MODIFIED SCATTERING STATES FOR SUBCRITICAL DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider the derivative nonlinear Schrödinger equations
\[
\begin{aligned}
   iu_t + \frac{1}{2} u_{xx} &= a(t) F(u, u_x), & (t, x) \in \mathbb{R}^2 \\
   u(0, x) &= \varepsilon u_0(x), & x \in \mathbb{R},
\end{aligned}
\]
where the coefficient \( a(t) \) satisfies the condition \( |a(t)| \leq C (1 + |t|)^{1-\delta} \), \( 0 < \delta < 1 \), \( \varepsilon \) is a sufficiently small constant and
\[
F(u, u_x) = \lambda_1 |u|^2 u + i\lambda_2 |u|^2 u_x + i\lambda_3 u^2 \overline{u}_x + \lambda_4 |u_x|^2 u + \lambda_5 \overline{u} u_x^2 + i\lambda_6 |u_x|^2 u_x,
\]
\( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{R} \), \( \lambda_2 - \lambda_3 \in \mathbb{C}, \lambda_2 - \lambda_3 \in \mathbb{R}, \lambda_4 - \lambda_5 \in \mathbb{R} \). We suppose that the initial data satisfy the exponential decay conditions. Then in the case \( \lambda_1 = 0, \lambda_2 - \lambda_3 = 0, \lambda_4 - \lambda_5 = 0, \lambda_6 = 0 \) we show that the usual scattering states exist, and in the other case we construct the modified scattering states.

1. Introduction

In this paper we continue to study the Cauchy problem for the cubic derivative nonlinear Schrödinger equation
\[
\begin{aligned}
   iu_t + \frac{1}{2} u_{xx} &= a(t) F(u, u_x), & x \in \mathbb{R}, t > 0, \\
   u(0, x) &= \varepsilon u_0(x), & x \in \mathbb{R},
\end{aligned}
\]
which was studied in the previous paper [11], where the coefficient \( a(t) \) satisfies the time growth condition \( |a(t)| \leq C (1 + |t|)^{1-\delta} \), \( \delta \in (0, 1), \varepsilon > 0 \) is a sufficiently small constant and the nonlinear interaction term \( F(u, u_x) \) consists of the cubic nonlinearities containing derivatives of the unknown function and satisfying the self-conjugate property \( F(e^{i\theta} u, e^{i\theta} u_x) = e^{i\theta} F(u, u_x) \) for any \( \theta \in \mathbb{R} \)
\[
F(u, u_x) = \lambda_1 |u|^2 u + i\lambda_2 |u|^2 u_x + i\lambda_3 u^2 \overline{u}_x + \lambda_4 |u_x|^2 u + \lambda_5 \overline{u} u_x^2 + i\lambda_6 |u_x|^2 u_x,
\]
where the coefficients \( \lambda_1, \lambda_6 \in \mathbb{R}, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{C}, \lambda_2 - \lambda_3 \in \mathbb{R}, \lambda_4 - \lambda_5 \in \mathbb{R} \). In general the nonlinearity in the problem (1.1) is sub critical from the point of view of the large time asymptotics, see [11] for the classification of nonlinear terms. The study of the large time behavior of solutions in the sub critical cases \( \delta \in (0, 1) \) is difficult since the nonlinearity in the uniform norm decays more slowly than the linear part of the equation and can not be considered as a small perturbation and therefore can not be treated by the usual methods of the perturbation theory in spite of smallness of the initial data.

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There are few works devoted to sub critical cases for the nonlinear Schrödinger type equations. It is known (see [4], [12]) that in the sub critical case $\delta \in (0, 1)$ the usual scattering states of nontrivial solutions to Hartree type equations

\[
i u_t + \frac{1}{2} u_{xx} = f \left( |u|^2 \right) u,
\]

where $f \left( |u|^2 \right) = \lambda \int |x - y|^{-\delta} |u|^2 (t, y) \, dy$, $\lambda \in \mathbb{R}$, do not exist in any reasonable sense, and must be replaced by modified scattering states including a rapidly oscillating factor in their definition, similarly to the case of the linear Schrödinger equation with a long-range potential. The asymptotics for large time of solutions to the Cauchy problem for nonlinear Schrödinger equation $i u_t + \frac{1}{2} u_{xx} = |t|^{1-\delta} |u|^2 u$, with $\delta \in \left( \frac{15}{16}, 1 \right)$ obtained in paper [13] by using the Gevrey classes. In paper [8] we studied the scattering problem for the Hartree type equation in the sub critical case $\delta \in (0, 1)$

\[
\begin{aligned}
i u_t + \frac{1}{2} \Delta u &= f \left( |u|^2 \right) u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \\
&\quad u (0, x) = \varepsilon u_0 (x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

where

\[
f \left( |u|^2 \right) = \lambda \int |x - y|^{-\delta} |u|^2 (t, y) \, dy, \quad \lambda \in \mathbb{R}, \ n \geq 1
\]

and $\varepsilon > 0$ is small. Under the supposition that initial data $u_0 \in H_n^{n+2,0} (\mathbb{R}^n) \cap H_0^{n+2} (\mathbb{R}^n)$ in the case $n \geq 2$ and $e^{\beta |x|} u_0 \in L^2 (\mathbb{R})$, $\beta > 0$ in the one-dimensional case $n = 1$ we proved the existence of a unique global solution to the Cauchy problem for the Hartree type equation (1.3) satisfying the sharp time decay estimate

\[
\| u (t) \|_{L^p} \leq C \epsilon \tilde{t}^{\frac{\frac{n}{2} - \frac{\beta}{2}}}{t^{n-p}}
\]

for all $t > 0$, where $2 \leq p \leq \infty$,

\[
H_n^{m,s} = \left\{ \phi \in L^2; \| \phi \|_{m,s} = \| \langle x \rangle^s \langle \partial_x \rangle^m \phi \|_{L^2} < \infty \right\}
\]

is the weighted Sobolev space, $m, s \in \mathbb{R}^+$, $\langle x \rangle = (1 + x^2)^{1/2}$. Moreover we showed in [8] that there exists a unique final state $\hat{u}_+ \in H^{n,0}$ such that the following asymptotics

\[
u (t, x) = M D \hat{u}_+ \exp \left( - \frac{i t^{1-\delta}}{1-\delta} f \left( |\hat{u}_+|^2 \right) + O \left( 1 + t^{1-2\delta} \right) \right) + O \left( t^{-\frac{n}{2} - \frac{\beta}{2}} \right)
\]

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}^n$, where $M = e^{\frac{\beta}{2n}}$,

\[
(D (t) \phi) (x) = \left( \frac{1}{t} \right)^{n/2} \phi \left( \frac{x}{t} \right)
\]

is the dilation operator. In the region $0 < \delta \leq \frac{1}{2}$ the value of the phase in the asymptotic formula (1.5) is determined with accuracy of the growing as $t^{1-2\delta}$ summand. For the region $\delta \in \left( \frac{1}{2}, 1 \right)$ the phase in the asymptotic formula (1.5) was evaluated in [8] more precisely

\[
u (t, x) = M D \hat{u}_+ \exp \left( - \frac{i t^{1-\delta}}{1-\delta} f \left( |\hat{u}_+|^2 \right) \right) + O \left( t^{-\frac{n}{2} + 1-2\delta} \right)
\]
for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}^n$, whence the existence of the modified scattering states follows

$$\begin{equation}
(1.7) \quad \left\| u(t) - \exp \left( \frac{it^{1-\delta}}{1-\delta} f \left( |u| \right) \right) U(t) u_+ \right\|_{L^2} \leq Ct^{1-2\delta}
\end{equation}$$

for all $t > 0$. In paper [9] we removed the regularity conditions on the initial data for the Hartree type equations (1.3) in space dimension $n \geq 2$. More precisely, under the condition that the initial data $u_0 \in H^{l,0}$, where $l$ is an integer satisfying $l \geq \frac{n}{2} + 3$, we proved the results similar to that of paper [8]: there exists a unique global solution of the Hartree type equation (1.3) such that $U(-t)u(t) \in C([0, \infty); H^{l,0})$ with decay estimate (1.4) and $\|U(-t)u(t)\|_{H^l} \leq C \varepsilon (1+t)^{(1-\delta)l}$. Moreover we proved that there exists a unique final state $\tilde{u}_+ \in H^{l-\gamma,0}$, $0 < \gamma \leq 1$ such that asymptotics (1.5) for $\delta \in (0, 1)$ and asymptotics (1.6) for the values $\delta \in (\frac{1}{2}, 1)$ are valid as $t \to \infty$. In particular estimate (1.7) is true for all $t > 0$, whence the existence of the modified scattering states follows in the case $\delta \in (\frac{1}{2}, 1)$. In paper [5] we studied the asymptotic behavior for large time of solutions to the Cauchy problem for the subcritical cubic nonlinear Schrödinger with a growing in time coefficient and Hartree type equation (1.3) with nonlinearity

$$f(|u|^2) = \lambda \int |x - y|^{-\delta} |u(y)|^2 dy + \mu |t|^{1-\delta} |u|^2,$$

where $\delta \in (0, 1)$, $\lambda, \mu \in \mathbb{R}$ in one dimensional case $n = 1$. We removed the regularity condition on the data which was assumed in the previous paper [8] and also we discussed a smoothing effect for the solutions in an analytic function space

$$\mathcal{H}_\alpha^s = \left\{ \phi \in L^2; \| \langle x \rangle^s e^{\alpha |x|} \phi(x) \|_{L^2} < \infty \right\}, s \in \mathbb{R}, \alpha > 0,$$

where $\alpha$ determines the analytic domain of function in $\mathcal{H}_\alpha^s$. We derived in paper [7] more exact exponential decay conditions on the initial data which lead to the Gevrey function spaces $G_\sigma^s$ of order $\frac{1}{\sigma}$, where $\sigma \in \left( 1 - \frac{\delta}{2-\delta}, 1 \right)$,

$$G_\sigma^s = \left\{ \phi \in L^2; \| \langle x \rangle^s e^{\alpha |x|} \phi(x) \|_{L^2} < \infty \right\}, s \in \mathbb{R}, \alpha > 0.$$  

We note that $\mathcal{H}_\alpha^s = G_\alpha^s$ when $\sigma = 1$.

In paper [6] we also studied the asymptotic behavior in time of solutions to the Cauchy problem for the generalized derivative nonlinear Schrödinger equation under strong assumptions on the data

$$\begin{cases}
\dot{u} + \frac{1}{4} u_{xx} = \lambda |u|^{2\delta} u - i \mu |u|^{2\delta} u_x, \quad x \in \mathbb{R}, \quad t > 1, \\
u(1, x) = \varepsilon u_0(x), \quad x \in \mathbb{R},
\end{cases}$$

where $\lambda, \mu \in \mathbb{R}, \varepsilon$ is sufficiently small, in the sub critical case $\delta \in (0, 1)$. We assumed that the initial data satisfy

$$\psi^m u_0 e^{-ix^2/2} \in \mathcal{H}_\beta^1, \psi^{m+1} \left( u_0 e^{-ix^2/2} \right)_{xx} \in \mathcal{H}_\beta^1,$$

and

$$\psi^{m+\frac{1}{4}} |u_0| \geq 1,$$

and the norm

$$\| \psi^m u_0 e^{-ix^2/2} \|_{\mathcal{H}_\beta^1} + \left\| \psi^{m+1} \left( u_0 e^{-ix^2/2} \right)_{xx} \right\|_{\mathcal{H}_\beta^1} \leq 1.$$
where
\[ \beta > 0, \psi = x^2 + \frac{1}{|x|}, m > \frac{4}{\delta}, \]
then we proved the sharp decay estimate (1.4), asymptotics formulas (1.5) and (1.6) and existence of modified scattering states (1.7) with \( f \left( |\hat{u}_+|^2 \right) \) replaced by \( (\lambda x + \mu) |\hat{u}_+ (x) |^{2s} \).

In paper [14], it was proved that if the initial data are such that \( \langle x \rangle^s u_0 (x) \in L^2 \) with \( s > 1 + \frac{5}{2} \) and the norm \( \| \langle \cdot \rangle^s u_0 \|_{L^2} \) is sufficiently small, then the solution \( u \) of the Hartree type equation satisfies the estimate
\[ \| u (t) \|_p \leq C |t|^{-\frac{5}{2} + \frac{5}{p}} \| \langle \cdot \rangle^s u_0 \|_{L^2} \]
for \( 2 \leq p \leq \frac{2n}{n - 2s} \) if \( s < \frac{n}{2} \) and for \( 2 \leq p \leq \infty \) if \( s > \frac{n}{2} \). In the case \( s \geq 2 \) there exist modified final states \( u_{\pm} \in L^2 \) such that the solution \( u \) satisfies the estimate
\[ \| U (-t) u (t) - e^{\mp iS_\pm (t; i\nabla)} u_{\pm} \|_{L^2} = O \left( |t|^{-1 - \delta} \right) \]
as \( t \to \pm \infty \), where
\[ S_\pm (t, \xi) = (1 - \delta)^{-1} |t|^{1 - \delta} f \left( |\hat{u}_{\pm}|^2 \right) (\xi). \]

The asymptotic profile in the \( L^\infty \) sense was found in paper [9] under the assumption that \( \langle x \rangle^s u_0 (x) \in L^2 \) with \( s > 2 + \left\lceil \frac{n}{2} \right\rceil \). The assumptions of [14] are weaker than that of paper [9] in the case of space dimensions \( n > 3 \).

In papers [1], [2], via some modification of the energy method it was proved the existence of modified wave operators for the Hartree type equation (1.3) with nonlinearity \( f \left( |u|^2 \right) = t^{\gamma - \delta} |\nabla|^{\gamma - n} |u|^2 \). Also the modified wave operators was constructed, with no size restriction on the final data if \( \delta \in \left( \frac{1}{2}, 1 \right) \) in [1] and then for all \( \delta \in (0, 1) \) in paper [2]. The space of the final data is the Sobolev space of finite order if \( \gamma \leq n - 2 \) which in particular restricts the space dimension to \( n \geq 3 \). In paper [3] the existence of modified local wave operators with no size restrictions of the final data was proved by using Gevrey classes for \( \gamma \in (0, n) \) and in particular for the cases of space dimensions 1 and 2 and the case of the cubic nonlinear Schrödinger equations with time growing nonlinearity.

In our previous paper [11] we proved that if the initial data are such that \( e^{\beta (x)} u_0 \in H^{\frac{5}{2}, 2} \), where \( \beta > 0 \), then there exists a unique global solution \( u \in C \left( (0, \infty) ; H^{\frac{5}{2}, 0} \right) \) of the Cauchy problem for the cubic derivative nonlinear Schrödinger equation (1.1) such that \( \mathcal{F} U (-t) u (t) \in C \left( (0, \infty) ; H^{\frac{5}{2}, 2} \right) \), and time decay estimate (1.4) is valid. Moreover we proved in [11] that there exists a unique final state \( \hat{u}_+ \in H^{\frac{3}{2}, 2} \) such that the following asymptotics for \( t \to \infty \)
\[ u (t, x) = MD \hat{u}_+ \exp \left( -i \int_0^t a (\tau) d\tau - b |\hat{u}_+|^2 + O \left( 1 + t^{1 - 2\delta} \right) \right) + O \left( t^{-1/2 - \delta} \right) \]
is true uniformly with respect to \( x \in \mathbb{R} \), where
\[ b (\xi) = \lambda_1 - (\lambda_2 - \lambda_3) \xi + (\lambda_4 - \lambda_5) \xi^2 - \lambda_6 \xi^3. \]
For the values $\delta \in \left( \frac{1}{2}, 1 \right)$ we obtained the existence of the modified scattering states: there exists a unique final state $\hat{u}_+ \in \mathcal{H}_{\beta/2}^{\delta}$ such that the asymptotics

$$u(t, x) = M D \hat{u}_+ \exp \left( -i \int_0^t \left( \frac{a(\tau)}{\tau + 1} b |\hat{u}_+|^2 \right) d\tau \right) + O(t^{1-2\delta})$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$ and the estimate

$$\left\| u(t) - \exp \left( -i \int_0^t \left( \frac{a(\tau)}{\tau + 1} b \left( \frac{\cdot}{\tau} \right) \right) d\tau \right) \hat{U}(t) \hat{u}_+ \right\|_{L^2} \leq C t^{1-2\delta}$$

is true for all $t > 0$. Therefore our result in [11] says that nonlinearity $a(t) F(u, u_x)$ is the sub critical one if $b(\xi) \neq 0$ and $\delta \in (0, 1)$, and if $\delta \in \left( \frac{1}{2}, 1 \right)$, then the modified scattering states exist.

The aim of the present work is to prove the existence of the modified scattering states for the whole region $\delta \in (0, 1)$. To state our result precisely, we now give

**Notation and function spaces.** Let $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of $\phi$ defined by

$$(\mathcal{F}\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} \phi(x) \, dx$$

and $\mathcal{F}^{-1}\phi(x)$ be the inverse Fourier transform of $\phi$, i.e.

$$(\mathcal{F}^{-1}\phi)(x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} \phi(\xi) \, d\xi.$$ 

We introduce some function spaces. The usual Lebesgue space is

$$L^p = \{ \phi \in S'; \|\phi\|_p < \infty \},$$

where the norm $\|\phi\|_p = (\int |\phi(x)|^p \, dx)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess. sup} \{ |\phi(x)|; x \in \mathbb{R} \}$ if $p = \infty$. For simplicity we write $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space is

$$H^{m,s} = \left\{ \phi \in L^2; \|\phi\|_{m,s} = \| (x)^s (i\partial_x)^m \phi \| < \infty \right\}, m, s \in \mathbb{R}^+,$$

where $\langle x \rangle = \sqrt{1 + x^2}$. Also we define the analytic function space

$$H_a^{m,s} = \{ \phi \in L^2; \|\phi\|_{H_a^{m,s}} = \| e^{\sigma|p|} (p)^m (i\partial_p)^s \hat{\phi}(p) \| < \infty \}, m, s \in \mathbb{R}^+,$$

which can be expressed in $x$ - representation in terms of the analyticity in the strip $-\sigma \leq \Im z \leq \sigma$ via the following norm

$$\|\phi(\cdot + i\sigma)\|_{H^{m,s}} + \|\phi(\cdot - i\sigma)\|_{H^{m,s}}.$$ 

Indeed we have the inequalities

$$\frac{1}{C} \|\phi\|_{H_a^{m,s}} \leq \|\phi(\cdot + i\sigma)\|_{H^{m,s}} + \|\phi(\cdot - i\sigma)\|_{H^{m,s}} \leq C \|\phi\|_{H_a^{m,s}}$$

with $C > 0$. We denote the inner product in $L^2$ by $\langle \psi, \varphi \rangle = \int \psi(x) \cdot \overline{\varphi(x)} \, dx$. We define the free Schrödinger evolution group $\hat{U}(t)$ by

$$\hat{U}(t) \hat{\phi} = \frac{1}{\sqrt{2\pi it}} \int e^{i(x-y)^2/2t} \phi(y) \, dy = \mathcal{F}^{-1} e^{itx^2/2} \mathcal{F} \hat{\phi} = M D(t) \mathcal{F} M \hat{\phi},$$
where \( M = e^{3/2} \) and \( \mathcal{D}(t) \) is the dilation operator
\[
(\mathcal{D}(t) \varphi)(x) = \frac{1}{\sqrt{t^2}} \varphi \left( \frac{x}{t} \right).
\]

Then since \( \mathcal{D}^{-1}(t) = i\mathcal{D}(1/t) \) we have \( \mathcal{U}(-t) = i\mathcal{M}\mathcal{F}^{-1}\mathcal{D}(1/t)\mathcal{M} \), where \( \mathcal{M} = e^{-3/2} \). By \( \mathcal{C}(I; \mathcal{B}) \) we denote the space of continuous functions from an interval \( I \) to a Banach space \( \mathcal{B} \). Different positive constants might be denoted by the same letter \( C \).

In this paper we prove

**Theorem 1.1.** We assume that \( \delta \in (0, 1) \) and the initial data are such that \( e^{\beta(x)} u_0 \in H^{3,5/2} \) with \( \beta > 0 \). Then there exists a unique modified final state \( W \in \mathcal{H}^{2,3}_{3/2} \), and unique real-valued phase function \( G(t) \in \mathcal{C}([1, \infty); \mathcal{H}^{2,3}_{3/2}) \), such that
\[
(1.8) \quad \left\| u(t) - MDW e^{iG(t)} \right\|_{p} \leq C\varepsilon t^{-\delta(1-\frac{3}{p})-\delta}
\]
for all \( t \geq 1 \), where \( 2 \leq p \leq \infty \).

**Remark 1.1.** From the proof of Theorem 1.1 we can see that \( G(t) = 0 \) in the case \( b = 0 \), i.e. \( \lambda_1 = 0, \lambda_2 - \lambda_3 = 0, \lambda_4 - \lambda_5 = 0, \lambda_6 = 0 \). Therefore this exceptional nonlinearity appears to be super critical since the usual scattering states exist. In the opposite case estimate (1.8) shows the existence of the modifies scattering states in the whole region \( \delta \in (0, 1) \).

In the next section we will prove Theorem 1.1 by constructing the higher order asymptotic expansion for the functions \( w \) and \( g \) (see systems (2.6) and (2.7)).

2. PROOF OF THEOREM 1.1

We define a new function \( v(t, \xi) = \mathcal{B}(t) u(t) \), where \( \mathcal{B}(t) = \mathcal{F} M(t) \mathcal{U}(-t) \).

Then \( v \) satisfies
\[
(2.1) \quad \left( i\partial_t + \frac{1}{2t^2} \partial_{\xi}^2 \right) v = \frac{1}{t} F \left( v, i\xi v + \frac{1}{t} v \xi \right)
\]
where
\[
F \left( v, i\xi v + \frac{1}{t} v \xi \right) = \bar{b} |v|^2 v + \frac{i\mu_2}{t} |v|^2 v \xi + \frac{i\mu_3}{t^2} v^2 \bar{v} \xi + \frac{\mu_4}{t^2} |v\xi|^2 v + \frac{\mu_5}{t^2} \bar{v}^2 v \xi + \frac{i\lambda_6}{t^5} |v\xi|^2 v \xi
\]
and
\[
\bar{b} = \lambda_1 - (\lambda_2 - \lambda_3) \xi + (\lambda_4 - \lambda_5) \xi^2 - \lambda_6 \xi^3,
\]
\[
\mu_2 = \lambda_2 - (\lambda_4 - 2\lambda_5) \xi + 2\lambda_6 \xi^2, \quad \mu_3 = \lambda_3 + \lambda_4 \xi - \lambda_6 \xi^2;
\]
\[
\mu_4 = \lambda_4 - 2\lambda_6 \xi, \quad \mu_5 = \lambda_5 + \lambda_6 \xi.
\]

In order to remove the nonlinear terms which do not have sufficient decay in time in the right hand side of (2.1) we represent \( v = we^{i\varphi} \), where the phase function \( g \) is a solution of the Cauchy problem
\[
\begin{cases}
g_t = -\frac{4}{7} g |v|^2 + \frac{1}{2t^2} \partial_{\xi}^2 g, & t > 1, \\
g(1) = 0,
\end{cases}
\]
where
\[ q = b - (\mu_2 - \mu_3) \frac{g_2}{t} + (\mu_4 - \mu_5) \left( \frac{g_6}{t} \right)^2 - \lambda_6 \left( \frac{g_6}{t} \right)^3. \]

Note that if \( b \equiv 0 \) then \( q \equiv 0 \). We now multiply both sides of (2.1) by \( e^{-ig} \) to get
\[ (2.2) \]
\[
\begin{cases}
(i\partial_t + \frac{1}{2i\tau} \partial^2_{\xi}) w = -\frac{1}{2i\tau} (2ig\xi w + ig\xi w) + \frac{g_6}{t} Q, \ t > 1, \\
g_t = -\frac{q_6}{t} |w|^2 + \frac{1}{2i\tau} g_6^2, \ t > 1, \\
w (1) = v (1), \ g (1) = 0,
\end{cases}
\]

where
\[ Q = Q (w, w_\xi, g_\xi) = \frac{iv_2}{t} |w|^2 w_\xi + \frac{iv_3}{t} - \frac{v_4}{t^2} |w_\xi|^2 w + \frac{v_5}{t^2} \bar{w}w_\xi + \frac{i\lambda_6}{\tau^2} |w_\xi|^2 w_\xi \]

and the coefficients \( \nu_j, j = 2, \ldots, 5 \) are defined as follows
\[
\begin{align*}
\nu_2 &= \mu_2 - (\lambda_4 - 2\lambda_5 - 4\lambda_6) \frac{g_2}{t} + 2\lambda_6 \left( \frac{g_6}{t} \right)^2, \\
\nu_3 &= \mu_3 + (\lambda_4 - 2\lambda_6) \frac{g_6}{t} - \lambda_6 \left( \frac{g_6}{t} \right)^2, \\
\nu_4 &= \mu_4 - 2\lambda_6 \frac{g_6}{t}, \nu_5 = \mu_5 + \lambda_6 \frac{g_6}{t}.
\end{align*}
\]

In the same way as in the proof of Theorem 2.1 from [11] we have

Proposition 2.1. Suppose that \( |a| \leq C t^{1-\delta} \) and the initial data \( u_0 \) satisfy the condition of Theorem 1.1. Then the following estimates are valid for the solutions of (2.2)
\[
\begin{align*}
\| w (t) \|_\mathcal{H}^{\beta/2, 3} &+ t^{\delta - 1 - \gamma} \| g_\xi (t) \|_\mathcal{H}^{\beta/2, 3} + t^{\delta - 1} \| g (t) \|_\mathcal{H}^{\beta/2, 3} \\
&\leq \| w (1) \|_\mathcal{H}^{\beta/2, 3} \leq \sup_{0 \leq t \leq 1} \| \mathcal{F} M U (-t) u (t) \|_\mathcal{H}^{\beta/2, 3}
\end{align*}
\]

for all \( t \geq 1 \), where \( \sigma (t) = \frac{\hat{\beta}}{t} (1 + (1 + t)^{-\gamma}), \ 0 < \gamma \leq \delta/2. \)

In Lemma 3.1 of [10] we showed that the solution \( u \) of (1.1) satisfy the estimate
\[
\begin{align*}
\sup_{0 \leq t \leq 1} \| \mathcal{F} M U (-t) u (t) \|_\mathcal{H}^{\beta/2, 3} \\
&\leq \sup_{0 \leq t \leq 1} \| \mathcal{F} U (-t) u (t) \|_\mathcal{H}^{\beta/2, 3} \leq C \varepsilon \left( e^{\hat{\beta} (t)} u_0 \right)_{\mathcal{H}^{3, 5/2}} \leq C \varepsilon.
\end{align*}
\]

Therefore we have the following estimates of solutions to system (2.2)
\[ (2.3) \]
\[
\| w (t) \|_\mathcal{H}^{\beta/2, 3} + t^{\delta - 1 - \gamma} \| g_\xi (t) \|_\mathcal{H}^{\beta/2, 3} + t^{\delta - 1} \| g (t) \|_\mathcal{H}^{\beta/2, 3} \leq C \varepsilon.
\]

In what follows we use freely the Sobolev type estimates
\[ (2.4) \]
\[
\| \phi \|_\mathcal{H}^{\beta/2, -(m+2)\alpha} \leq C \| \phi \|_\mathcal{H}^{\beta/2, -(m+1)\alpha}.
\]
for any \( n \in \mathbb{N} \), where \( \varkappa > 0 \) is sufficiently small \( (\Delta_{\varkappa}) > \frac{1}{2} \). By virtue of (2.2), (2.3) and (2.4) we have with a sufficiently small positive constant \( \varkappa \)

\[
\|w(t) - w(s)\|_{\mathcal{H}^{3/2}_{y/2 - \varkappa}} \\
\leq \int_s^t \|w(t)\|_{\mathcal{H}^{3/2}_{y/2 - \varkappa}} \, dt \leq C \int_s^t \left( \|g\|_{\mathcal{H}^{3/2}_{y/2 - \varkappa}} + \|w\|_{\mathcal{H}^{3/2}_{y/2 - \varkappa}} \right) \, dt \\
+ \|w_{y\xi}\|_{\mathcal{H}^{3/2}_{y/2 - \varkappa}} \frac{d\tau}{\tau^2} + \int_s^t |a| \|Q\|_{\mathcal{H}^{3/2}_{y/2 - \varkappa}} \, d\tau \\
\leq C \varepsilon \int_s^t \tau^{-1 - \delta} \, d\tau \leq C \varepsilon s^{-\delta}
\]

for all \( t > s > 1 \). Hence there exists a unique limit \( w_+ \in \mathcal{H}^{0,3}_{y/2 - \varkappa} \) such that

\[
\lim_{t \to \infty} w(t) = w_+ \text{ in } \mathcal{H}^{0,3}_{y/2 - \varkappa}
\]

and

\[
\|w(t) - w_+\|_{\mathcal{H}^{0,3}_{y/2 - \varkappa}} \leq C \varepsilon t^{-\delta}.
\]

By virtue of \( w_+ \), we define \( w_0(t) = w_+ \), so that \( i\partial_tw_0 = 0 \), and \( \partial_yg_0 = -\frac{a(t)}{4} |w_0|^2 \) with initial condition \( g_0(1) = 0 \). According to the idea from [2] and [3] we denote by \( w_{m+1}(t) \), successively in \( m = 0, 1, 2, \ldots \), the solutions to the final state problem

\[
\begin{cases}
  i\partial_tw_{m+1} = -\frac{1}{2\pi} \sum_{j=0}^{m} \left( 2i (\partial_yg_j) \partial_x + i \left( \frac{\partial_y^2}{\partial_x^2} g_j \right) \right) w_{m-j} + \frac{a(t)}{t} Q_m, \\
  \lim_{t \to \infty} w_{m+1}(t) = 0,
\end{cases}
\]

and \( g_{m+1} \) as a solution to the Cauchy problem

\[
\begin{cases}
  \partial_yg_{m+1} = -\frac{a(t)}{4} \sum_{j=0}^{m+1} w_j w_{m+1-j} + \frac{1}{2\pi} \sum_{j=0}^{m} \left( \partial_yg_j \right) \left( \partial_yg_{m-j} \right), \\
  g_{m+1}(1) = 0,
\end{cases}
\]

where

\[
Q_m = \sum_{j=0}^{m} \sum_{l=0}^{j} \left( \frac{i2l}{t} w_l w_{m-j-l} \partial_x w_{m-j} + \frac{i2l}{t} w_l w_{m-j-l} \partial_x w_{m-j} - \frac{i4}{t^2} \partial_x w_l \partial_x w_{m-j-l} w_{m-j} + \frac{i4}{t^2} \partial_x w_l \partial_x w_{m-j-l} w_{m-j} \\
+ \frac{i\lambda_6}{t^2} \partial_x w_l \partial_x w_{m-j-l} w_{m-j} \right).
\]

To prove the desired result of Theorem 1.1 we need two lemmas. In the next lemma we estimate the solutions \( w_m \) and \( g_m \) of systems (2.6) and (2.7). Denote \( k = \left[ \frac{1}{y} \right] \), where \( \delta \in (0, 1) \). Let \( \varkappa > 0 \) be sufficiently small \((\Delta_{\varkappa}) > \frac{1}{2} \).

**Lemma 2.2.** We have the estimates

\[
\|w_m(t)\|_{\mathcal{H}^{0,3}_{y/2 - (m+1)\varkappa}} \leq Ce^{2m+1} \max(t^{-m\delta}, t^{-1}),
\]

and

\[
\|g_m(t)\|_{\mathcal{H}^{0,3}_{y/2 - (m+1)\varkappa}} \leq Ce^{2m+2} \max(1, t^{1-(m+1)\delta}),
\]

for all \( t \geq 1 \) and \( 0 \leq m \leq k \).
Proof. When \( m = 0 \) then the result follows from the definition of \( w_0 \) and \( g_0 \). By induction we assume that (2.8) and (2.9) are valid for some \( m \geq 0 \), then by virtue of (2.3), (2.4) we obtain from (2.6)

\[
\|w_{m+1}(t)\|_{H^{\beta,3/2-(m+2)\infty}} \leq C \sum_{j=0}^{m} \int_{t}^{\infty} \left\| (g_j)_{\xi} w_{m-j} \right\|_{H^{\beta,3/2-(m+2)\infty}} \frac{ds}{s^2}
\]

\[
+ C \sum_{j=0}^{m} \int_{t}^{\infty} \left\| (g_j)_{\xi} (w_{m-j})_{\xi} \right\|_{H^{\beta,3/2-(m+2)\infty}} \frac{ds}{s^2} + C \sum_{j=0}^{m} \int_{t}^{\infty} |a| \left\| Q_m \right\|_{H^{\beta,3/2-(m+2)\infty}} \frac{ds}{s}
\]

\[
\leq C \sum_{j=0}^{m} \int_{t}^{\infty} \left\| g_j \right\|_{H^{\beta,3/2-(m+2)\infty}} \left\| w_{m-j} \right\|_{H^{\beta,3/2-(m+2)\infty}} \frac{ds}{s^2}
\]

\[
+ C \sum_{j=0}^{m} \sum_{l=0}^{j} \int_{t}^{\infty} \left\| w_l \right\|_{H^{\beta,3/2-(m+2)\infty}} \left\| w_{j-l} \right\|_{H^{\beta,3/2-(m+2)\infty}} \left\| w_{m-j} \right\|_{H^{\beta,3/2-(m+2)\infty}} \frac{ds}{s^{1+\delta}}
\]

\[
\leq C \varepsilon^{2m+3} \sum_{j=0}^{m} \int_{t}^{\infty} \max \left(1, s^{-(j+1)\delta} \right) \max \left(s^{-m-j}\delta, s^{-1}\right) \frac{ds}{s^{1+\delta}}
\]

\[
+ C \varepsilon^{2m+3} \sum_{j=0}^{m} \int_{t}^{\infty} \max \left(s^{-j\delta}, s^{-1}\right) \max \left(s^{-1}\delta, s^{-1}\right) \frac{ds}{s^{1+\delta}}
\]

\[
\leq C \varepsilon^{2m+3} \max \left(t^{-1}, t^{-1}\right).
\]

Therefore we have (2.8). Via (2.7), (2.3), (2.4) we get

\[
\|g_{m+1}(t)\|_{H^{\beta,3/2-(m+2)\infty}} \leq C \int_{1}^{t} |a| \sum_{j=0}^{m+1} \left\| qw_j w_{m+1-j} \right\|_{H^{\beta,3/2-(m+2)\infty}} \frac{ds}{s}
\]

\[
+ C \int_{1}^{t} \sum_{j=0}^{m+1} \left\| (g_j)_{\xi} (g_{m-j})_{\xi} \right\|_{H^{\beta,3/2-(m+2)\infty}} \frac{ds}{s^2}
\]

\[
\leq C \int_{1}^{t} \sum_{j=0}^{m+1} \left\| w_j \right\|_{H^{\beta,3/2-(j+1)\infty}} \left\| w_{m+1-j} \right\|_{H^{\beta,3/2-(m+2-j)\infty}} \frac{ds}{s^{1+\delta}}
\]

\[
+ C \int_{1}^{t} \sum_{j=0}^{m+1} \left\| g_j \right\|_{H^{\beta,3/2-(j+1)\infty}} \left\| g_{m-j} \right\|_{H^{\beta,3/2-(m+1-j)\infty}} \frac{ds}{s^2}
\]

\[
\leq C \varepsilon^{2m+4} \int_{1}^{t} |a| \sum_{j=0}^{m+1} \max \left(s^{-j\delta}, s^{-1}\right) \max \left(s^{-m+1-j}\delta, s^{-1}\right) \frac{ds}{s^{1+\delta}}
\]

\[
+ C \varepsilon^{2m+4} \int_{1}^{t} \sum_{j=0}^{m} \max \left(1, s^{1-(j+1)\delta}\right) \max \left(1, s^{1-(m-j)\delta}\right) \frac{ds}{s^{1}}
\]

\[
\leq C \varepsilon^{2m+4} \max \left(1, s^{1-(m+2)\delta}\right)
\]

which shows estimate (2.9). This completes the proof of the lemma. \(\blacksquare\)

We now estimate the differences \( W_m = w - \sum_{j=0}^{m} w_j \) and \( G_m = g - \sum_{j=0}^{m} g_j \), where \((w,g), w_j, g_j\) are the solutions of systems (2.2),(2.6),(2.7) respectively, \(0 \leq m \leq k, k = \left[ \frac{t}{T_k} \right] \).
Lemma 2.3. We have the estimates

\[ \|W_m(t)\|_{H^{\beta/2-(m+2)\infty}} \leq C\varepsilon \max \left( t^{-(m+1)\delta}, t^{-1} \right) \tag{2.10} \]

and

\[ \|G_m(t)\|_{H^{\beta/2-(m+2)\infty}} \leq C\varepsilon^2 \max \left( 1, t^{1-(m+2)\delta} \right) \tag{2.11} \]

for all \( t \geq 1, 0 \leq m \leq k. \) Moreover

\[ \|G_k(t) - G_k(s)\|_{H^{\beta/2-(k+2)\infty}} \leq C\varepsilon^2 s^{-\delta} \tag{2.12} \]

for all \( t > s \geq 1. \)

Proof. For \( m = 0, \) (2.10) follows from (2.5). By the definition of \( g_0 \) and (2.2) we have

\[ \partial_t G_0 = -\frac{a}{t} q |w|^2 + \frac{1}{2\varepsilon^2} \partial_\xi^2 + \frac{a}{t} q |w_0|^2. \]

Therefore by virtue of (2.3) and (2.4)

\[
\begin{align*}
\|G_0\|_{H^{\beta/2-2\infty}} & \leq C \int_1^t \left\| q \left( |w|^2 - |w_0|^2 \right) \right\|_{H^{\beta/2-2\infty}} \frac{ds}{s^3} + C \int_1^t \|g_0\|_{H^{\beta/2-2\infty}} \frac{ds}{s^2} \\
& \leq C \int_1^t \left( \|w\|_{H^{\beta/2-\infty}} + \|w_0\|_{H^{\beta/2-\infty}} \right) \|W_0\|_{H^{\beta/2-2\infty}} \frac{ds}{s^3} + C \int_1^t \|g_0\|_{H^{\beta/2-2\infty}} \frac{ds}{s^2} \\
& \leq C\varepsilon^2 \max \left( 1, t^{1-2\delta} \right).
\end{align*}
\]

This implies (2.11) for \( m = 0. \) By induction we assume that (2.10) and (2.11) hold for some \( m-1. \) By (2.2), (2.6) we have for \( 1 \leq m \leq k \)

\[ i\partial_t W_m = F + \frac{a}{t} \left( Q - \sum_{l=0}^{m-1} Q_l \right) - \frac{1}{2\varepsilon^2} w_0 \]

with

\[ F = -\frac{1}{2\varepsilon^2} \left( 2ig_0 \partial \xi + ig_0 \xi \right) w - \sum_{l=0}^{m-1} \sum_{j=0}^l \left( 2i (\partial \xi g_j) \partial \xi + i (\partial^2 \xi g_j) \right) w_{l-j}. \]

Via the identity

\[ \sum_{l=0}^{m-1} \sum_{j=0}^l a_j b_{l-j} = \left( a - \sum_{l=0}^{m-1} a_l \right) \sum_{l=0}^{m-1} b_l + \sum_{l=0}^{m-1} a_l \left( b - \sum_{l=0}^{m-1} b_l \right) \]

(2.14) + \left( a - \sum_{l=0}^{m-1} a_l \right) \left( b - \sum_{l=0}^{m-1} b_l \right) + \sum_{l+j=m}^{2m-2} a_i b_j \]
by the assumption of the induction, estimate (2.4) and Lemma 2.2 it follows that

\[ \|F\|_{H^0_{β/2-(m+2)\infty}} \leq Ct^{-2} \left( \|G_m-1\|_{H^0_{β/2-(m+1)\infty}} \sum_{l=0}^{m-1} \|w_l\|_{H^0_{β/2-(m+1)\infty}} \\
+ \|W_m-1\|_{H^0_{β/2-(m+1)\infty}} \sum_{l=0}^{m-1} \|g_l\|_{H^0_{β/2-(m+1)\infty}} \right) \]

\[ + \sum_{j+l=m} \|w_j\|_{H^0_{β/2-(m+1)\infty}} \|g_l\|_{H^0_{β/2-(m+1)\infty}} \]

\[ \leq C t^{-2} \varepsilon^3 \left( \max \left(1, t^{1-(m+1)\delta} \right) + t^{1-\delta} \max \left(t^{-m\delta}, t^{-1} \right) \right) \]

(2.15)

Similarly we have

\[ \left\| \frac{d}{dt} \left( Q - \sum_{l=0}^{m-1} Q_l \right) \right\|_{H^0_{β/2-(m+2)\infty}} \leq C \varepsilon^2 t^{-1-\delta} \max \left(t^{-m\delta}, t^{-1} \right). \]

(2.16)

By virtue of (2.3), (2.15) and (2.16) we find from (2.13)

\[ \|W_m\|_{H^0_{β/2-(m+2)\infty}} \leq C \varepsilon \max \left(t^{-(m+1)\delta}, t^{-1} \right) \]

which implies (2.10). In view of (2.2) and (2.7) we get

\[ \partial_t G_m = -\frac{a}{t} q \left( |w|^2 - \sum_{l=0}^{m} \sum_{j=0}^{l} w_j w_{l-j} \right) \]

\[ + \frac{1}{2t^2} \left( g_{\xi}^2 - \sum_{l=0}^{m-1} \sum_{j=0}^{l} \left( \partial_{\xi} g_j \right) \left( \partial_{\xi} g_{l-j} \right) \right), \]

whence via (2.14) we obtain

\[ \|G_m\|_{H^0_{β/2-(m+2)\infty}} \]

\[ \leq C \int_{1}^{t} \left\| q \left( |w|^2 - \sum_{l=0}^{m} \sum_{j=0}^{l} w_j w_{l-j} \right) \right\|_{H^0_{β/2-(m+1)\infty}} \frac{ds}{s^3} \]

\[ + C \int_{1}^{t} \left\| \left( g_{\xi} \right)^2 - \sum_{l=0}^{m-1} \sum_{j=0}^{l} \left( \partial_{\xi} g_j \right) \left( \partial_{\xi} g_{l-j} \right) \right\|_{H^0_{β/2-(m+2)\infty}} \frac{ds}{s^2}. \]

(2.17)
In the same way as in the proof of (2.15) we have by (2.3), (2.10), (2.14) and estimates of Lemma 2.2
\[
\left\| q \left[ |w|^2 - \sum_{l=0}^m \sum_{j=0}^l w_j \overline{w}_{l-j} \right] \right\|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \\
\leq C \| W_m \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \left( \sum_{l=0}^m \| w_l \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} + \| W_m \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \right) \\
+ C \sum_{j+l=m+1}^{2m} \| w_j \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \| w_l \|_{H^{0,3}_{\beta/2 - (m+1)\infty}}
\]
(2.18) \leq C \varepsilon^2 \max \left( t^{-(m+1)\delta}, t^{-1} \right)
and
\[
\left\| (\partial_t g)^2 - \sum_{l=0}^{m-1} \sum_{j=0}^l (\partial_t g_j) \left( \partial_t g_{l-j} \right) \right\|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \\
\leq C \| G_{m-1} \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \left( \sum_{l=0}^{m-1} \| g_l \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} + \| G_{m-1} \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \right) \\
+ C \sum_{j+l \geq m}^{2m-2} \| g_j \|_{H^{0,3}_{\beta/2 - (m+1)\infty}} \| g_l \|_{H^{0,3}_{\beta/2 - (m+1)\infty}}
\]
(2.19) \leq C \varepsilon^4 \max \left( 1, t^{1-(m+1)\delta} \right) t^{1-\delta}.

Substitution of estimates (2.18) and (2.19) to (2.17) yields (2.11). Thus by induction estimates (2.10) and (2.11) are proved for all \( 0 \leq m \leq k \).

In the same way as in the proof of (2.11) by virtue of (2.18) and (2.19) we get
\[
\| G_k (t) - G_k (s) \|_{H^{0,3}_{\beta/2 - (k+2)\infty}} \leq C \varepsilon^2 \int_s^t \max \left( 1, t^{1-(k+1)\delta} \right) \frac{dt}{\tau^{1+\delta}} \leq C \varepsilon^2 s^{-\delta}
\]
for all \( t > s > 1 \), since \((k+1)\delta > 1\). Lemma 2.3 is proved. \( \blacksquare \)

We are now in a position to prove Theorem 1.1. By estimate (2.12) we see that there exists a unique limit \( \Phi_+ \in H^{0,3}_{\beta/2 - (k+2)\infty} \) such that
\[
\lim_{t \to \infty} \| G_k (t) - \Phi_+ \|_{H^{0,3}_{\beta/2 - (k+2)\infty}} = 0
\]
and
\[
(2.20) \quad \| G_k (t) - \Phi_+ \|_{H^{0,3}_{\beta/2 - (k+2)\infty}} \leq C \varepsilon^2 t^{-\delta}
\]
for all \( t \geq 1 \). Denote
\[
G (t) = \sum_{m=0}^k g_m (t), \quad W = w_+ e^{i \Phi_+}, \quad k = \left\lfloor \frac{1}{\delta} \right\rfloor.
\]
We have in view of estimates (2.5) and (2.20)
\[
\left\| u(t) - M D W e^{iG(t)} \right\|_p
\]
\[
= \left\| M D F M U (-t) u(t) - M D w_+ e^{i\Phi_+ + \sum_{m=0}^{k} g_m(t)} \right\|_p
\]
\[
= C t^{-\frac{1}{2}(1-\frac{1}{p})} \left\| F M U (-t) u(t) - w_+ e^{i\Phi_+ + \sum_{m=0}^{k} g_m(t)} \right\|_p
\]
\[
\leq C t^{-\frac{1}{2}(1-\frac{1}{p})} \left( \| w(t) - w_+ \|_p + \left\| g(t) - \Phi_+ - \sum_{m=0}^{k} g_m(t) \right\|_p \right) \leq C t^{-\frac{1}{2}(1-\frac{1}{p})-\delta}
\]
for all $t \geq 1$, where $2 \leq p \leq \infty$. Thus estimate (1.8) is true. Theorem 1.1 is proved.

**References**


