W_i)\big|\\ &\leq C(1+M^{-1})\|\mathrm{op}(\lambda_i^{1/2})\mathrm{op}(l_{ij\alpha\beta})W_j\|\|\mathrm{op}(k^{\epsilon}_{ij\alpha\beta})\mathrm{op}(\lambda_i^{1/2})W_i\|\\ &\leq C'(1+M^{-1})\|\mathrm{op}(\lambda_j^{1/2})W_j\|\|\mathrm{op}(k^{\epsilon}_{ij\alpha\beta})\mathrm{op}(\lambda_i^{1/2})W_i\|\end{aligned}$$

because $\lambda_i^{1/2}\#l_{ij\alpha\beta}\in S(\lambda_j^{1/2},g)$ in view of Lemma 12.1. On the other hand, taking Lemma 12.1 into account, it follows from the sharp Gårding inequality (e.g. [6, Theorem 18.1.14])

$$\begin{aligned}\|\mathrm{op}(k^{\epsilon}_{ij\alpha\beta})\mathrm{op}(\lambda_i^{1/2})W_i\|&\leq C\nu\|\mathrm{op}(\lambda_i^{1/2})W_i\|\\ &+C(M,\nu,\epsilon)\|\mathrm{op}(\lambda_i^{1/2})\langle D\rangle_{\gamma}^{-1/2}W_i\|.\end{aligned}$$

Therefore applying the above obtained estimates one can find $C>0$ independent of ϵ, ν and M such that

$$\begin{aligned}\big|\mathsf{Re}\big(\mathrm{op}(\Lambda)\mathrm{op}(\tilde H_{\epsilon})W,W\big)\big|&\leq C(\nu+M^{-1/2})\|\mathrm{op}(\Lambda^{1/2})W\|^2\\ &+C(M,\nu,\epsilon)\|\mathrm{op}(\Lambda^{1/2})\langle D\rangle_{\gamma}^{-1/4}W\|^2\\ &\leq C(\nu+M^{-1/2})\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\|^2\\ &+C'(M,\nu,\epsilon)\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\langle D\rangle_{\gamma}^{-1/4}V\|^2.\end{aligned}$$

Since it follows from the same reasoning that

$$\begin{aligned}&\big|\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(f_{\epsilon}^{-1}\Phi_{\epsilon})V,\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon})V\big)\\ &-\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V,\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\big)\big|\\ &\leq c(M,\epsilon)\mathcal{N}(\langle D\rangle_{\gamma}^{-1/4}V)\end{aligned}$$

we obtain finally

$$
(12.4)\qquad \begin{aligned}
&-\mathsf{Re}\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(f_\epsilon^{-1}\Phi_\epsilon)V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V\big)\\
&+\mathsf{Re}\big(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})[\mathrm{op}(\Phi_\epsilon), \mathrm{op}(i\mathcal{A})]V, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V\big)\\
&\le -\big(1-C(\nu+M^{-1/2})\big)\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V\|^2\\
&\qquad +c(M,\nu,\epsilon)\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V).
\end{aligned}
$$

We fix M_0 and ν_0 such that $1-C(\nu_0+M_0^{-1/2})\ge 0$ and Proposition 10.1 holds, and from now on $\gamma = M_0^5$ and $\delta = 1/M_0^4$ are assumed to be fixed. Applying Proposition 10.1 to $\mathrm{op}(\Phi)V$ instead of V one obtains, in view of (12.3) and (12.4) that

Proposition 12.1. *For any $0<\nu\le\nu_0$ and any $\epsilon>0$ one can find $C>0$ such that*

$$
\begin{aligned}
\mathcal{E}_1(\mathrm{op}(\Phi_\epsilon)V)+\int_0^t \tau^{-2n}\mathcal{N}(\mathrm{op}(\Phi_\epsilon)V)d\tau&\\
\le C\int_0^t \tau^{-2n}\|\mathrm{op}(\kappa^{-1/2}\Lambda^{1/2})\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)\tilde{L}V\|^2 d\tau&\\
+C\int_0^t \tau^{-2n}\mathcal{N}(\langle D\rangle_\gamma^{-1/4}V)d\tau.&
\end{aligned}
$$

Applying $\langle D\rangle_\gamma^s$ to (12.3) and repeating similar arguments proving Propositions 11.1 and 12.1 one obtains

Proposition 12.2. *For any $s\in\mathbb{R}$, any $0<\nu\le\nu_0$ and any $\epsilon>0$ one can find $C>0$ such that*

$$
\begin{aligned}
\mathcal{E}_1(\langle D\rangle_\gamma^s\mathrm{op}(\Phi_\epsilon)V)+\int_0^t \tau^{-2n}\mathcal{N}(\langle D\rangle_\gamma^s\mathrm{op}(\Phi_\epsilon)V)d\tau&\\
\le C\int_0^t \tau^{-2n+1}\|\langle D\rangle_\gamma^{n+s}\mathrm{op}(\Phi_\epsilon)\tilde{L}V\|^2 d\tau + C\int_0^t \tau^{-2n}\mathcal{N}(\langle D\rangle_\gamma^{s-1/4}V)d\tau&
\end{aligned}
$$

for $0\le t\le\delta$.

12.2 Micro support propagates with finite speed

Lemma 12.2. *Assume $t^{-n}\langle D\rangle_\gamma^{l_1}V\in L^2((0,\delta)\times\mathbb{R}^d)$ and $t^{-n+1/2}\langle D\rangle_\gamma^{l_2}\tilde{L}V\in L^2((0,\delta)\times\mathbb{R}^d)$ and $t^{-n+1/2}\langle D\rangle_\gamma^{n+s_0}\mathrm{op}(\Phi_{\epsilon_0})\tilde{L}V\in L^2((0,\delta)\times\mathbb{R}^d)$ with some $l_1,l_2\in\mathbb{R}$ and $s_0\in\mathbb{R}$. Then for every $0<\epsilon_0<\epsilon$ we have*

$$
t^{-n}\langle D\rangle_\gamma^s\Phi_\epsilon V\in L^2((0,\delta)\times\mathbb{R}^d)
$$

for all $s\le s_0-5/4$. Moreover

$$
\begin{aligned}
\int_0^t \tau^{-2n}\|\langle D\rangle_\gamma^s\mathrm{op}(\Phi_\epsilon)V(\tau)\|^2 d\tau \le C\int_0^t\Big(\tau^{-2n}\|\langle D\rangle_\gamma^{l_1}V(\tau)\|^2&\\
+\tau^{-2n+1}\|\langle D\rangle_\gamma^{l_2}\tilde{L}V(\tau)\|^2\Big)d\tau + C\int_0^t \tau^{-2n+1}\|\langle D\rangle_\gamma^{n+s_0}\mathrm{op}(\Phi_{\epsilon_0})\tilde{L}V(\tau)\|^2 d\tau&
\end{aligned}
$$

for $0 < t \leq \delta$.

Proof. We may assume $l_1 \leq s_0$ otherwise nothing to be proved. Let J be the largest integer such that $l_1 + J/4 \leq s_0$. Take $\epsilon_j > 0$ such that $\epsilon_0 < \epsilon_1 < \cdots < \epsilon_J = \epsilon$. We write $\Phi_{\epsilon_j} = \Phi_j$ and $f_j = f_{\epsilon_j}$ in this proof. Inductively we show that

$$\begin{aligned}\int_0^t \tau^{-2n}\mathcal{N}(\langle D\rangle_\gamma^{l_1+j/4}\mathrm{op}(\Phi_j)V)d\tau \leq C\int_0^t \tau^{-2n}\|\langle D\rangle_\gamma^{l_1}V(\tau)\|^2 d\tau \\ +C\int_0^t \tau^{-2n+1}\big\{\|\langle D\rangle_\gamma^{l_2}\tilde{L}V(\tau)\|^2 + \|\langle D\rangle_\gamma^{l_1+n+j/4}\mathrm{op}(\Phi_0)\tilde{L}V(\tau)\|^2\big\}d\tau.\end{aligned} \tag{12.5}$$

Note that from (10.12) and (10.11) it follows

$$\|\langle D\rangle_\gamma^{-1}V\|/C \leq \mathcal{N}(V) \leq C\|\langle D\rangle_\gamma^{n}V\|. \tag{12.6}$$

Choose $\psi_j(x,\xi) \in S(1,g_0)$ so that $\mathrm{supp}\,\psi_j \subset \{f_j < 0\}$ and $\{f_{j+1} < 0\} \subset \{\psi_j = 1\}$. Noting that

$$\mathrm{op}(\Phi_{j+1})\tilde{L}\,\mathrm{op}(\psi_j) = \mathrm{op}(\Phi_{j+1}\#\psi_j)\tilde{L} + \mathrm{op}(\Phi_{j+1})[\tilde{L},\mathrm{op}(\psi_j)]$$

we apply Proposition 12.2 with $s = l_1 + (j+1)/4$, $\Phi = \Phi_{j+1}$ and $V = \mathrm{op}(\psi_j)V$. Since $\Phi_{j+1}\#\psi_j - \Phi_{j+1} \in S^{-\infty}$ then $\|\langle D\rangle_\gamma^{l_1+(j+1)/4+n}\mathrm{op}(\Phi_{j+1})\tilde{L}\,\mathrm{op}(\psi_j)V\|^2$ is bounded by

$$c\|\langle D\rangle_\gamma^{l_1+(j+1)/4+n}\mathrm{op}(\Phi_{j+1})\tilde{L}V\|^2 + C(j)\|\langle D\rangle_\gamma^{l_1}V\|^2$$

and hence by

$$C(j)\big\{\|\langle D\rangle_\gamma^{l_1+(j+1)/4+n}\mathrm{op}(\Phi_{\epsilon_0})\tilde{L}V\|^2 + \big\{\|\langle D\rangle_\gamma^{l_2}\tilde{L}V\|^2 + \|\langle D\rangle_\gamma^{l_1}V\|^2\big\} \tag{12.7}$$

because $\Phi_{j+1} - k_j\#\Phi_{\epsilon_0} \in S^{-\infty}$ with some $k_j \in S(1,g_0)$. Since $\psi_j - \tilde{k}_j\#\Phi_j \in S^{-\infty}$ with some $\tilde{k}_j \in S(1,g_0)$ it follows that

$$\mathcal{N}(\langle D\rangle_\gamma^{l_1+j/4}\mathrm{op}(\Phi_{j+1})\mathrm{op}(\psi_j)V) \leq C\mathcal{N}(\langle D\rangle_\gamma^{l_1+j/4}\mathrm{op}(\Phi_j)V) + C\|\langle D\rangle_\gamma^{l_1}V\|^2.$$

Consider $\mathcal{N}(\langle D\rangle_\gamma^{l_1+(j+1)/4}\mathrm{op}(\Phi_{j+1})\mathrm{op}(\psi_j)V)$. Noting that $\Phi_{j+1}\#\psi_j - \Phi_{j+1} \in S^{-\infty}$ the same reasoning shows that

$$\begin{aligned}\mathcal{N}(\langle D\rangle_\gamma^{l_1+(j+1)/4}\mathrm{op}(\Phi_{j+1})V) \\ \leq C\mathcal{N}(\langle D\rangle_\gamma^{l_1+(j+1)/4}\mathrm{op}(\Phi_{j+1})\mathrm{op}(\psi_j)V) + C\|\langle D\rangle_\gamma^{l_1}V\|^2.\end{aligned} \tag{12.8}$$

Multiply (12.8) and (12.7) by t^{-2n} and t^{-2n+1} respectively and integrate it from 0 to t we conclude from Proposition 12.2 that (12.5) holds for $j+1$ and hence for $j = J$. Since $l_1 + J/4 \leq s_0$ and $l_1 + J/4 > s_0 - 1/4$ we conclude the assertion by (12.6). □

Let Γ_i $(i = 1,2,3)$ be open conic sets in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ with relatively compact basis such that $\Gamma_1 \Subset \Gamma_2 \Subset \Gamma_3$. Take $h_i(x,\xi) \in S(1,g_0)$ with $\operatorname{supp} h_1 \subset \Gamma_1$ and $\operatorname{supp} h_2 \subset \Gamma_3 \backslash \Gamma_2$. Consider a solution V with $t^{-n}\langle D\rangle_\gamma^l V \in L^2((0,\delta)\times\mathbb{R}^d)$ to the equation

$$\tilde{L}V = \mathrm{op}(h_1)F, \quad t^{-n+1/2}\langle D\rangle_\gamma^s F \in L^2((0,\delta)\times\mathbb{R}^d).$$

Proposition 12.3. *Notations being as above. There exists $\delta' = \delta'(\Gamma_i) > 0$ such that for any $r \in \mathbb{R}$ there is $C > 0$ such that*

$$\begin{aligned}&\int_0^t \tau^{-2n}\|\langle D\rangle_\gamma^r \mathrm{op}(h_2)V(\tau)\|^2 d\tau\\ &\leq C\int_0^t \big\{\tau^{-2n+1}\|\langle D\rangle_\gamma^s F(\tau)\|^2 + \tau^{-2n}\|\langle D\rangle_\gamma^l V(\tau)\|^2\big\}d\tau, \quad 0 < t \leq \delta'.\end{aligned}$$

Proof. Let $f_\epsilon = t - \nu_0\hat{\tau} + \nu_0 d_\epsilon(x,\xi;y,\eta)$ with a small $\hat{\tau} > 0$. It is clear that there is $\hat{\epsilon} > 0$ such that

$$\{t \geq 0\} \cap \{f_{\hat{\epsilon}} \leq 0\} \cap \big(\mathbb{R}\times \operatorname{supp} h_1\big) = \emptyset$$

for any $(y,\eta) \notin \Gamma_2$. Take $\hat{\epsilon} < \tilde{\epsilon} < \hat{\tau}$. It is also clear that one can find a finite number of $(y_i,\eta_i) \in \Gamma_3 \setminus \Gamma_2$, $i = 1,\ldots,M$ such that with $\delta' = \nu_0(\hat{\tau} - \tilde{\epsilon})/2$

$$\begin{gathered}\Gamma_3 \setminus \Gamma_2 \Subset \Big(\bigcup_{i=1}^M \{f_{\tilde{\epsilon}}(\delta',x,\xi;y_i,\eta_i) \leq 0\}\Big),\\ \{t \geq 0\} \cap \{f_{\tilde{\epsilon}}(t,x,\xi;y_i,\eta_i) \leq 0\} \cap \big(\mathbb{R}\times \operatorname{supp} h_1\big) = \emptyset.\end{gathered}$$

Now $\Phi_{i\epsilon}$ is defined by (12.2) with $f_\epsilon(t,x,\xi;y_i,\eta_i)$. Then since $\sum \Phi_{i\tilde{\epsilon}} > 0$ on $[0,\delta'] \times \operatorname{supp} h_2$ there is $k \in S(1,g_0)$ such that $h_2 - k\sum \Phi_{i\tilde{\epsilon}} \in S^{-\infty}$. Noting that $t^{-n+1/2}\langle D\rangle_\gamma^r \mathrm{op}(\Phi_{i\hat{\epsilon}})\mathrm{op}(h_1)F \in L^2((0,\delta)\times\mathbb{R}^d)$ for any $r \in \mathbb{R}$ we apply Lemma 12.2 with $\Phi_{\epsilon_0} = \Phi_{\hat{\epsilon}}$, $\Phi_\epsilon = \Phi_{i\tilde{\epsilon}}$ and $s_0 = r + 5/4$ to obtain

$$\begin{aligned}\int_0^t \tau^{-2n}\|\langle D\rangle_\gamma^r \mathrm{op}(\Phi_{i\tilde{\epsilon}})V(\tau)\|^2 d\tau \leq C\int_0^t \tau^{-2n}\|\langle D\rangle_\gamma^l V(\tau)\|^2 d\tau\\ + \int_0^t \tau^{-2n+1}\big(\|\langle D\rangle_\gamma^{2n+r+5/4}\mathrm{op}(\Phi_{i\hat{\epsilon}})\mathrm{op}(h_1)F(\tau)\|^2 + \|\langle D\rangle_\gamma^s F(\tau)\|\big)d\tau\end{aligned}$$

for $\|\langle D\rangle_\gamma^s \tilde{L}V(\tau)\| \leq C\|\langle D\rangle_\gamma^s F(\tau)\|$. Since $\Phi_{i\hat{\epsilon}}\# h_1 \in S^{-\infty}$ summing up the above estimates over $i = 1,\ldots,M$ one concludes the desired assertion. □

Lemma 12.3. *The same assertion as Proposition 12.3 holds for L.*

Proof. Assume that U satisfies

$$LU = \mathrm{op}(h_1)F, \quad t^{-n}\langle D\rangle_\gamma^l U \in L^2((0,\delta)\times\mathbb{R}^d)$$

where $t^{-n+1/2}\langle D\rangle_\gamma^s F \in L^2((0,\delta)\times\mathbb{R}^d)$. Choose $\tilde{\Gamma}_i$ such that $\Gamma_1 \Subset \tilde{\Gamma}_1 \Subset \tilde{\Gamma}_2 \Subset \Gamma_2 \Subset \Gamma_3 \Subset \tilde{\Gamma}_3$ and $\tilde{h}_i \in S(1,g_0)$ such that $\operatorname{supp}\tilde{h}_1 \subset \tilde{\Gamma}_1$, $\operatorname{supp}\tilde{h}_2 \subset \tilde{\Gamma}_3 \setminus \tilde{\Gamma}_2$

and $\tilde{h}_i = 1$ on the support of h_i. Recall that $L\,\text{op}(T) = \text{op}(T)\,\tilde{L}$. Then with $U = \text{op}(T)V$ one has

$$\tilde{L}V = (I + \text{op}(K))\text{op}(T^{-1})\text{op}(h_1)F.$$

Since there is $\tilde{T} \in S(1,g)$ such that $(I+K)\#T^{-1}\#h_1 - \tilde{h}_1\tilde{T} \in S^{-\infty}$ it follows from Proposition 12.3 (or rather its proof) that

$$\int_0^t \tau^{-2n}\|\langle D\rangle_\gamma^r \text{op}(\tilde{h}_2)V(\tau)\|^2 d\tau$$
$$\leq C\int_0^t \{\tau^{-2n}\|\langle D\rangle_\gamma^l V(\tau)\|^2 + \tau^{-2n+1}\|\langle D\rangle_\gamma^s F(\tau)\|^2\}d\tau.$$

Similarly since there is $\tilde{T} \in S(1,g)$ such that $h_2\#T - \tilde{h}_2\tilde{T} \in S^{-\infty}$ repeating the same arguments we conclude the assertion. □

Returning to $\hat{P}$ we have

Proposition 12.4. *Notations being as above. Then there exists $\delta' = \delta'(\Gamma_i) > 0$ such that for any s, $r \in \mathbb{R}$ there is C such that for any solution u to*

$$\hat{P}u = \text{op}(h_1)f, \quad t^{-n}\langle D\rangle^{l+2-j}D_t^j u \in L^2((0,\delta')\times\mathbb{R}^d), \;\; j = 0,1,2$$

with $t^{-n+1/2}\langle D\rangle^s f \in L^2((0,\delta')\times\mathbb{R}^d)$ one has

$$\int_0^t \tau^{-2n}\sum_{j=0}^2 \|\langle D\rangle^{r+2-j}\text{op}(h_2)D_t^j u(\tau)\|^2 d\tau$$
$$\leq C\int_0^t \{\tau^{-2n+1}\|\langle D\rangle^s f(\tau)\|^2 + \tau^{-2n}\sum_{j=0}^2 \|\langle D\rangle^{l+2-j}D_t^j u(\tau)\|^2\}d\tau, \;\; 0 < t \leq \delta'.$$

Denote by $\mathcal{H}_{n,s}((0,\delta)\times\mathbb{R}^d)$ the set of all u such that

$$\int_0^\delta \tau^{-2n}\|\langle D\rangle^s f(\tau,\cdot)\|^2 d\tau < +\infty.$$

Thanks to Theorem 11.1 for any $f \in \mathcal{H}_{-n+1/2,n+s}((0,\delta)\times\mathbb{R}^d)$ there is a unique solution $u \in \mathcal{H}_{-n,s+1}((0,\delta)\times\mathbb{R}^d)$ to $\hat{P}u = f$ satisfying (11.6). Denote this map by

$$\hat{G} : \mathcal{H}_{-n+1/2,n+s}((0,\delta)\times\mathbb{R}^d) \ni f \mapsto u \in \mathcal{H}_{-n,s+1}((0,\delta)\times\mathbb{R}^d)$$

then it follows from Proposition 12.4 and Theorem 11.1 that

$$\int_0^t \tau^{-2n}\sum_{j=0}^2 \|\langle D\rangle^{r+2-j}\text{op}(h_2)D_t^j\hat{G}\,\text{op}(h_1)f(\tau)\|^2 d\tau$$
$$\leq C\int_0^t \{\tau^{-2n+1}\|\langle D\rangle^{n+s}f(\tau)\|^2 + \tau^{-2n}\sum_{j=0}^2 \|\langle D\rangle^{s+1-j}D_t^j u(\tau)\|^2\}d\tau$$
$$\leq C\int_0^t \tau^{-2n+1}\|\langle D\rangle^{n+s}f(\tau)\|^2 d\tau.$$

Replacing $n+s$ by s and $r+2$ by r we obtain

$$\int_0^t \tau^{-2n}\sum_{j=0}^{2}\|\langle D\rangle^{r-j}\mathrm{op}(h_2)D_t^j\hat{G}\,\mathrm{op}(h_1)f(\tau)\|^2 d\tau$$
$$\leq C\int_0^t \tau^{-2n+1}\|\langle D\rangle^s f(\tau)\|^2 d\tau.$$

Summarizing we conclude

Proposition 12.5. *Notations being as above and let Γ_i $(i=1,2,3)$ be open conic sets in $\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$ with relatively compact basis such that $\Gamma_1\Subset\Gamma_2\Subset\Gamma_3$ and $h_i(x,\xi)\in S(1,g_0)$ with $\mathrm{supp}\,h_1\subset\Gamma_1$ and $\mathrm{supp}\,h_2\subset\Gamma_3\setminus\Gamma_2$. Then there exists $\delta'=\delta'(\Gamma_i)>0$ such that for any r, s one can find $C>0$ such that*

$$\int_0^t \tau^{-2n}\sum_{j=0}^{2}\|\langle D\rangle^{r-j}\mathrm{op}(h_2)D_t^j\hat{G}\,\mathrm{op}(h_1)f(\tau)\|^2 d\tau$$
$$\leq C\int_0^t \tau^{-2n+1}\|\langle D\rangle^s f(\tau)\|^2 d\tau,\quad 0<t\leq\delta'$$

for any $f\in\mathcal{H}_{-n+1/2,s}((0,\delta')\times\mathbb{R}^d)$.

Denote by $\mathcal{H}^*_{n,s}((0,\delta]\times\mathbb{R}^d)$ the set of all f with $\int_0^\infty t^{2n}\|\langle D\rangle^s f\|^2 dt<+\infty$ such that $f=0$ for $t\geq\delta$. Thanks to Theorem 11.2 for any $f\in\mathcal{H}^*_{n+1/2,n+s}((0,\delta]\times\mathbb{R}^d)$ there is a unique solution $u\in\mathcal{H}^*_{n,s+1}((0,\delta]\times\mathbb{R}^d)$ to $\hat{P}^*u=f$ satisfying (11.6). Denote this map by

$$\hat{G}^*:\mathcal{H}^*_{n+1/2,n+s}((0,\delta]\times\mathbb{R}^d)\ni f\mapsto u\in\mathcal{H}^*_{n,s+1}((0,\delta]\times\mathbb{R}^d).$$

Repeating similar arguments one obtains

Proposition 12.6. *Notations being as in Proposition 12.5. Then there exists $\delta'=\delta'(\Gamma_i)>0$ such that for any r, s one can find $C>0$ such that*

$$\int_t^{\delta'} \tau^{2n}\sum_{j=0}^{2}\|\langle D\rangle^{r-j}\mathrm{op}(h_2)D_t^j\hat{G}^*\,\mathrm{op}(h_1)f(\tau)\|^2 d\tau$$
$$\leq C\int_t^{\delta'} \tau^{2n+1}\|\langle D\rangle^s f(\tau)\|^2 d\tau,\quad 0<t\leq\delta'$$

for any $f\in\mathcal{H}^*_{n+1/2,s}((0,\delta']\times\mathbb{R}^d)$.

Remark 12.1. It is clear from the proof that for any $n'\geq n$, Propositions 12.5 and 12.6 hold.

13 Proof of Theorem 1.1

Applying the fact that the micro support of $u(t,\cdot)$, solution to $\hat{P}u=f$ obtained by Theorem 11.1, propagates with a finite speed (Proposition 12.5) we prove Theorem 1.1 following [22], [24].

13.1 Parametrix with finite propagation speed

Consider

$$P = D_t^m + \sum_{j=1}^{m} a_j(t,x,D)\langle D\rangle^j D_t^{m-j} \tag{13.1}$$

which is differential operator in t with coefficients $a_j \in S^0$. We say that G is a parametrix for P with finite propagation speed of micro supports (which we abbreviate to "parametrix with fps" from now on) with loss of (n,l) derivatives if G satisfies the following conditions:

(i) There exists $\delta > 0$ such that for any $s \in \mathbb{R}$ there is $C > 0$ such that we have $PGf = f$ and

$$\sum_{j=0}^{m-1} \int_0^t \tau^{-2n} \|\langle D\rangle^{-l+s+m-j} D_t^j G f(\tau)\|^2 d\tau$$
$$\leq C \int_0^t \tau^{-2n+1} \|\langle D\rangle^s f(\tau)\|^2 d\tau, \quad f \in \mathcal{H}_{-n+1/2,s}((0,\delta)\times\mathbb{R}^d).$$

(ii) For any $h_j(x,\xi) \in S(1,g_0)$, $j = 1,2$ with $\operatorname{supp} h_2 \Subset (\mathbb{R}^d \times \mathbb{R}^d) \setminus \operatorname{supp} h_1$ there exists $\delta' > 0$ such that for any r, $s \in \mathbb{R}$ there is $C > 0$ such that

$$\begin{aligned}&\sum_{j=0}^{m-1} \int_0^t \tau^{-2n} \|\langle D\rangle^{r-j} \mathrm{op}(h_2) D_t^j G \mathrm{op}(h_1) f(\tau)\|^2 d\tau \\ &\qquad \leq C \int_0^t \tau^{-2n+1} \|\langle D\rangle^s f(\tau)\|^2 d\tau, \quad 0 < t \leq \delta'\end{aligned} \tag{13.2}$$

holds for any $f \in \mathcal{H}_{-n+1/2,s}((0,\delta')\times\mathbb{R}^d)$.

Let P_1 and P_2 be two operators of the form (13.1). We say

$$P_1 \equiv P_2 \quad \text{at} \ (\hat{x},\hat{\xi})$$

if there exist $\delta' > 0$ and a conic neighborhood W of $(\hat{x},\hat{\xi})$ such that

$$P_1 - P_2 = \sum_{j=1}^{m} R_j(t,x,D)\langle D\rangle^j D_t^{m-j} \tag{13.3}$$

with $R_j \in S^0$ which are in $S^{-\infty}(W)$ uniformly in $0 \leq t \leq \delta'$.

Theorem 13.1. *Assume that for any $(\hat{x},\eta)$, $|\eta| = 1$ one can find P_η of the form* (13.1) *having a parametrix with fps with loss of $(n,\ell(\eta))$ derivatives such that $P \equiv P_\eta$ at $(\hat{x},\eta)$. Then there exist $\delta > 0$, $\ell \geq 0$ and a neighborhood U of $\hat{x}$ such that for every $f \in \mathcal{H}_{-n+1/2,s+\ell}((0,\delta)\times\mathbb{R}^d)$ there exists u with $D_t^j u \in \mathcal{H}_{-n,s+m-j}((0,\delta)\times\mathbb{R}^d)$, $0 \leq j \leq m-1$, satisfying*

$$Pu = f \quad \text{in} \quad (0,\delta)\times U$$

where $\ell = \sup_{|\eta|=1} \ell(\eta)$.

Proof. By assumption P_η has a parametrix G_η with fps with loss of $(n, \ell(\eta))$ derivatives. There are finite open conic neighborhood W_i of $(\hat{x}, \eta_i)$ such that $\cup_i W_i \supset \Omega \times (\mathbb{R}^d \setminus \{0\})$, where Ω is a neighborhood of $\hat{x}$, and $P \equiv P_{\eta_i}$ at $(\hat{x}, \eta)$ with $W = W_i$ in (13.3). Now take another open conic covering $\{V_i\}$ of $\Omega \times (\mathbb{R}^d \setminus \{0\})$ with $V_i \Subset W_i$, and a partition of unity $\{\alpha_i(x, \xi)\}$ subordinate to $\{V_i\}$ so that

$$\sum_i \alpha_i(x, \xi) = \alpha(x)$$

where $\alpha(x)$ is equal to 1 in a neighborhood of $\hat{x}$. Define

$$G = \sum_i G_{\eta_i} \alpha_i.$$

Then denoting $P - P_{\eta_i} = R_i$ we have

$$PGf = \sum_i PG_{\eta_i} \alpha_i f = \sum_i P_{\eta_i} G_i \alpha_i f + \sum_i R_i G_i \alpha_i f = \alpha(x) f - Rf$$

where $R = \sum_i R_i G_{\eta_i} \alpha_i$. Then

$$\int_0^t \tau^{-2n} \|\langle D \rangle^{s+\ell} Rf(\tau)\|^2 d\tau \le C \int_0^t \tau^{-2n+1} \|\langle D \rangle^{s+\ell} f(\tau)\|^2 d\tau$$

for $0 \le t \le \delta''$ with some $\delta'' > 0$ in view of (13.2) where $\ell = \max_i \ell(\eta_i)$. Choosing $\delta_1 > 0$ small such that

$$\int_0^t \tau^{-2n} \|\langle D \rangle^{s+\ell} Rf(\tau)\|^2 d\tau \le \frac{1}{2} \int_0^t \tau^{-2n} \|\langle D \rangle^{s+\ell} f(\tau)\|^2 d\tau, \quad 0 < t \le \delta_1$$

for $f \in \mathcal{H}_{-n,s+\ell}((0,\delta) \times \mathbb{R}^d))$. With $S = \sum_{k=0}^\infty R^k$ one has $Sf \in \mathcal{H}_{-n,s+\ell}((0,\delta_1) \times \mathbb{R}^d)$ and

$$\int_0^t \tau^{-2n} \|\langle D \rangle^{s+\ell} Sf(\tau)\|^2 d\tau \le 2 \int_0^t \tau^{-2n} \|\langle D \rangle^{s+\ell} f(\tau)\|^2 d\tau, \quad 0 < t \le \delta_1.$$

Let $\gamma(x) \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 near $\hat{x}$ such that $\operatorname{supp} \gamma \Subset \{\alpha = 1\}$. Since $\gamma(\alpha - R)S = \gamma(I - R)S = \gamma$ it follows that

$$\gamma(x) PGSf = \gamma(x) f.$$

With $u = GSf$ one has

$$\sum_{j=0}^{m-1} \int_0^t \tau^{-2n} \|\langle D \rangle^{s+m-j} D_t^j u(\tau)\|^2 d\tau \le C \int_0^t \tau^{-2n+1} \|\langle D \rangle^{s+\ell} Sf(\tau)\|^2 d\tau$$

$$\le C' \int_0^t \tau^{-2n} \|\langle D \rangle^{s+\ell} f(\tau)\|^2 d\tau$$

which proves the assertion. □

We define a parametrix with fps for P^* with obvious modifications then

Theorem 13.2. *Assume that for any $(\hat{x}, \eta)$, $|\eta| = 1$ one can find P^*_η of the form (13.1) such that $P^* \equiv P^*_\eta$ for which parametrix with fps exists. Then there exist $\delta > 0$, $\ell \geq 0$ and a neighborhood U of $\hat{x}$ such that for every $f \in \mathcal{H}^*_{n+1/2,s+\ell}((0,\delta] \times \mathbb{R}^d)$ there exists u with $D_t^j u \in \mathcal{H}^*_{n,s+m-j}((0,\delta] \times \mathbb{R}^d)$, $0 \leq j \leq m-1$, satisfying*

$$P^* u = f \quad in \quad (0,\delta) \times U.$$

13.2 Local existence and uniqueness

First consider a third order operator P of the form (2.1). To reduce P to the case $a_1(t,x,D) = 0$ we apply a Fourier integral operator, which is actually the solution operator $S(t',t)$ of the Cauchy problem

$$D_t u + a_1(t,x,D_x)u = 0, \quad u(t',x) = \phi(x)$$

such that $S(t',t) : \phi \mapsto u(t)$ then it is clear that $S(t,0)(D_t + a_1)S(0,t) = D_t$. Let

$$S(t,0)PS(0,t) = \tilde{P}$$

and assume that $\tilde{P}$ has a parametrix with fps $\tilde{G}$ with loss of (n,ℓ) derivatives. Then one can show that $G = S(0,t)\tilde{G}S(t,0)$ is a parametrix of P with fps with loss of (n,ℓ) derivatives.

Let $|\eta| = 1$ be given. Assume that p has a triple characteristic root $\bar{\tau}$ at $(0,0,\eta)$ and $(0,0,\bar{\tau},\eta)$ is effectively hyperbolic. Theorem 11.1 and Proposition 12.4 imply that $\hat{P}$, which coincides with the original P in W_M, given by (4.3), has a parametrix with fps with loss of $(n,n+2)$ derivatives. Now assume that p has a double characteristic root $\bar{\tau}$ at $(0,0,\eta)$ such that $(0,0,\bar{\tau},\eta)$ is effectively hyperbolic characteristic if it is a critical point. Note that one can write

$$p(t,x,\tau,\xi) = \big(\tau + b(t,x,\xi)\big)\big(\tau^2 + a_1(t,x,\xi)\tau + a_2(t,x,\xi)\big) = p_1 p_2$$

in a conic neighborhood of $(0,0,\eta)$ where $p_1(0,0,\bar{\tau},\eta) \neq 0$. Note that there exist $\hat{P}_i$ such that

$$P \equiv \hat{P}_1 \cdot \hat{P}_2 \quad \text{at} \ (0,\eta)$$

where the principal symbol of $\hat{P}_j$ coincides with p_j in a conic neighborhood of $(0,0,\eta)$. Note that if $\hat{P}_i$ has a parametrix with fps G_i with loss of (n,ℓ_i) derivatives then one can see that G_2G_1 is a parametrix with fps for $\hat{P}_1 \cdot \hat{P}_2$ with loss of $(n,\ell_1+\ell_2)$ derivatives.

First assume that $(0,0,\bar{\tau},\eta)$ is a critical point. Then it is easy to see that

$$F_p(0,0,\bar{\tau},\eta) = cF_{p_2}(0,0,\bar{\tau},\eta)$$

with some $c \neq 0$ and hence $(0,0,\bar{\tau},\eta)$ is effectively hyperbolic characteristic of p_2. Then following [22, 24] there is a parametrix with fps for $\hat{P}_2$. Since $\hat{P}_1$ is a first order operator with real principal symbol p_1 it is easy to see that $\hat{P}_1$ has a parametrix with fps. Therefore P has a parametrix with fps. Turn to the case that $(0,0,\bar{\tau},\eta)$ is not a critical point. Writing p_2 as

$$p_2(t,x,\tau,\xi) = \tau^2 - a(t,x,\xi)|\xi|^2$$

it is easily seen that $(0,0,\bar{\tau},\eta)$ is not a critical point implies that $\partial_t a(0,0,\eta) > 0$, which is the case that $\hat{P}_2$ is a hyperbolic operator of principal type and some detailed discussion is found in [6, Chapter 23.4]. It is easily proved that $\hat{P}_2$ has a parametrix with fps, because it suffices to employ the weight t^{-n} (ϕ^{-n} is now absent) in order to obtain weighted energy estimates.

Turn to the general case. Let $|\eta| = 1$ be arbitrarily fixed. Write $p(0,0,\tau,\eta) = \prod_{j=1}^{r} (\tau - \tau_j)^{m_j}$ where $\sum m_j = m$ and τ_j are real and distinct from each other, where $m_j \le 3$ which follows from the assumption. There exist $\delta > 0$ and a conic neighborhood U of $(0,\eta)$ such that one can write

$$p(t,x,\tau,\xi) = \prod_{j=1}^{r} p^{(j)}(t,x,\tau,\xi),$$
$$p^{(j)}(t,x,\tau,\xi) = \tau^{m_j} + a_{j,1}(t,x,\xi)\tau^{m_j-1} + \cdots + a_{j,m_j}(t,x,\xi)$$

for $(t,x,\xi) \in (-\delta,\delta) \times U$ where $a_{j,k}(t,x,\xi)$ are real valued, homogeneous of degree k in ξ and $p^{(j)}(0,0,\tau,\eta) = (\tau-\tau_j)^{m_j}$ and $p^{(j)}(t,x,\tau,\xi) = 0$ has only real roots in τ for $(t,x,\xi) \in [0,\delta) \times U$. If $(0,0,\tau_j,\eta)$ is a critical point of p, and necessarily $m_j \ge 2$, then $(0,0,\tau_j,\eta)$ is a critical point of $p^{(j)}$ and it is easy to see

$$F_p(0,0,\tau_j,\eta) = c_j F_{p^{(j)}}(0,0,\tau_j,\eta)$$

with some $c_j \ne 0$ and hence $F_{p^{(j)}}(0,0,\tau_j,\eta)$ has non-zero real eigenvalues if $F_p(0,0,\tau_j,\eta)$ does and vice versa. It is well known that one can find $P^{(j)}$ such that

$$P \equiv P^{(1)}P^{(2)} \cdots P^{(r)} \quad \text{at} \;\; (0,\eta)$$

where $P^{(j)}$ are operators of the form (13.1) with $m = m_j$ whose principal symbol coincides with $p^{(j)}$ in some conic neighborhood of $(0,0,\eta)$. Since each $P^{(j)}$ has a parametrix with fps thanks to Theorem 11.1 and Proposition 12.5 hence so does P. Therefore Theorem 1.1 results from Theorem 13.1 noting Remark 12.1.

Repeating a parallel arguments to the existence proof for P above we obtain

Theorem 13.3. *Under the same assumption as in Theorem 1.1 there exist $\delta > 0$, a neighborhood U of the origin and $n > 0$ such that for any $s \in \mathbb{R}$ and any $f \in \mathcal{H}^*_{n+1/2,s}((0,\delta]\times\mathbb{R}^d)$ there exists u with $D_t^j u \in \mathcal{H}^*_{n,-n-2+s+m-j}((0,\delta]\times\mathbb{R}^d)$, $j = 0,1,\ldots,m-1$ satisfying*

$$P^* u = f \quad \text{in} \quad (0,\delta) \times U.$$

Now we prove a local uniqueness result for the Cauchy problem for P applying Theorem 13.3. From the assumption one can find a neighborhood W of the origin of $\mathbb{R}^d$ and $T > 0$ such that every multiple characteristic of p on

$(t, x, \xi) \in (0, T) \times W$ is at most double and double characteristic is effectively hyperbolic. Let $f \in C_0^\infty((0, \delta') \times \{|x| < \varepsilon\})$ $(\delta' \le T)$ and let v be a solution to $P^* v = f$ vanishing in $t \ge \delta'$. Then thanks to [15, Main Theorem] there exists $\hat{c} > 0$ such that

$$\mathrm{supp}_x v(t, \cdot) \subset \{|x| \le \varepsilon + \hat{c}\,\delta'\}, \quad 0 < t \le \delta'.$$

Now assume that u satisfies $Pu = 0$ in $(0, \delta) \times U$ and $\partial_t^k u(0, x) = 0$ for all k. Choose $\varepsilon > 0$ and $\delta' > 0$ such that $\{|x| \le \varepsilon + \hat{c}\,\delta'\} \subset U$, $\delta' \le \delta$. Then we see

$$0 = \int_0^{\delta'} \big(Pu, v\big) dt = \int_0^{\delta'} \big(u, P^* v\big) dt = \int_0^{\delta'} \big(u, f\big) dt.$$

Since $f \in C_0^\infty((0, \delta') \times \{|x| < \varepsilon\})$ is arbitrary, we conclude that

$$u(t, x) = 0, \quad (t, x) \in (0, \delta') \times \{|x| \le \varepsilon\}.$$

Theorem 13.4. *If $u(t, x) \in C^\infty([0, \delta) \times U)$ satisfies $Pu = 0$ in $[0, \delta) \times U$ and $\partial_t^k u(0, x) = 0$ for all k then $u = 0$ in a neighborhood of $(0, 0)$.*

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