A note on time functions associated with effectively hyperbolic double characteristics

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1 Introduction

Consider

$$P = -D_t^2 + A_2(t,x,D) + A_0(t,x,D)D_t + A_1(t,x,D) \tag{1.1}$$

where $A_j(t,x,D)$ are classical pseudodifferential operators of order j. Denote the principal symbol by

$$p(t,x,\tau,\xi) = -\tau^2 + a(t,x,\xi)$$

where $a(t,x,\xi)$ is positively homogeneous of degree 2 in ξ, smooth in $(-T,T)\times U\times(\mathbb{R}^d\setminus 0)$ and satisfies

$$a(t,x,\xi)\geq 0, \quad (t,x,\xi)\in[0,T_1)\times U\times\mathbb{R}^d \tag{1.2}$$

with some $T_1>0$ and some neighborhood U of $0\in\mathbb{R}^d$. Note that if $(0,0,\tau,\xi)$, $(\tau,\xi)\neq 0$ is a singular point of $p=0$ then $\tau=0$ and $a(0,0,\xi)=0$. Assume that a singular point $(0,0,0,\bar\xi)$, $\bar\xi\neq 0$ is effectively hyperbolic, that is the Hamilton map $H_p(0,0,0,\bar\xi)$ has nonzero real eigenvalues (e.g. [1], [2]). Assuming a slightly stronger assumption than (1.2) such that

$$a(t,x,\xi)\geq 0, \quad (t,x,\xi)\in(-\delta_1,T_1)\times U\times\mathbb{R}^d$$

with some $\delta_1>0$ one can find a smooth function $\psi(x,\xi)$ in some conic neighborhood V of $(0,\bar\xi)$, homogeneous of degree 0, and constants $0<\kappa<1$, $c>0$, $\delta>0$ satisfying

$$a(t,x,\xi)\geq c(t-\psi(x,\xi))^2|\xi|^2, \quad \{\psi,a\}^2\leq 4\kappa a.$$

This is the key to proving that the Cauchy problem for P with Cauchy data on $t=0$ is C^∞ well-posed for any lower order term ([4, 5]).

To apply the same arguments as in [4, 5] under the present assumption we need some modifications. Our aim in this note is to improve [5, Lemma 1.2.2] to

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Proposition 1.1. *Assume* (1.2). *If a singular point* $(0,0,0,\bar{\xi})$ *of* $p=0$ *is effectively hyperbolic then there exist a smooth function* $\psi(x,\xi)$ *in a conic neighborhood* V *of* $(0,\bar{\xi})$*, homogeneous of degree* 0 *and constants* $0<\kappa<1$, $c>0$, $\delta>0$ *such that*

$$a(t,x,\xi)\geq c\min\big\{t^2,(t-\psi(x,\xi))^2\big\}|\xi|^2,\quad \{\psi,a\}^2\leq 4\kappa a \tag{1.3}$$

for $(t,x,\xi)\in[0,\delta)\times V$ *where* $\psi(x,\xi)$ *satisfies* $\big|\partial_x^\alpha\partial_\xi^\beta\psi\big|\precsim\langle\xi\rangle^{-|\beta|}$ *and* $\{\psi,a\}$ *denotes the Poisson bracket of* ψ *and* a.

Once Proposition 1.1 is examined, applying the same pseuododifferential weight as in [6], one can prove that the Cauchy problem for P with Cauchy data on $t=0$ is C^∞ well-posed for any lower order term.

2 Proof of Proposition 1.1

In what follows we denote $x^{(p)}=(x_p,\ldots,x_d)$, $\xi^{(p)}=(\xi_p,\ldots,\xi_d)$, $1\leq p\leq d$ and $x_0=t$. Note that $(0,0,0,\bar{\xi})$ is a singular point of $p=0$ implies

$$\partial_t^k\partial_x^\alpha\partial_\xi^\beta a(0,0,\bar{\xi})=0,\quad k+|\alpha+\beta|=1. \tag{2.1}$$

Proposition 2.1. *Assuming the same assumption as in Proposition 1.1 one can find a homogeneous symplectic coordinates* (x,ξ) *around* $(0,\bar{\xi})$ *such that* $(0,e_d)=(0,\bar{\xi})$ *and* $a(t,x,\xi)$ *takes the following form* (2.2) *with* (2.3) *or* (2.4) *with* (2.5) *and* (2.6)*;*

$$\begin{aligned}&\sum_{i=1}^p(x_{i-1}-x_i)^2q_i(t,x,\xi)+\sum_{i=1}^p\xi_i^2r_i(t,x,\xi)\\&+\big\{(x_p-\phi_p(x^{(p+1)},\xi^{(p+1)}))^2+\psi_p(x^{(p+1)},\xi^{(p+1)})\big\}q_{p+1}(t,x,\xi),\end{aligned} \tag{2.2}$$

$$\{\phi_p,\{\phi_p,\psi_p\}\}(0,0,e_d)=0, \tag{2.3}$$

where $0\leq p\leq d-1$ *and*

$$\sum_{i=1}^p(x_{i-1}-x_i)^2q_i(t,x,\xi)+\sum_{i=1}^p\xi_i^2r_i(t,x,\xi)+g_p(x^{(p)},\xi^{(p+1)})r_p(t,x,\xi), \tag{2.4}$$

$$\{\xi_p,\{\xi_p,g_p\}\}(0,0,e_d)=0, \tag{2.5}$$

$$\sum_{i=1}^p r_i^{-1}(0,0,e_d)>1 \tag{2.6}$$

where $1\leq p\leq d-1$. *In both cases* $q_i(t,x,\xi)$, $r_i(t,x,\xi)$ *are homogeneous of degree* 2, 0 *respectively, positive at* $(0,0,e_d)$ *and* ϕ_p, ψ_p, g_p *are homogeneous of degree* 0, 0, 2 *respectively vanishing at* $(0,0,e_d)$.

Proof. It is enough to repeat exactly the same arguments as proving [3, Theorem 1.1]. After a linear change of coordinates x one can assume that $(0,\bar{\xi})=(0,e_d)$. If $\partial_t^2 a(0,0,e_d)=0$ then the Taylor expansion of $a(t,x,\xi+e_d)$ at $(t,x,\xi)=(0,0,0)$ gives

$$a(\epsilon^2 t,\epsilon^3 x,\epsilon^3\xi+e_d)=\epsilon^5 t\sum_{|\alpha+\beta|=1}\partial_t\partial_x^\alpha\partial_\xi^\beta a(0,0,e_d)x^\alpha\xi^\beta+O(\epsilon^6)$$

which proves that $\partial_t\partial_x^\alpha\partial_\xi^\beta p(0,0,0,e_d)=0$ for $|\alpha+\beta|=1$. Then it is easy to see that $F_p(0,0,0,e_d)$ has only pure imaginary eigenvalues, contradicting to that $(0,0,0,e_d)$ is effectively hyperbolic. Thus we conclude $\partial_t^2 a(0,0,e_d)\neq 0$.

We first prove that we have either (2.2) with (2.3) or (2.4) with (2.5). After that we show (2.6) if the case (2.4) with (2.5) occurs. From the Malgrange preparation theorem one can write

$$a(t,x,\xi)=\big\{(t-\phi_0(x^{(1)},\xi^{(1)}))^2+\psi_0(x^{(1)},\xi^{(1)})\big\}q_1(t,x^{(1)},\xi^{(1)})$$

where ϕ_0, ψ_0 are homogeneous of degree 0 vanishing at $(0,e_d)$ and q_1 is homogeneous of degree 2, $q_1(0,0,e_d)\neq 0$. Since $a(t,0,e_d)=t^2q_1(t,0,e_d)$ it is clear that $q_1(0,0,e_d)>0$. Hence if $\{\phi_0,\{\phi_0,\psi_0\}\}(0,e_d)=0$ this is just (2.2) with (2.3) with $p=0$. We go on to the induction on p. Assume that (2.2) holds with $p-1$ while (2.3) with $p-1$ fails. Set $X_p(x^{(p)},\xi^{(p)})=\phi_{p-1}(x^{(p)},\xi^{(p)})$. Note that $d\phi_{p-1}$ and $\sum_{j=p}^d\xi_j dx_j$ are linearly independent at $(0,e_d)$. In fact if not we would have $\{\phi_{p-1},\{\phi_{p-1},\psi_{p-1}\}\}(0,e_d)=0$ thanks to the Euler's identity, which contradicts the assumption. Thus we can find a homogeneous symplectic coordinates $\{X_j(x^{(p)},\xi^{(p)}),\Xi_j(x^{(p)},\xi^{(p)})\}_{j=p}^d$ (e.g. [2, Theorem 21.1.9]) such that

$$\text{(2.7)}\quad X_j(0,e_d)=0,\ p\leq j\leq d,\ \ \Xi_j(0,e_d)=0,\ p\leq j\leq d-1,\ \Xi_d(0,e_d)\neq 0.$$

Denoting $\{X_j,\Xi_j\}_{j=p}^d$ by $\{x_j,\xi_j\}_{j=p}^d$ again and noting that $\partial_{\xi_p}^2\psi_{p-1}(0,e_d)\neq 0$ thanks to the Malgrange preparation theorem one can write

$$\psi_{p-1}(x^{(p)},\xi^{(p)})=\big\{(\xi_p-h_p(x^{(p)},\xi^{(p+1)}))^2+g_p(x^{(p)},\xi^{(p+1)})\big\}b_p(x^{(p)},\xi^{(p)})$$

where b_p is of homogeneous degree -2 with $b_p(0,e_d)\neq 0$ and h_p and g_p are homogeneous of degree 1, 2 respectively, vanishing at $(0,e_d)$. Take $x=0$ and $\xi_j=0$ unless $j=p,d$ and $\xi_d=1$ then we have

$$a(0,0,\xi)=\xi_p^2 b_p(0,e_d)q_p(0,0,\xi)\geq 0$$

which implies that $b_p(0,e_d)>0$ for $q_p(0,0,e_d)>0$. Set

$$\Xi_p(x^{(p)},\xi^{(p)})=\xi_p-h_p(x^{(p)},\xi^{(p+1)}),\quad X_p(x^{(p)},\xi^{(p)})=x_p.$$

It is clear that $\{\Xi_p,X_p\}=1$ and the differentials $\sum_{j=p}^d\xi_j dx_j$, $d\Xi_p$ and dX_p are linearly independent at $(0,e_d)$. Indeed if $dx_d=\alpha d\Xi_p+\beta dX_p$ with some α,β then applying H_{X_p} to this relation we conclude $\alpha=0$ hence $dx_d=$

βdX_p which is a contradiction because $p \le d-1$. Again from [2, Theorem 21.1.9] one can extend Ξ_p, X_p to a homogeneous symplectic coordinates $\{X_j(x^{(p)},\xi^{(p)}),\Xi_j(x^{(p)},\xi^{(p)})\}_{j=p}^d$ verifying (2.7). Since $0=\{\xi_j,x_p\}=\{\xi_j,X_p\}=-\partial\xi_j/\partial\Xi_p$, $p+1\le j\le d$ and $0=\{x_j,x_p\}=\{x_j,X_p\}=-\partial x_j/\partial\Xi_p$ we see that $\xi_j(X^{(p)},\Xi^{(p)})$, $p+1\le j\le d$ and $x_j(X^{(p)},\Xi^{(p)})$, $p\le j\le d$ are independent of Ξ_p. Thus we obtain (2.4) with p where $r_p=b_pq_p$ which is positive at $(0,e_d)$. Now assume that (2.5) with p fails. Then one can write

$$g_p(x^{(p)},\xi^{(p+1)})=\big\{(x_p-\phi_p(x^{(p+1)},\xi^{(p+1)})^2+\psi_p(x^{(p+1)},\xi^{(p+1)})\big\}c_p(x^{(p)},\xi^{(p+1)})$$

where c_p is homogeneous of degree 2 with $c_p(0,e_d)\ne 0$ and ϕ_p, ψ_p are homogeneous of degree 0, vanishing at $(0,e_d)$. Choose $x^{(p+1)}=0$, $\xi=e_d$ and $x_p=\cdots=x_1=x_0(=t)\ge 0$ then

$$a(t,x,\xi)=x_p^2c_p(x_p,0,e_d)r_p(t,x,\xi)\ge 0$$

hence c_p is positive at $(0,e_d)$ and so is $q_{p+1}=c_pr_p$ because $r_p(0,0,e_d)>0$. Thus we conclude that (2.2) with p holds. Therefore the induction on p proves the asertion.

We now show that if the case (2.4) with (2.5) occurs then we have (2.6). Choosing $(t=)x_0=\cdots=x_p\ge 0$ and $\xi_1=\cdots=\xi_p=0$ in (2.4) we have $a(t,x,\xi)=g_p(x^{(p)},\xi^{(p+1)})r_p(t,x,\xi)\ge 0$ hence

$$g_p(x_p,x^{(p+1)},\xi^{(p+1)})\ge 0,\quad x_p\ge 0$$

because r_p is positive at $(0,0,e_d)$. Since (2.5) implies that $\partial_{x_p}^2g_p(0,e_d)=0$ repeating the same argument as in the beginning of the proof of Proposition 2.1 we see that

$$\partial_{x_\mu}^k\partial_{\xi_\nu}^l\partial_{x_p}g_p(0,e_d)=0,\ \ p+1\le\mu,\nu\le d,\quad k+l=1.$$

This shows that the corresponding quadratic form at $(0,e_d)$ is

$$\sum_{i=1}^p\bar q_i(x_{i-1}-x_i)^2+\sum_{i=1}^p\bar r_i\xi_i^2+\bar r_p\sum_{p+1\le\mu,\nu\le d}\partial_{x_\mu}^k\partial_{\xi_\nu}^lg_p(0,e_d)x_\mu^k\xi_\nu^l$$

where $\bar q_i=q_i(0,0,e_d)$ and the same for $\bar r_i$. Denote the sum of the first two terms by Q_1 and the third term by Q_2. Since $g_p(0,x^{(p+1)},\xi^{(p+1)})\ge 0$ then Q_2 is positive semi-definite and hence the eigenvalues of F_{Q_2} are pure imaginary. It is not difficult to see that

$$\det(\lambda+F_p(0,e_d))=\det(\lambda+F_{Q_1})\det(\lambda+F_{Q_2}).$$

On the other hand, a direct computation gives

$$\det(\lambda+F_{Q_2})=\lambda^2\psi(\lambda),\quad \psi(0)=-\big(\prod_{j=1}^p4\bar q_j\big)\big(\prod_{j=1}^p\bar r_j\big)\big(\sum_{j=1}^p\bar r_j^{\,-1}-1\big).$$

From [1] the equation $\psi(\lambda)=0$ has only pure imaginary roots except possibly a pair of nonzero real simple roots $\pm\mu$. Therefore $F_p(0,0,0,e_d)$ has nonzero real eigenvalues if and only if $\psi(0)<0$, that is (2.6). □

Proof of Proposition 1.1: Without restrictions we may assume that $(0,\bar{\xi}) = (0,e_d)$. Taking the homogeneity in ξ it suffices to show the assertion for (x,ξ) in a small neighborhood of $(0,e_d)$. Let $\chi(s) \in C^\infty(\mathbb{R})$ be such that $\chi(s) = s$ for $|s| \le 1$, $|\chi(s)| = 2$ for $|s| \ge 2$ and $\chi'(s) \ge 0$. First consider the case (2.2). Extend q_j and r_j to $\hat{q}_j$, $\hat{r}_j$ replacing x_k, ξ_k by $\delta\chi(x_k/\delta)$, $\delta\chi(\xi_k/\delta)$ for $k = 1,\dots,p$ where $\delta > 0$ is chosen suitably small such that q_j and r_j are defined for $(x_{(p)},\xi_{(p)}) = (x_1,\dots,x_p,\xi,\dots,\xi_p) \in \mathbb{R}^{2p}$ and $|\hat{q}_j(t,x,\xi) - q_j(0,e_d)|$ and $|\hat{r}_j(t,x,\xi) - r_j(0,e_d)|$ are enough small. Such a $\delta > 0$ is fixed from now on. By an obvious abuse of notation we often write $q(t,x,\xi) = q(t,x_{(p)},\xi_{(p)},x^{(p+1)},\xi^{(p+1)})$. Considering $a/\hat{q}_{p+1}$ we may assume that $\hat{q}_{p+1} = 1$. Writing $(x^{(p+1)},\xi^{(p+1)}) = z + (0,e_d)$ and $w = (y_{(p)},\eta_{(p)})$ consider

$$Q(w,t,z,\epsilon) = \sum_{j=1}^{p}(y_{j-1} - y_j)^2\tilde{q}_j(w,t,z,\epsilon) + (y_p+1)^2 + \sum_{j=1}^{p}\eta_j^2\,\tilde{r}_j(w,t,z,\epsilon)$$

where $y_0 = 0$ and $\tilde{q}_j(w,t,z,\epsilon) = \hat{q}_j(t,\epsilon y + t,\epsilon\eta,z + (0,e_d))$ and the same for $\tilde{r}_j$. Note that if we choose $\epsilon = t - \phi(z)$ with $\phi(z) = \phi_p(z + (0,e_d))$ we have

$$a(t,\epsilon y + t,\epsilon\eta,z + e_d) = \epsilon^2 Q(w,t,z,\epsilon) + \psi(z), \quad \epsilon = t - \phi(z)$$

where $\psi(z) = \psi_p(z + (0,e_d))$. It is clear that for small $|\theta| = |(t,z,\epsilon)|$ there is a positive minimum of $Q(w,\theta)$ when w varies in $\mathbb{R}^{2p}$ and which is attained at $|w| < R$ with some $R > 0$. Denote

$$\min_{w\in\mathbb{R}^{2p}} Q(w,\theta) = m(\theta).$$

Note that the Hessian $\nabla_w^2 Q(w,\theta)$ of $Q(w,\theta)$ with respect to w can be written $\nabla_w^2 Q(w,\theta) = H + R(w,\theta)$ with a nonsingular constant matrix H such that for any $\epsilon > 0$ there is $\delta_1 > 0$ such that $\|R(w,\theta)\| \le \epsilon$ if $|t| + |\epsilon| + |z| < \delta_1$ and $|w| < R$. Therefore there exists a smooth $\bar{w}(\theta)$ near $\theta = (0,0,0)$ such that

$$m(\theta) = Q(\bar{w}(\theta),\theta).$$

Taking $\epsilon = t - \phi(z)$ we have

$$\begin{aligned} a(t,\epsilon y + t,\epsilon\eta,z + (0,e_d)) &= (t - \phi(z))^2 Q(w,t,z,t - \phi(z)) + \psi(z) \\ &\ge m(t,z,t - \phi(z))(t - \phi(z))^2 + \psi(z) = m_1(t,z)(t - \phi(z))^2 + \psi(z) \end{aligned} \tag{2.8}$$

where we have set $m_1(t,z) = m(t,z,t - \phi(z))$.

Assume $\phi(z) < 0$ and hence $\epsilon = t - \phi(z) > 0$ for $t \ge 0$. Then choosing $w = (y,\eta)$ so that $(\epsilon y + t,\epsilon\eta) = (x_{(p)},\xi_{(p)})$ one concludes that

$$a(t,x,\xi) = a(t,\epsilon y + t,\epsilon\eta,z + (0,e_d)) \ge m_1(t,z)(t - \phi(z))^2 + \psi(z).$$

Moreover choosing $w = \bar{w}(t,z,t - \phi(z))$ in (2.8) we see

$$m_1(t,z)(t - \phi(z))^2 + \psi(z) \ge 0, \quad t \ge 0.$$

In particular, taking $t=0$ we have

$$m_1(0,z)\phi^2(z)+\psi(z)\geq 0. \tag{2.9}$$

Noting that $m_1(t,z)\geq c_1>0$ we also have

$$\begin{aligned} a(t,x,\xi) &\geq m_1(t,z)\big(t-\phi(z)\big)^2+\psi(z)\\ &= m_1(t,z)\big(t^2+2t|\phi(z)|+\phi^2(z)\big)+\psi(z)\\ &\geq c_1t^2+2c_1t|\phi(z)|+m_1(0,z)\phi^2(z)+\psi(z)\\ &\quad +\big(m_1(t,z)-m_1(0,z)\big)\phi^2(z). \end{aligned}$$

Since $|m_1(t,z)-m_1(0,z)|\leq Ct$, in view of (2.9) we see

$$a(t,x,\xi)\geq c_1t^2+t|\phi(z)|\big(2c_1-C|\phi(z)|\big)\geq c_1t^2,\quad \phi(z)<0 \tag{2.10}$$

in a neighborhood of $(0,0)$ because $\phi(0)=0$. Next if $\phi(z)\geq 0$ then choosing $x_j=\phi(z)\geq 0$, $0\leq j\leq p$ and $\xi_j=0$, $1\leq j\leq p$ in (2.2) it follows that $\psi(z)\geq 0$ hence

$$a(t,x,\xi)\geq \big(t-\phi(z)\big)^2,\quad \phi(z)\geq 0 \tag{2.11}$$

is obvious. Thus from (2.10) and (2.11), returning to the original a, we conclude

$$a(t,x,\xi)\geq c\,\min\{t^2,(t-\phi_p(x,\xi))^2\}|\xi|^2.$$

Note that for any $\epsilon>0$ one can find a neighborhood U of $(0,e_d)$ such that

$$\big|H_\phi^2\,a\big|=\big|\{\phi,\{\phi,a\}\}\big|\leq\epsilon,\quad (x,\xi)\in U$$

uniformly in $t\geq 0$ for $\{\phi,\{\phi,a\}\}(0,e_d)=0$. Since $a\geq 0$ for $t\geq 0$ thanks to the Glaeser's inequality we see that

$$\big|H_\phi a\big|^2=\big|\{\phi,a\}\big|^2\leq 2\,\epsilon\,a,\quad t\geq 0$$

which finishes the proof for the case (2.2).

Turn to the case (2.4). Extend q_j and r_j to $\hat{q}_j$, $\hat{r}_j$ just as in the case (2.2) where $\hat{q}_j$ is obtained from q_j replacing x_k by $\delta\chi(x_k/\delta)$, $k=1,\ldots,p-1$. Considering $a/\hat{r}_p$ we may assume $\hat{r}_p=1$ as before. Consider

$$\begin{aligned} Q(w,t,z,x_p,\epsilon) &= (y_1+1)^2\tilde{q}_1(w,t,z,x_p,\epsilon)\\ &+\sum_{j=1}^{p-1}(y_j-y_{j+1})^2\tilde{q}_j(w,t,z,x_p,\epsilon)+\sum_{j=1}^{p}\eta_j^2\,\tilde{r}_j(w,t,z,x_p,\epsilon) \end{aligned}$$

where $y_p=0$, $w=(y_1,\ldots,y_{p-1},\eta_1,\ldots,\eta_p)\in\mathbb{R}^{2p-1}$ and $\tilde{q}_j(w,t,z,x_p,\epsilon)=\hat{q}_j(t,x_p-\epsilon y,x_p,\epsilon\eta,z+(0,e_d))$ and the same for $\tilde{r}_j$. Note that if we choose $\epsilon=t-x_p$ then

$$a(t,x_p-\epsilon y,x_p,\epsilon\eta,z+(0,e_d))=\epsilon^2Q(w,t,z,x_p,\epsilon)+g(x_p,z),\ \ \epsilon=t-x_p$$

where $g(x_p, z) = g_p(x_p, x^{(p+1)}, \xi^{(p+1)})$. Denoting $\theta = (t, z, x_p, \epsilon)$ consider

$$\min_{w \in \mathbb{R}^{2p-1}} Q(w, \theta) = m(\theta).$$

As before there exists a smooth $\bar{w}(\theta)$ near $\theta = 0$ such that

$$m(\theta) = Q(\bar{w}(\theta), \theta).$$

Choosing $\epsilon = t - x_p$ we have

$$\begin{aligned} & a(t, x_p - \epsilon y, x_p, \epsilon\eta, z + (0, e_d)) \\ &= (t - x_p)^2 Q(w, t, z, x_p, t - x_p) + g(x_p, z) \\ &\geq m(t, z, x_p, t - x_p)(t - x_p)^2 + g(x_p, z) \\ &= m_1(t, x_p, z)(t - x_p)^2 + g(x_p, z) \end{aligned} \tag{2.12}$$

where $m_1(t, x_p, z) = m(t, z, x_p, t - x_p)$. When $x_p < 0$ repeating the same arguments as above one can find $c_1 > 0$

$$a(t, x, \xi) \geq c_1 t^2 + t|x_p|(2c_1 - C|x_p|) \geq c_1 t^2, \quad x_p < 0 \tag{2.13}$$

in a neighborhood of $(0, e_d)$. Assume $x_p \geq 0$. Thanks to Proposition 2.1 one has $\sum_{i=1}^p \hat{r}_i^{-1}(0, e_d) > 1$ then one can find $\epsilon_i > 0$ such that $\sum_{i=1}^p \epsilon_i^2 \hat{r}_i(0, e_d) = \rho < 1$ with $\sum_{i=1}^p \epsilon_i = 1$. Define

$$\varphi(x) = \sum_{i=1}^p \epsilon_i x_i.$$

Since $x_{i-1} - x_i = 0$, $i = 1, \ldots, p$ implies that $t - \varphi(x) = x_0 - x_0 = 0$ hence $t - \varphi(x)$ is a linear combination of $x_{i-1} - x_i$ so that

$$t - \varphi(x) = \sum_{i=1}^p \alpha_i (x_{i-1} - x_i), \quad \alpha_i \in \mathbb{R}.$$

It is clear that there is $C > 0$ such that

$$(t - \varphi(x))^2 \leq C \sum_{i=1}^p (x_{i-1} - x_i)^2 \hat{q}_i.$$

Since $g(x_p, z) \geq 0$ for $x_p \geq 0$ which follows from (2.12) taking $x_p = t \geq 0$ hence $a(t, x, \xi) \geq \sum_{i=1}^p (x_{i-1} - x_i)^2 \hat{q}_i$ which implies that there is $c' > 0$ such that

$$a(t, x, \xi) \geq c' \, (t - \varphi(x))^2, \quad x_p \geq 0. \tag{2.14}$$

Thus from (2.13) and (2.14) we have

$$a(t, x, \xi) \geq c \, \min\{t^2, (t - \varphi(x, \xi))^2\} |\xi|^2.$$

It is clear that for any $\epsilon > 0$ one can find a neighborhood U of $(0, e_d)$ such that

$$\left|H_\varphi^2 a\right| \leq 2\sum_{i=1}^{p} \epsilon_i^2\, \hat{r}_i(0,0,e_d) + \epsilon = 2\rho + \epsilon, \quad (x,\xi) \in U$$

uniformly in $t \geq 0$ small because $H_\varphi^2 a(0,0,e_d) = 2\sum_{i=1}^{p} \epsilon_i^2\, \hat{r}_i(0,0,e_d)$. Since $a \geq 0$ for $t \geq 0$ we obtain from the Glaeser's inequality that

$$\left|H_\varphi a\right|^2 = \left|\{\varphi, a\}\right|^2 \leq 2(2\rho + \epsilon)a, \quad (x,\xi) \in U$$

for $t \geq 0$ small where one can assume $2\rho + \epsilon < 2$. Thus we have completed the proof for the case (2.4).

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