On adelic Hurwitz zeta measures

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ABSTRACT. In this paper we construct a $\hat{\mathbb{Z}}$-valued measure on $\hat{\mathbb{Z}}$ which interpolates p-adic Hurwitz zeta functions for all p.

CONTENTS

1. **Introduction**

Let $m \geq 1$, $0 < a < m$ be integers such that a is prime to m, and let p be a rational prime. Set $q := 4$, $q := p$ according to whether $p = 2$ or $p > 2$ respectively, and $e := |(\mathbb{Z}/q\mathbb{Z})^\times|$. When $p \nmid m$, let $\langle ap^{-1} \rangle$ denote the least positive integer such that $\langle ap^{-1} \rangle p \equiv a$ mod m. Define the Bernoulli polynomials $B_k(T)$ ($k \in \mathbb{N}$) by $\sum_{k=0}^{\infty} B_k(T) \frac{w^k}{k!} = \frac{we^{Tw}}{e^w - 1}$ and set the Bernoulli numbers $B_k := B_k(0)$.

In [Sh], Shiratani constructed p-adic Hurwitz zeta functions $\zeta_p^{Sh}(s; a, m)$ ($s \in \mathbb{Z}_p$, $s \neq 1$) characterized by the interpolation property:

$$\zeta_p^{Sh}(1-k; a, m) = \begin{cases} -\frac{m^{k-1}}{k} B_k(\frac{a}{m}), & (p \mid m); \\ -\frac{m^{k-1}}{k} B_k(\frac{a}{m}) + p^{k-1} \frac{m^{k-1}}{k} B_k(\frac{\langle ap^{-1} \rangle}{m}), & (p \nmid m) \end{cases} \tag{1.1}$$

for all integers $k > 1$ with $k \equiv 0$ mod e. In [W3], assuming $p \nmid m$, the second author introduced a p-adic Hurwitz L-function $L_p^\beta(s; a, m)$ for $\beta \in (\mathbb{Z}/e\mathbb{Z})$ which satisfies

$$L_p^\beta(1-k; a, m) = \frac{1}{k} B_k\left(\frac{a}{m}\right) - \frac{p^{k-1}}{k} B_k\left(\frac{\langle ap^{-1} \rangle}{m}\right) \tag{1.2}$$

for all integers $k > 1$ with $k \equiv \beta$ mod e using certain p-adic measures arising in the study of Galois actions on paths on $\mathbf{P}^1 - \{0, 1, \infty\}$ (see also [W4]). The purpose of this paper is to complete the construction to include the case $p \mid m$ and to lift it over $\hat{\mathbb{Z}} = \varprojlim_N (\mathbb{Z}/N\mathbb{Z})$.

Throughout this paper, we fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. For any subfield $F \subset \mathbb{C}$, denote by G_F the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$.

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Theorem 1.1. *Let m and a be mutually prime integers with $m > 1$, $0 < a < m$. Then, for every $\sigma \in G_{\mathbb{Q}(\mu_m)}$, there exists a certain measure $\hat{\zeta}_{a,m}(\sigma)$ in $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$ such that for every prime p, its image $\hat{\zeta}_{p,a,m}(\sigma)$ in $\mathbb{Z}_p[[\mathbb{Z}_p]]$ has the following integration properties over $\mathbb{Z}_p^\times$:*

$$\int_{\mathbb{Z}_p^\times} b^{k-1} d\hat{\zeta}_{p,a,m}(\sigma)(b) = \begin{cases} (1-\chi_p(\sigma)^k)\cdot m^{k-1}\cdot \frac{1}{k}B_k(\frac{a}{m}) & (p \mid m); \\ (1-\chi_p(\sigma)^k)\cdot m^{k-1}\left(\frac{1}{k}B_k(\frac{a}{m}) - \frac{p^{k-1}}{k}B_k(\frac{\langle ap^{-1}\rangle}{m})\right) & (p \nmid m) \end{cases}$$

for all integers $k \geq 1$, where $\chi_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ denotes the p-adic cyclotomic character, and $\langle ap^{-1}\rangle$ represents the least positive integer such that $\langle ap^{-1}\rangle p \equiv a \mod m$.

Remark 1.2. Note that, in the above theorem, the case $m = 1$ is excluded. In fact, the case $m = a = 1$ corresponds to the $\hat{\mathbb{Z}}$-zeta function treated in [W2]. This separation of treatment is necessary for the appearance of tangential base point $\overrightarrow{10}$ in the construction of measure, which causes replacements of both $B_k(\frac{a}{m})$, $B_k(\frac{\langle ap^{-1}\rangle}{m})$ of RHS by $B_k(1)$.

Remark 1.3. More generally, we construct the measure $\hat{\zeta}_{a,m}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$ for $m > 1$ and $m \nmid a$ which satisfies the above integration property for all primes $p|m$ with $p \nmid a$ (cf. Remark 5.6).

Remark 1.4. Using any $\sigma \in G_{\mathbb{Q}(\mu_m)}$ with $\chi_p(\sigma)^e \neq 1$, we obtain from $\hat{\zeta}_{p,a,m}$ a set of p-adic Hurwitz functions $\{L_p^{[\beta]}(s,a,m)\}_{\beta\in(\mathbb{Z}/e\mathbb{Z})}$ by the standard integral

$$L_p^{[\beta]}(s;a,m) = \frac{1}{1-\omega(\chi_p(\sigma))^\beta[\chi_p(\sigma)]^{1-s}}\int_{\mathbb{Z}_p^\times}[b]^{1-s}b^{-1}\omega(b)^\beta d\hat{\zeta}_{p,a,m}(\sigma)(b)$$

where $\omega : \mathbb{Z}_p^\times \to \mu_e$ is the Teichmüller character, and for every $b \in \mathbb{Z}_p^\times$, $[b] \in 1 + q\mathbb{Z}_p$ is defined by $b = [b]\omega(b)$. Note that the above integral converges in $s \in \mathbb{Z}_p$ except when it has a pole at $s = 1$ in the case $\beta \equiv 0 \pmod e$. It follows from Theorem 1.1 that, for each $\beta \in \mathbb{Z}/e\mathbb{Z}$, the L-function $L_p^{[\beta]}(s;a,m)$ has the interpolation property:

$$L_p^{[\beta]}(1-k;a,m) = \begin{cases} \frac{m^{k-1}}{k}B_k(\frac{a}{m}) & (p \mid m); \\ \frac{m^{k-1}}{k}\left(B_k(\frac{a}{m}) - p^{k-1}B_k(\frac{\langle ap^{-1}\rangle}{m})\right) & (p \nmid m) \end{cases} \tag{1.3}$$

for all $k \geq 1$ with $k \equiv \beta \mod e$. Since $\mathbb{Z}_{>0,\equiv\beta(\mathrm{mod}\, e)}$ is dense in the space $\beta + \frac{q}{p}\mathbb{Z}_p$ ($= \mathbb{Z}_p$ $(p > 2)$, $2\mathbb{Z}_2$ or $1 + 2\mathbb{Z}_2$), the above interpolation property shows that $L_p^{[\beta]}(s,a,m)$ is determined independently of σ (at least) on that space. In particular when $\beta \equiv 0 \pmod e$ and $p > 2$, $L_p^{[0]}(s;a,m) = -\zeta_p^{Sh}(s;a,m)$ for $s \in \mathbb{Z}_p - \{1\}$. See also Appendix A for relations of $L_p^{[\beta]}(s,a,m)$ with Cohen's Hurwitz zeta functions $\zeta_p(s,x)$.

In the present paper, we hope to make a small step towards the quest of Coates about existence of zeta functions on $\hat{\mathbb{Z}}$ with values in $\hat{\mathbb{Z}}$ [W2, Introduction].

The mapping $\hat{\zeta}_{a,m}$ in Theorem 1.1 gives a 1-cocycle $G_{\mathbb{Q}(\mu_m)} \to \hat{\mathbb{Z}}(1)[[\hat{\mathbb{Z}}(-1)]]$ whose $(k-1)$st moment integral gives rise to a cohomology class in $H^1(G_{\mathbb{Q}(\mu_m)}, \hat{\mathbb{Z}}(k))$ for $k \geq 2$. In fact, we will show in Corollary 5.5:

$$\int_{\mathbb{Z}_p} b^{k-1} d\hat{\zeta}_{p,a,m}(\sigma)(b) = \frac{m^{k-1}}{k}B_k\Big(\frac{a}{m}\Big)(1-\chi_p(\sigma)^k) \quad (\sigma \in G_{\mathbb{Q}(\mu_m)},\ k \geq 2)$$

which implies that the p-adic image of the above cohomology class is torsion with order calculated explicitly by Bernoulli values. It is noteworthy that this cohomology class is closely related to the ξ_m^a-component of the $\mathbb{Z}(k)$-torsor '$P_{m,k}+(-1)^k\epsilon P_{m,k}$' over μ_m studied by Deligne in [De, Proposition 3.14, Lemma 18.5].

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2. The Kummer-Heisenberg measure $\varkappa_1$

2.1. **Cyclic coverings.** Let $F\subset\mathbb{C}$ be a finite extension of $\mathbb{Q}$ with the algebraic closure $\overline{F}\subset\mathbb{C}$. For any (normal) algebraic variety V over F and F-rational points $x,y\in V(F)$, we write $\pi_1^{\text{ét}}(V;y,x)$ for the set of étale paths from x to y on the geometric variety $V\otimes\overline{F}$, and $\pi_1^{\text{ét}}(V;x)=\pi_1^{\text{ét}}(V;x,x)$ for the étale fundamental group with base point x. Denote by $\pi_1^{\text{pro-}p}(V,x)$ the maximal pro-p quotient of $\pi_1^{\text{ét}}(V,x)$, and by $\pi_1^{\text{pro-}p}(V;y,x)$ the natural push forward of $\pi_1^{\text{ét}}(V;y,x)$ induced from the projection $\pi_1^{\text{ét}}(V,x)\twoheadrightarrow\pi_1^{\text{pro-}p}(V,x)$.

For each $n\geq 1$, write $\xi_n:=\exp(\frac{2\pi i}{n})$ so that $\mu_n:=\{1,\xi_n,\xi_n^2,\dots,\xi_n^{n-1}\}$. Let

$$V_n:=\mathbf{P}^1\setminus\{0,\mu_n,\infty\},$$

where we understand $\{0,\mu_n,\infty\}$ is the abbreviation of $\{0,\infty\}\cup\mu_n$. Regard $V_n(\mathbb{C})=\mathbb{C}^\times\setminus\mu_n$. Let $\overrightarrow{01}_n$ be the tangential base point on V_n represented by the unit tangent vector and denote for simplicity $\overrightarrow{01}$. Then, for each $n\geq 1$, there is a standard cyclic étale cover $p_n:V_n\to V_1$ given by $z\mapsto z^n$ which sends $\overrightarrow{01}_n$ to a Galois functor equivalent to $\overrightarrow{01}_1$ on V_1. Thus, without ambiguity, we may omit the index of $\overrightarrow{01}$ on V_n and regard $(V_n,\overrightarrow{01})$ as a pointed étale cover over $(V_1,\overrightarrow{01})$. By standard Galois theory, it allows us to identify $\pi_1^{\text{ét}}(V_n,\overrightarrow{01})$ as a subgroup of $\pi_1^{\text{ét}}(V_1,\overrightarrow{01})$.

Let x,y be the generators of $\pi_1^{\text{ét}}(V_1,\overrightarrow{01})$ given by the loops based at $\overrightarrow{01}$ on $V_1=\mathbf{P}^1-\{0,1,\infty\}$ running around $0,1$ once anti-clockwise respectively. Then, it is easy to see that, as a subgroup of it, $\pi_1^{\text{ét}}(V_n,\overrightarrow{01})$ is freely generated by $x_n:=x^n$ and $y_{b,n}:=x^{-b}yx^b$ $(0\leq b<n)$.

2.2. **Galois associators and Kummer-Heisenberg measure.** Now, let z be an F-point of $V_1=\mathbf{P}^1-\{0,1,\infty\}$. We have the canonical comparison map

$$\pi_1(V_1(\mathbb{C});z,\overrightarrow{01})\longrightarrow\pi_1^{\text{ét}}(V_1;z,\overrightarrow{01})$$

from the set of homotopy classes of paths from $\overrightarrow{01}$ to z on $V_1(\mathbb{C})$ to the étale paths from $\overrightarrow{01}$ to z on $V_1\otimes\bar{F}$. The Galois group G_F acts on the profinite group $\pi_1^{\text{ét}}(V_1,\overrightarrow{01})$ and its torsor of paths $\pi_1^{\text{ét}}(V_1;z,\overrightarrow{01})$.

Let us fix an étale path $\gamma\in\pi_1^{\text{ét}}(V_1(\mathbb{C});z,\overrightarrow{01})$. For $\sigma\in G_F$, define the *Galois associator* for the path γ by

$$f_\gamma(\sigma):=\gamma^{-1}\cdot\sigma(\gamma)\in\pi_1^{\text{ét}}(V_1,\overrightarrow{01}),\tag{2.1}$$

where $\sigma(\gamma):=\sigma\circ\gamma\circ\sigma^{-1}$.

Write π' for the commutator subgroup of a profinite group π. The abelianization of $f_\gamma(\sigma)$ is known (cf. [NW1, Proposition 1]) to be expressed as:

$$f_\gamma(\sigma)\equiv x^{\rho_{z,\gamma}(\sigma)}y^{\rho_{1-z,\gamma}(\sigma)}\mod \pi_1^{\text{ét}}(V_1,\overrightarrow{01})',\tag{2.2}$$

with the $\hat{\mathbb{Z}}$-valued functions

$$\rho_{z,\gamma},\ \rho_{1-z,\gamma} : G_F \to \hat{\mathbb{Z}}$$

the Kummer 1-cocycles associated with the roots of z and $1-z$. They are respectively calculated along γ with the above chosen base of the Tate module

$$(\xi_n)_{n\geq 1} \in \hat{\mathbb{Z}}(1) := \varprojlim_n \mu_n. \tag{2.3}$$

For the latter $\rho_{1-z,\gamma}$, we understand the points $\overrightarrow{01}$ and $1-z$ are connected by the unit segment $[0,1]$ on $\mathbf{P}^1$ followed with the reversed path of γ by $(* \mapsto 1-*)$. We sometimes omit the mention to γ when it is obvious from context.

Definition 2.1. Let $\sigma \in G_F$ and set

$$f_\gamma^\flat(\sigma) := x^{-\rho_{z,\gamma}(\sigma)} f_\gamma(\sigma) \quad (\sigma \in G_F).$$

which belongs to the subgroup $\pi_1(V_n, \overrightarrow{01}) \subset \pi_1(V_1, \overrightarrow{01})$ by (2.2) for every $n \geq 1$. Given $0 \leq b < n$, we define $\kappa_{z,\gamma}^{(n)}(\sigma)(b) \in \hat{\mathbb{Z}}$ by the congruence

$$f_\gamma^\flat(\sigma) \equiv \prod_{b=0}^{n-1} {y_{b,n}}^{\kappa_{z,\gamma}^{(n)}(\sigma)(b)}$$

modulo $\pi_1^{\text{ét}}(V_n, \overrightarrow{01})'$: the commutator subgroup of $\pi_1^{\text{ét}}(V_n, \overrightarrow{01})$.

Proposition 2.2 (See [NW1] Lemma 1)**.** *For each $\sigma \in G_F$, the system of functions*

$$\left\{ \mathbb{Z}/n\mathbb{Z} \ni b \mapsto \kappa_{z,\gamma}^{(n)}(\sigma)(b) \in \hat{\mathbb{Z}} \right\}_{n\in\mathbb{N}}$$

running over $n \geq 1$ defines a $\hat{\mathbb{Z}}$-valued measure on $\hat{\mathbb{Z}}$. □

We shall denote the above measure by

$$\varkappa_1(\gamma : \overrightarrow{01} \dashrightarrow z)(\sigma) \text{ or } \varkappa_1(z)_\gamma(\sigma)$$

and call it the *Kummer-Heisenberg measure* associated with the path $\gamma : \overrightarrow{01} \dashrightarrow z$. We view it as an element of the Iwasawa algebra $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$. Recall that $\hat{\mathbb{Z}}(1)$ in (2.3) is the Galois module $\hat{\mathbb{Z}}$ acted on by G_F by multiplication by the cyclotomic character. Let $\hat{\mathbb{Z}}(-1)$ be its dual.

Proposition 2.3. *The function*

$$\varkappa_1(\gamma : \overrightarrow{01} \dashrightarrow z) : G_F \to \hat{\mathbb{Z}}(1)[[\hat{\mathbb{Z}}(-1)]]$$

is a cocycle. Namely it holds that

$$\kappa_{z,\gamma}^{(n)}(\sigma\tau)(b) = \kappa_{z,\gamma}^{(n)}(\sigma)(b) + \chi(\sigma)\cdot \kappa_{z,\gamma}^{(n)}(\tau)(\chi(\sigma)^{-1}b)$$

for $\sigma, \tau \in G_F$, $n \geq 1$, $b \in \mathbb{Z}/n\mathbb{Z}$.

Proof. By the definition of f_γ (2.1), we have $f_\gamma(\sigma\tau) = f_\gamma(\sigma)\cdot\sigma(f_\gamma(\tau))$, hence $f_\gamma^\flat(\sigma\tau) \equiv f_\gamma^\flat(\sigma)\cdot\sigma(f_\gamma^\flat(\tau))$ modulo $\pi_1(V_n)'$. The assertion follows from this and the observation

$$\sigma(y_{b,n}) \equiv x^{-\chi(\sigma)b} y^{\chi(\sigma)} x^{\chi(\sigma)b} \equiv (y_{\chi(\sigma)b,n})^{\chi(\sigma)}$$

modulo $\pi_1(V_n)'$. □

Remark 2.4. In [NW1, Lemma 1], we introduced a compatible sequence $(\kappa_n)_n$ in the projective system $\varprojlim_n \hat{\mathbb{Z}}[\mathbb{Z}/n\mathbb{Z}]$ which forms a measure $\hat{\kappa} \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$. We call $\hat{\kappa}$ (resp. $\varkappa_1$) the Kummer-Heisenberg measure in e-form (resp. t-form) in the terminology of Appendix B. These two measures are 'oppositely directed' mainly because of different choice of path conventions as follows. After identification $\hat{\mathbb{Z}} \xrightarrow{\sim} \hat{\mathbb{Z}}(1)$ by $1 \mapsto (\xi_n = \exp(2\pi i/n))_n$, let ϵ denote the involution on $\hat{\mathbb{Z}}(1)$ induced by $\xi \mapsto \xi^{-1}$. Then, we have $\varkappa_1(\sigma) = \epsilon \cdot \hat{\kappa}(\sigma)$ $(\sigma \in G_F)$ as elements of $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}(1)]]$.

3. **Adelic Hurwitz measure**

3.1. **Paths to roots of unity.** Fix $m \in \mathbb{N}_{>1}$ and let a be an integer with $m \nmid a$. Let $\iota : V_1 \to V_1$ be the involution given by $\iota(z) = z^{-1}$. For any path γ on $V_1(\mathbb{C})$ from $\overrightarrow{01}$ to ξ_m^a, we set another path $\bar{\gamma}$ from $\overrightarrow{01}$ to ξ_m^{-a} by

$$\bar{\gamma} := \iota(\gamma) \cdot \Gamma_\infty$$

where Γ_∞ is a path on $V_1(\mathbb{C})$ from $\overrightarrow{01}$ to $\overrightarrow{\infty 1}$ as in Figure 1.

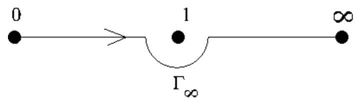

FIGURE 1. Γ_∞ is a path from $\overrightarrow{01}$ to $\overrightarrow{\infty 1}$

Write $\frac{a}{m} = \lfloor \frac{a}{m} \rfloor + \{\frac{a}{m}\}$ so that $0 \leq \{\frac{a}{m}\} < 1$, and define the path $\Gamma_{a/m} : \overrightarrow{01} \dashrightarrow \xi_m^a$ to be the composition $\Gamma_{\{a/m\}} \cdot x^{\lfloor a/m \rfloor}$, where $\Gamma_{\{a/m\}}$ is the path illustrated as in Figure 2.

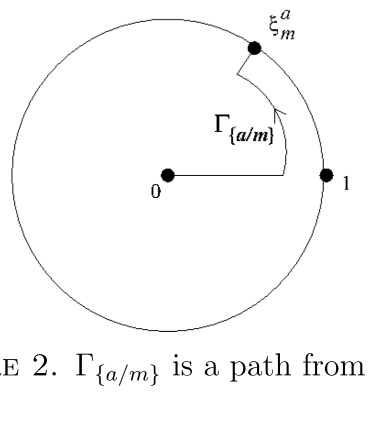

FIGURE 2. $\Gamma_{\{a/m\}}$ is a path from $\overrightarrow{01}$ to ξ_m^a

It is easy to see the following lemma.

Lemma 3.1. *Along the above paths $\Gamma_{a/m} : \overrightarrow{01} \dashrightarrow \xi_m^a$ and $\bar{\Gamma}_{a/m} : \overrightarrow{01} \dashrightarrow \xi_m^{-a}$, the associated Kummer 1-cocycles are coboundaries satisfying*

$$\rho_{\xi_m^a, \Gamma_{a/m}}(\sigma) = \frac{a}{m}(\chi(\sigma) - 1), \quad \rho_{\xi_m^{-a}, \bar{\Gamma}_{a/m}}(\sigma) = -\frac{a}{m}(\chi(\sigma) - 1)$$

for $\sigma \in G_{\mathbb{Q}(\mu_m)}$.

Proof. The first formula is immediate from the definition and the identification $\hat{\mathbb{Z}} \cong \hat{\mathbb{Z}}(1)$ by $1 \mapsto (\xi_n)_{n\geq 1}$. For the second, it suffices to note that the image of $\bar{\Gamma}_{a/m}$ by $\mathbf{P}^1 - \{0,1,\infty\} \hookrightarrow \mathbf{P}^1 - \{0,\infty\}$ is topologically homotopic to the complex conjugate of $\Gamma_{a/m}$. $\square$

Remark 3.2. It is worth noting that $\Gamma_\alpha x^n = \Gamma_{\alpha+n}$ for any $\alpha \in \mathbb{Q}$ and $n \in \mathbb{Z}$. This additivity property does not hold for $\bar{\Gamma}_\alpha$ in general. Still, if $0 \leq \alpha \leq 1$, then it holds that $\bar{\Gamma}_\alpha = \Gamma_{-\alpha} = \Gamma_{n-\alpha} x^{-n}$ for every $n \in \mathbb{Z}$. This last point will play a crucial role later in Lemma 5.7.

3.2. **Translation of a measure.** Let $p_n : V_n \to V_1$ be the cyclic cover of degree n considered in §2.1. For an étale path $\gamma : \overrightarrow{01} \dashrightarrow z$ on V_1, we shall write

$$\gamma_n\bigl(= (\gamma)_n\bigr) : \overrightarrow{01}_n \dashrightarrow z^{1/n}$$

to denote the lift of γ to V_n for all $n \geq 1$. Let us fix $\sigma \in G_F$. Note that the end point $z^{1/n}$ may or may not be fixed by the σ. By (2.2), we have

$$f^\flat_\gamma(\sigma) = (\gamma \cdot x^{\rho_{z,\gamma}(\sigma)})^{-1}\sigma(\gamma) \in \pi_1^{\text{ét}}(V_1, \overrightarrow{01})'. \tag{3.1}$$

But since V_n is an abelian cover of V_1, $\pi_1^{\text{ét}}(V_1, \overrightarrow{01})'$ is contained in $\pi_1(V_n, \overrightarrow{01})$. Therefore (3.1) implies that the lifts of $\gamma \cdot x^{\rho_{z,\gamma}(\sigma)}$ and of $\sigma(\gamma)$ departing at $\overrightarrow{01}_n$ on V_n end at the same point $\sigma(z^{1/n}) = \xi_n^{\rho_{z,\gamma}(\sigma)} z^{1/n}$. Since the lift $(x^{\rho_{z,\gamma}(\sigma)})_n$ of $x^{\rho_{z,\gamma}(\sigma)}$ from $\overrightarrow{01}_n$ ends at $\xi_n^{\rho_{z,\gamma}(\sigma)}\overrightarrow{01}_n$, the subsequent path $\gamma_{n,\sigma}$ should lift γ so as to start from that point $\xi_n^{\rho_{z,\gamma}(\sigma)}\overrightarrow{01}_n$ with ending at the point $\sigma(z^{1/n})$ on V_n:

$$\begin{tikzcd} \overrightarrow{01}_n \arrow[r, dashed, "x^{\rho_{z,\gamma}(\sigma)}"] \arrow[rr, dashed, bend right, "(\sigma(\gamma))_n"'] & \xi_n^{\rho_{z,\gamma}(\sigma)}\overrightarrow{01}_n \arrow[r, dashed, "\gamma_{n,\sigma}"] & \sigma(z^{1/n}) \end{tikzcd}. \tag{3.2}$$

In summary, writing $(\sigma(\gamma))_n$ for the lift of $\sigma(\gamma)$ from $\overrightarrow{01}_n$ on V_n, we may express $f^\flat_\gamma(\sigma)$ as the composition of those three paths

$$f^\flat_\gamma(\sigma) = (x^{\rho_{z,\gamma}(\sigma)})_n^{-1} \cdot \gamma_{n,\sigma}^{-1} \cdot (\sigma(\gamma))_n$$

on V_n.

Below, we shall see magnification of the base space $\hat{\mathbb{Z}}$ on a coset $s + r\hat{\mathbb{Z}}$ ($s, r \in \mathbb{Z}$, $r \geq 1$) under the measure $\varkappa_1(z)_\gamma(\sigma)$ can be interpreted as a twisted lifting of the reference path $\gamma : \overrightarrow{01} \dashrightarrow z$ to V_r followed with 's-rotated' embedding by $V_r \hookrightarrow V_1$.

Set an 's-modified' path $\gamma_{\langle -s\rangle} : \overrightarrow{01} \dashrightarrow z$, for the given path $\gamma : \overrightarrow{01} \dashrightarrow z$ on V_1, by

$$\gamma_{\langle -s\rangle} := \gamma \cdot x^{-s}. \tag{3.3}$$

It follows easily that

$$\rho_{z,\gamma_{\langle -s\rangle}}(\sigma) = \rho_{z,\gamma}(\sigma) - s(\chi(\sigma) - 1) \quad (\sigma \in G_F). \tag{3.4}$$

Suppose that $\xi_r, z^{1/r} \in F$. Write

$$\gamma_r = (\gamma)_r : \overrightarrow{01}_r \dashrightarrow z^{1/r},$$
$$\gamma_{\langle -s\rangle, r} = (\gamma_{\langle -s\rangle})_r : \overrightarrow{01}_r \dashrightarrow \xi_r^{-s}\, z^{1/r}$$

for the lifts of the paths γ and $\gamma_{\langle -s\rangle}$ by $p_r : V_r \to V_1$ respectively, and

$$\gamma_{r*} = \jmath_r(\gamma_r) : \overrightarrow{01} \dashrightarrow z^{1/r},$$
$$\gamma_{\langle -s\rangle,r*} = \jmath_r(\gamma_{\langle -s\rangle,r}) : \overrightarrow{01} \dashrightarrow \xi_r^{-s}\, z^{1/r}$$

for the images of paths γ_r, $\gamma_{\langle -s\rangle,r}$ on V_r by the immersion $\jmath_r : (V_r, \overrightarrow{01}_r) \hookrightarrow (V_1, \overrightarrow{01})$ respectively. It follows that

$$\begin{aligned}\rho_{z^{1/r},\gamma_{\langle -s\rangle,r*}}(\sigma) &= \rho_{z^{1/r},(\gamma)_{r*}}(\sigma) - \frac{s}{r}(\chi(\sigma)-1) \\ &= \frac{1}{r}\rho_{z,\gamma}(\sigma) - \frac{s}{r}(\chi(\sigma)-1)\end{aligned} \tag{3.5}$$

for every $\sigma \in G_F$.

Lemma 3.3. *Notations being as above, with assumptions $\xi_r, z^{1/r} \in F$ and $\sigma \in G_F$.*
(i) For every $n \geq 1$, it holds that

$$\kappa_{z,\gamma}^{(nr)}(\sigma)(vr + s\chi(\sigma)) = \kappa^{(n)}_{\xi_r^{-s} z^{1/r}, \gamma_{\langle -s\rangle,r*}}(\sigma)(v) \qquad (v = 0,\dots,n-1),$$

where $vr + s\chi(\sigma)$ in LHS is regarded $\in (\mathbb{Z}/nr\mathbb{Z})$.
(ii) For any continuous function φ on $\hat{\mathbb{Z}}$, we have

$$\int_{s\chi(\sigma)+r\hat{\mathbb{Z}}} \varphi(b)\, d\varkappa_1(\gamma : \overrightarrow{01} \dashrightarrow z)(\sigma)(b) = \int_{\hat{\mathbb{Z}}} \varphi(rv+s\chi(\sigma))\, d\varkappa_1(\gamma_{\langle -s\rangle,r*} : \overrightarrow{01} \dashrightarrow \xi_r^{-s} z^{1/r})(\sigma)(v).$$

Proof. In this proof, for $n \geq 1$, we denote $\pi(n) := \pi_1^{\text{ét}}(V_n, \overrightarrow{01})$ and write

$$\varpi_{nr} : \pi(nr) \twoheadrightarrow \pi(n) \tag{3.6}$$

for the surjection induced from the open immersion $(V_{nr}, \overrightarrow{01}_{nr}) \hookrightarrow (V_n, \overrightarrow{01}_n)$. Note that, among the standard generators x^{nr}, $y_{b,nr}$ $(b = 0,\dots,nr-1)$ of $\pi(nr)$, only x^{nr} and $y_{vr,nr}$ $(v = 0,\dots,n-1)$ survive via ϖ_{nr} to be x^n, $y_{v,n}$ $(v = 0,\dots n-1)$ in $\pi(n)$.

Noting that $x^{-u}yx^u = y_{u,nr} \equiv y_{u+nrk,nr} \bmod \pi(nr)'$ for $u, k \in \mathbb{Z}$, we see from Definition 2.1 that

$$\begin{aligned} x^{s\chi(\sigma)} \cdot f_\gamma^\flat(\sigma) \cdot x^{-s\chi(\sigma)} &\equiv \prod_{u=0}^{nr-1} {y_{-s\chi(\sigma)+u,nr}}^{\kappa_{z,\gamma}^{(nr)}(\sigma)(u)} \mod \pi(nr)' \\ &\equiv \prod_{v=0}^{nr-1} {y_{v,nr}}^{\kappa_{z,\gamma}^{(nr)}(\sigma)(v+s\chi(\sigma))} \mod \pi(nr)' \end{aligned} \tag{3.7}$$

which should map via ϖ_{nr} to the product over those v multiples of r:

$$\varpi_{nr}\left(x^{s\chi(\sigma)} \cdot f_\gamma^\flat(\sigma) \cdot x^{-s\chi(\sigma)}\right) \equiv \prod_{v=0}^{n-1} {y_{v,n}}^{\kappa_{z,\gamma}^{(nr)}(\sigma)(rv+s\chi(\sigma))} \mod \pi(n)' \tag{3.8}$$

as $\pi(n)' \supset \varpi_{nr}(\pi(nr)')$. We shall interpret the LHS of the above expression (3.7) by applying the composition diagram (3.2) to the path $\gamma_{\langle -s\rangle} : \overrightarrow{01} \dashrightarrow z$ (3.3) on V_1 and its lift

$(\gamma_{\langle -s\rangle})_r = \gamma_{\langle -s\rangle,r}$ on V_r:

$$
\begin{tikzcd}
 & V_{nr} \arrow[r, hook] \arrow[d, two heads] & V_n \arrow[d, two heads] & \\
{[\gamma_{\langle -s\rangle,r} : \overrightarrow{01} \dashrightarrow \xi_r^{-s} z^{1/r}]} & V_r \arrow[l, dotted, hook'] \arrow[r, hook] \arrow[d, two heads] & V_1 \arrow[r, dotted, hook] & {[\gamma_{\langle -s\rangle,r*}]} \\
{[\gamma_{\langle -s\rangle} : \overrightarrow{01} \dashrightarrow z]} & V_1 \arrow[l, dotted, hook'] & &
\end{tikzcd}
.
$$

We first derive:

$$
\begin{aligned}
(3.9)\qquad x^{s\chi(\sigma)} \cdot f^\flat_\gamma(\sigma) \cdot x^{-s\chi(\sigma)} &= x^{s\chi(\sigma)} \cdot x^{-\rho_{z,\gamma}(\sigma)} \gamma^{-1} \sigma(\gamma) \cdot x^{-s\chi(\sigma)} \\
&= x^{s(\chi(\sigma)-1)-\rho_{z,\gamma}(\sigma)} \cdot (\gamma x^{-s})^{-1} \sigma(\gamma x^{-s}) \\
&= (\gamma_{\langle -s\rangle} \cdot x^{\rho_{z,\gamma_{\langle -s\rangle}}(\sigma)})^{-1} \cdot \sigma(\gamma_{\langle -s\rangle}).
\end{aligned}
$$

By (3.4), the former factor of path composition reads on V_r

$$
(\gamma_{\langle -s\rangle})_{r,\sigma} \cdot \left(x^{\rho_{z,\gamma_{\langle -s\rangle}}(\sigma)}\right)_r = (\gamma x^{-s})_{r,\sigma} \cdot \left(x^{\rho_{z,\gamma}(\sigma) - s(\chi(\sigma)-1)}\right)_r
$$

where $(\gamma_{\langle -s\rangle})_{r,\sigma}$ stands for a suitable lift of $\gamma_{\langle -s\rangle}$ on V_r, which arrives at the same end point on V_r as the latter σ-transformed factor

$$
(\sigma(\gamma_{\langle -s\rangle}))_r : \overrightarrow{01}_r \dashrightarrow \sigma(\xi_r^{-s}\, z^{1/r}).
$$

It turns out that $(\gamma_{\langle -s\rangle})_{r,\sigma}$ starts at $\xi_r^{\rho_{z,\gamma}(\sigma)-s(\chi(\sigma)-1)} \cdot \overrightarrow{01}_r$ which is equal to $\overrightarrow{01}_r$ by our assumption $\xi_r, z^{1/r} \in F$. Thus we conclude

$$
(3.10)\qquad (\gamma_{\langle -s\rangle})_{r,\sigma} = \gamma_{\langle -s\rangle,r} \left(= (\gamma \cdot x^{-s})_r\right).
$$

By virtue of this and (3.5), applying to (3.9) the surjection $\varpi_r : \pi(r) \twoheadrightarrow \pi(1)$ determined by $x^r \mapsto x$, $y_0 \mapsto y$ and $y_1, \dots, y_{r-1} \mapsto 1$ as the case $n = 1$ of (3.6), we obtain

$$
\begin{aligned}
\varpi_r\left((x^{-\rho_{z,\gamma_{\langle -s\rangle}}(\sigma)})_r \cdot (\gamma_{\langle -s\rangle})^{-1}_{r,\sigma} \cdot \sigma(\gamma_{\langle -s\rangle,r})\right) &= \varpi_r\left((x^r)^{-\rho_{z^{1/r},\gamma_r}(\sigma) + \frac{s}{r}(\chi(\sigma)-1)} \gamma^{-1}_{\langle -s\rangle,r} \cdot \sigma(\gamma_{\langle -s\rangle,r})\right) \\
&= x^{-\rho_{z^{1/r},\gamma_{\langle -s\rangle,r*}}(\sigma)} \cdot {\gamma_{\langle -s\rangle,r*}}^{-1} \cdot \sigma(\gamma_{\langle -s\rangle,r*}) \\
&= f^\flat_{\gamma_{\langle -s\rangle,r*}}(\sigma) \\
&\equiv \prod_{v=0}^{n-1} y_{v,n}^{\kappa^{(n)}_{\xi_r^{-s} z^{1/r},\gamma_{\langle -s\rangle,r*}}(\sigma)(v)} \quad \bmod \pi(n)'.
\end{aligned}
$$

This, combined with (3.8) and (3.9) and the compatibility $\varpi_{nr} = \varpi_r|_{\pi(nr)}$, proves (i). The assertion (ii) is just a formal consequence of (i). □

Suppose we are given a measure $\mu \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$. Let $m, a \in \mathbb{Z}$ be integers as in §3.1 and pick $\nu \in \hat{\mathbb{Z}}^\times$. Consider the coset $Q_{a\nu,m} := \frac{a\nu}{m} + \hat{\mathbb{Z}}$ of $\hat{\mathbb{Z}}$ in $\mathbb{Q}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$. Then, obviously,

$$
R_{a\nu,m} := m \cdot Q_{a\nu,m} = a\nu + m\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}.
$$

We define the measure $[m, a\nu]_*(\mu)$ on $R_{a\nu,m}$ by assigning to each open subset $U \subset R_{a\nu,m}$ the value $\mu(U')$, where U' is the inverse image of U by the affine map $t \mapsto mt + a\nu$ $(t \in \hat{\mathbb{Z}})$.

Note that Lemma 3.3 (ii) reads:

$$\varkappa_1(\gamma : \overrightarrow{01} \dashrightarrow z)(\sigma)|_{s\chi(\sigma)+m\hat{\mathbb{Z}}} = [m, s\chi(\sigma)]_* \Big(\varkappa_1(\gamma_{\langle -s\rangle, m*} : \overrightarrow{01} \dashrightarrow \xi_m^{-s} z^{1/m})(\sigma) \Big) \tag{3.11}$$

for $\sigma \in G_F$, $m, s \in \mathbb{Z}$, $m \geq 1$, where $*|_{s\chi(\sigma)+m\hat{\mathbb{Z}}}$ in LHS designates the restricted measure on $R_{s\chi(\sigma),m} = s\chi(\sigma) + m\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$.

Let ι denote the action of the complex conjugation on $\hat{\mathbb{Z}}(1)[[\hat{\mathbb{Z}}(-1)]]$, that is, the action of $-1 \in \hat{\mathbb{Z}}^\times$. It is straightforward to see

$$\iota \circ [m, a\nu]_* = [m, -a\nu]_* \circ \iota. \tag{3.12}$$

Now we are ready to introduce the fundamental object of our study. Let $m > 1$ and $a \in \mathbb{Z}$ as above, and let $\Gamma_{a/m} \in \pi(V_1(\mathbb{C}); \xi_m^a, \overrightarrow{01})$ be the path introduced in §3.1.

Definition 3.4 (**$\hat{\mathbb{Z}}$-Hurwitz and adelic Hurwitz measure).** For each $\sigma \in G_{\mathbb{Q}(\mu_m)}$ we define the $\hat{\mathbb{Z}}$-Hurwitz measure $\boldsymbol{\zeta}_{a/m}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$ and the adelic Hurwitz measure $\hat{\zeta}_{a,m}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$ by the formulas

$$\begin{aligned} \boldsymbol{\zeta}_{a/m}(\sigma) &:= \varkappa_1\big(\overrightarrow{01} \overset{\bar{\Gamma}_{a/m}}{\dashrightarrow} \xi_m^{-a}\big)(\sigma) + \iota \left(\varkappa_1\big(\overrightarrow{01} \overset{\Gamma_{a/m}}{\dashrightarrow} \xi_m^{a}\big)(\sigma) \right); \\ \hat{\zeta}_{a,m}(\sigma) &:= [m, a\chi(\sigma)]_* \, \boldsymbol{\zeta}_{a/m}(\sigma). \end{aligned}$$

4. Geometrical interpretation of translation of measure

In this section we address the fact that translation of Kummer-Heisenberg measure by $[m, a\chi]_*$ corresponds to path composition with the loop x^α. We work however only with p-adic measures.

4.1. ***p*-adic Galois polylogarithms.** Let γ be an étale path on V_1 from $\overrightarrow{01}$ to an F-rational (possibly tangential) point z. Let $\mathbb{Q}_p\langle\langle X, Y\rangle\rangle$ be the non-commutative power series ring in two variables X, Y, and write $\mathscr{E} : \pi_1^{\text{pro-}p}(V_1, \overrightarrow{01}) \hookrightarrow \mathbb{Q}_p\langle\langle X, Y\rangle\rangle$ for the embedding that sends the standard generators x, y to $\exp(X)$, $\exp(Y)$. We define I_Y to be the ideal of $\mathbb{Q}_p\langle\langle X, Y\rangle\rangle$ generated by monomials containing Y twice or more.

For $\sigma \in G_F$, set

$$\begin{aligned} \Lambda_\gamma(\sigma) &:= \mathscr{E}(f_\gamma(\sigma)); \\ \overline{\Lambda_\gamma}(\sigma) &:= \mathscr{E}(f_\gamma^\flat(\sigma)) = \exp(-\rho_{z,\gamma}(\sigma)X) \cdot \Lambda_\gamma(\sigma). \end{aligned}$$

Definition 4.1. Define p-adic Galois polylogarithms $\ell i_k(z)_\gamma, \mathrm{Li}_k(z)_\gamma : G_F \to \mathbb{Q}_p$ by the congruence expansion

$$\log \Lambda_\gamma(\sigma) \equiv \rho_{z,\gamma}(\sigma) + \sum_{k=1}^{\infty} (-1)^{k-1} \ell i_k(z)_\gamma(\sigma)\, (\mathrm{ad}X)^{k-1}(Y),$$

$$\log \overline{\Lambda_\gamma}(\sigma) \equiv \sum_{k=1}^{\infty} (-1)^{k-1} \mathrm{Li}_k(z)_\gamma(\sigma)\, (\mathrm{ad}X)^{k-1}(Y)$$

modulo the ideal I_Y.

Proposition 4.2. *The family of functions* $\{\rho_{z,\gamma},\ \ell i_k(z)_\gamma,\ \mathrm{Li}_k(z)_\gamma : G_F \to \mathbb{Q}_p\}_{k\geq 1}$ *satisfy*

$$\text{(i)}\qquad \mathrm{Li}_k(z)_\gamma = \sum_{i=1}^{k} \frac{\rho_{z,\gamma}^{k-i}}{(k+1-i)!}\,\ell i_i(z)_\gamma,$$

$$\text{(ii)}\qquad \ell i_k(z)_\gamma = \sum_{s=0}^{k-1} \frac{B_s}{s!}\rho_{z,\gamma}^{s}\,\mathrm{Li}_{k-s}(z)_\gamma$$

for $k = 1, 2, \ldots$.

In fact, this proposition is a formal consequence of the following lemma:

Lemma 4.3. *Let* K *be a field of characteristic zero, and suppose that two sequences* $\{b_i\}_{i\geq 0}$, $\{u_i\}_{i\geq 0}$ *in* K *satisfy the congruence*

$$e^{-u_0X} e^{u_0X+\sum_{k=0}^{\infty} u_{k+1}(\mathrm{ad}X)^k(Y)} \equiv e^{b_0X+\sum_{k=0}^{\infty} b_{k+1}(\mathrm{ad}X)^k(Y)} \mod I_Y$$

as non-commutative power series in X, Y. *Then,* $b_0 = 0$ *and, for* $k = 1, 2, \ldots,$

$$b_k = \sum_{i=1}^{k} \frac{(-u_0)^{k-i}}{(k+1-i)!}u_i, \qquad u_k = \sum_{s=0}^{k-1} \frac{B_s}{s!}(-u_0)^s b_{k-s},$$

where $B_0, B_1, \ldots$ *are Bernoulli numbers defined by* $\sum_{s=0}^{\infty} \frac{B_s}{s!}T^s = \frac{T}{e^T-1}$.

Proof. We use the classical Campbell-Hausdorff formula

$$\log(e^\alpha e^\beta) \equiv \beta + \sum_{n=0}^{\infty} \frac{B_n}{n!}(\mathrm{ad}\beta)^n(\alpha) \mod \deg(\alpha) \geq 2.$$

Set $-\alpha = \sum_{i=0}^{\infty} b_{i+1}(\mathrm{ad}X)^i(Y)$ and $-\beta = u_0X$ so that congruences mod $\deg(\alpha) \geq 2$ derive those mod I_Y. It follows that $\log(e^{u_0X}e^{b_1Y+b_2[X,Y]+\ldots})$ is congruent to $u_0X + \sum_{k=0}^{\infty}\big(\sum_{s=0}^{k} \frac{B_s}{s!}(-u_0)^s b_{k+1-s}\big)(\mathrm{ad}X)^k(Y)$ mod I_Y. This is equivalent to the equality

$$\sum_{k=0}^{\infty} u_{k+1}T^k = \left(\frac{-u_0T}{e^{-u_0T}-1}\right)\sum_{k=0}^{\infty} b_{k+1}T^k. \tag{4.1}$$

The assertion follows from this immediately. □

4.2. **Extension for $\mathbb{Q}_p$-paths.** Let $\pi_{\mathbb{Q}_p}(\overrightarrow{01})$ be the pro-algebraic hull of the image of the above embedding $\mathscr{E} : \pi_1^{\text{pro-}p}(V_1, \overrightarrow{01}) \hookrightarrow \mathbb{Q}_p\langle\langle X, Y\rangle\rangle$, and extend it to the inclusion of path torsors $\pi_1^{\text{pro-}p}(V_1; z, \overrightarrow{01}) \hookrightarrow \pi_{\mathbb{Q}_p}(z, \overrightarrow{01})$ naturally. The elements of $\pi_{\mathbb{Q}_p}(\overrightarrow{01})$, $\pi_{\mathbb{Q}_p}(z, \overrightarrow{01})$ will be simply called $\mathbb{Q}_p$-paths, and the action of the Galois group G_F on the pro-p paths extends to that on the $\mathbb{Q}_p$-paths in the obvious manner.

For each $\mathbb{Q}_p$-path $\gamma : \overrightarrow{01} \dashrightarrow z$ and $\sigma \in G_F$, we may define the Galois associator $f_\gamma(\sigma) := \gamma^{-1}\cdot\sigma(\gamma) \in \pi_{\mathbb{Q}_p}(\overrightarrow{01})$ extending (2.1). Then, define $\rho_{z,\gamma}, li_k(z,\gamma) : G_F \to \mathbb{Q}_p$ $(k = 1, 2, \ldots)$ as the coefficients in $\log(f_\gamma(\sigma))$ so as to extend the congruence in Definition 4.1 mod I_Y, and then, define $\mathrm{Li}_k(z)_\gamma : G_F \to \mathbb{Q}_p$ $(k = 1, 2, \ldots)$ as the coefficients of $\log(\exp(-\rho_{z,\gamma}(\sigma)X)\cdot f_\gamma(\sigma))$ again as the extension of Definition 4.1. Then, it is simple to see that the identities in Proposition 4.2 hold true for $\mathbb{Q}_p$-paths $\gamma : \overrightarrow{01} \dashrightarrow z$ in the same forms.

4.3. **Relation with $\varkappa_{1,p}$.** We now arrive at the stage to connect the ℓ-adic polylogarithms Li_k and the Kummer-Heisenberg measure $\varkappa_1$. In [NW1], we showed that, for pro-p paths $\gamma : \overrightarrow{01} \dashrightarrow z$, the function $\mathrm{Li}_k(z)_\gamma$ multiplied by $(k-1)!$ can be written by a certain polylogarithmic character $\tilde{\chi}_k(z)_\gamma : G_F \to \mathbb{Z}_p$ defined by Galois transformations of certain sequence of numbers of forms $\prod_{s=0}^{p^n-1}(1-\xi^s z^{1/p^n})^{s^{k-1}/p^n}$ $(\xi \in \mu_{p^n}, n \geq 1)$. This enabled us to express $\mathrm{Li}_k(z)_\gamma(\sigma)$ $(\sigma \in G_F)$ by the moment integral $\frac{1}{(k-1)!}\int_{\mathbb{Z}_p} b^{k-1} d\varkappa_{1,p}(\sigma)(b)$ over the p-adic measure $\varkappa_{1,p}(\sigma)$ which is by definition the image of the Kummer-Heisenberg measure $\varkappa_1(\sigma)$ (§2.2) by the projection $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \to \mathbb{Z}_p[[\mathbb{Z}_p]]$.

A generalization of this phenomenon has been investigated in [NW3] for some more general $\mathbb{Q}_p$-paths of the form $\gamma x^{a/m}$. We summarize the result as follows:

Proposition 4.4 ([NW3] §7). *Let $\gamma : \overrightarrow{01} \dashrightarrow z$ be a pro-p path. Then, for any $\alpha \in \mathbb{Q}_p$, we have*

$$\mathrm{Li}_k(z)_{\gamma x^\alpha}(\sigma) = \frac{1}{(k-1)!}\int_{\mathbb{Z}_p} (b + \alpha\chi(\sigma))^{k-1} d\varkappa_{1,p}(\overrightarrow{01} \overset{\gamma}{\dashrightarrow} z)(b) \qquad (\sigma \in G_F).$$

Proof. We just translate [NW3, §7] from e-form to t-form in the terminology of Appendix B. In e-form, it reads (with $\delta := \gamma$, $\alpha := -\frac{s}{n}$, $\hat{\kappa}_p := \boldsymbol{\kappa}_{z,\gamma}$ in [NW3, §7]):

$$\tilde{\boldsymbol{\chi}}_k^{\mathtt{x}^\alpha\delta}(\sigma) = \int_{\mathbb{Z}_p} (b - \alpha\chi(\sigma))^{k-1} d\hat{\kappa}_p(\sigma)(b).$$

Let γx^α be the t-path reciprocally corresponding to the e-path $\mathtt{x}^\alpha\delta$. In RHS, we regard the measure $\hat{\kappa}_p$ as the p-adic image of $\hat{\kappa}(\delta)$ of Remark 2.4 which can be switched into the e-form $\varkappa_{1,p}(\gamma)$ to obtain

$$\int_{\mathbb{Z}_p} (b - \alpha\chi(\sigma))^{k-1} d\hat{\kappa}_p(\sigma)(b) = \int_{\mathbb{Z}_p} (-b - \alpha\chi(\sigma))^{k-1} d\varkappa_{1,p}(\sigma)(b).$$

At the same time, we may convert the LHS to t-form by (B.11) and (B.13) as

$$\begin{aligned}\tilde{\boldsymbol{\chi}}_k^{\mathtt{x}^\alpha\delta}(\sigma) &= -(k-1)!\, \mathscr{L}i_k(z)_{\mathtt{x}^\alpha\delta : \overrightarrow{01} \rightsquigarrow z}(\sigma) \\ &= (-1)^{k-1}(k-1)!\, \mathrm{Li}_k(z)_{\gamma x^\alpha : \overrightarrow{01} \dashrightarrow z}(\sigma).\end{aligned}$$

The formula of the proposition follows from combination of these identities. □

5. **Consequence of Inversion Formula**

5.1. **Pro-p inversion formula.** We start this section with the main technical result.

Let a, m be integers with $m > 1$, $m \nmid a$, and fix the m-th root of unity $z := \xi_m^a \in \mu_m$ and set $F = \mathbb{Q}(z)$. Pick any path $\gamma : \overrightarrow{01} \dashrightarrow z$ in $\pi_1^{\text{pro-}p}(\mathbf{P}^1 - \{0, 1, \infty\}, z, \overrightarrow{01})$ and let $\bar{\gamma} : \overrightarrow{01} \dashrightarrow z^{-1}$ be the associated path defined in §3.1.

By the assumption $z \in \mu_m$, using the p-adic cyclotomic character $\chi_p : G_F \to \mathbb{Z}_p^\times$, we may suppose that the Kummer 1-cocycle $\rho_{z,\gamma} : G_F \to \mathbb{Z}_p$ (written just ρ_z for simplicity) is of a 1-coboundary form

$$\rho_{z,\gamma}(\sigma) = \rho_z(\sigma) = \alpha(\chi_p(\sigma) - 1) \quad (\sigma \in G_F) \tag{5.1}$$

with a unique constant $\alpha \in \frac{a}{m} + \mathbb{Z}_p$. Since we do not assume $p \nmid m$, the constant $\alpha \in \mathbb{Q}_p$ may generally have denominator, while $\rho_z(\sigma) \in \mathbb{Z}_p$.

Theorem 5.1. *Notations being as above, we have*

$$\mathrm{Li}_k(\xi_m^{-a})_{\bar{\gamma}x^{\alpha}}(\sigma)+(-1)^k\mathrm{Li}_k(\xi_m^{a})_{\gamma x^{-\alpha}}(\sigma)=\frac{1}{k!}B_k(\alpha)(1-\chi_p(\sigma)^k).$$

for $\sigma\in G_F$ and $k\geq 1$.

This result generalizes [W3, Theorem 10.2], where only the case $p\nmid m$ was considered. Here, we shall present a proof using the inversion formula for p-adic Galois polylogarithms [NW2]. For $\sigma\in G_F$, consider the ℓ-adic polylogarithmic characters (for $\ell=p$) $\tilde{\chi}_k(z)_\gamma(\sigma)$, $\tilde{\chi}_k(\frac{1}{z})_{\bar{\gamma}}(\sigma)$ along those pro-p paths γ and $\bar{\gamma}$. In [NW2, 6.3], we showed an inversion formula for γ and $\bar{\gamma}$ in the following form*

$$\tilde{\chi}_k(z)_\gamma(\sigma)+(-1)^k\tilde{\chi}_k(\tfrac{1}{z})_{\bar{\gamma}}(\sigma)=-\frac{1}{k}\{B_k(-\rho_z(\sigma))-B_k\cdot\chi_p(\sigma)^k\}\qquad(\sigma\in G_F),\tag{5.2}$$

where $B_k(T)$ is the Bernoulli polynomial defined by $\sum_{k=0}^{\infty}B_k(T)\frac{w^k}{k!}=\frac{we^{Tw}}{e^w-1}$ and $B_k=B_k(0)$. Apply to (5.2) the translation formula

$$\mathrm{Li}_k(z)_\gamma(\sigma)=\frac{(-1)^{k-1}}{(k-1)!}\,\tilde{\chi}_k(z)_\gamma(\sigma)\qquad(\sigma\in G_F,\ k\geq 1)\tag{5.3}$$

for which we refer the reader to (B.11), (B.13) and (B.14), and obtain

$$\mathrm{Li}_k(\tfrac{1}{z})_{\bar{\gamma}}(\sigma)+(-1)^k\mathrm{Li}_k(z)_\gamma(\sigma)=\frac{1}{k!}(B_k(-\rho_z(\sigma))-B_k\cdot\chi_p(\sigma)^k)\qquad(\sigma\in G_F).\tag{5.4}$$

Observe that this formula already gives a special case of Theorem 5.1 where $\alpha=0$ and $\rho_z(\sigma)=0$. What we shall do from now is to deform this formula into a form involved with the $\mathbb{Q}_p$-paths $\gamma x^{-\alpha}$ and $\bar{\gamma}x^{\alpha}$. In fact, it follows from Proposition 4.4, we generally have

$$\mathrm{Li}_k(z)_{\gamma x^{\alpha}}(\sigma)=\sum_{i=0}^{k-1}\mathrm{Li}_{k-i}(z)_\gamma(\sigma)\frac{(\alpha\chi_p(\sigma))^i}{i!},$$

hence the LHS of Theorem 5.1 can be written as:

$$\begin{aligned}&\mathrm{Li}_k(\tfrac{1}{z})_{\bar{\gamma}x^{\alpha}}(\sigma)+(-1)^k\mathrm{Li}_k(z)_{\gamma x^{-\alpha}}(\sigma)\\&=\sum_{i=0}^{k-1}\frac{(\alpha\chi_p(\sigma))^i}{i!}\Big(\mathrm{Li}_{k-i}(\tfrac{1}{z})_{\bar{\gamma}}(\sigma)+(-1)^{k-i}\mathrm{Li}_{k-i}(z)_\gamma(\sigma)\Big).\end{aligned}\tag{5.5}$$

To complete the proof of Theorem 5.1, by comparing (5.4) and (5.5), we are now reduced to the following core lemma:

Lemma 5.2. *Let k be a positive integer, and set $J_s:=-\frac{1}{s}\{B_s(-\rho_z(\sigma))-B_s\cdot\chi_p(\sigma)^s\}$ for $s=1,\dots,k$ and $\sigma\in G_F$. Then, we have*

$$\frac{1}{k}B_k(\alpha)\left(\chi_p(\sigma)^k-1\right)=\sum_{i=0}^{k-1}\binom{k-1}{i}\alpha^i\chi_p(\sigma)^iJ_{k-i}(\sigma).$$

*See (B.14). The path $\bar{\gamma}$ from $\overrightarrow{01}$ to $z^{-1}\in\mu_m$ in t-form here reciprocally corresponds to the path $\langle 0,1\rangle[1_0^{\infty}]\langle 1,\infty\rangle\cdot f_2(\gamma)$ in e-form with the notation of [NW2, §6.3].

Proof. For simplicity, we omit σ in this proof. To simplify the RHS of the lemma, we use the Bernoulli addition formula

$$B_k(y+x) = \sum_{s=0}^{k} \binom{k}{s} B_s(y) x^{k-s}. \tag{5.6}$$

Applying (5.6) with $x = \alpha\chi_p$, $y = -\rho_z$ so that $x + y = \alpha$ by (5.1) (resp. with $x = \alpha$, $y = 0$ so that $x + y = \alpha$), we obtain:

$$\begin{cases} B_k(\alpha) &= \sum_{i=0}^{k-1} \binom{k}{i} B_{k-i}(-\rho_z)(\alpha\chi_p)^i + (\alpha\chi_p)^k, \\ B_k(\alpha) &= \sum_{i=0}^{k-1} \binom{k}{i} \alpha^i B_{k-i} + \alpha^k. \end{cases}$$

Each of the above identities respectively simplifies the former and latter term of the following computation of the RHS of the lemma. In fact, noting $\frac{1}{k-i}\binom{k-1}{i} = \frac{1}{k}\binom{k}{i}$, one computes:

$$\begin{aligned} \mathrm{RHS} &= \left[-\sum_{i=0}^{k-1} \frac{(\alpha\chi_p)^i}{k} \binom{k}{i} B_{k-i}(-\rho_z) \right] + \left[\frac{\chi_p^k}{k} \sum_{i=0}^{k-1} \binom{k}{i} \alpha^i B_{k-i} \right] \\ &= \left[-\frac{1}{k} B_k(\alpha) + \frac{1}{k} \alpha^k \chi_p^k \right] + \left[\frac{\chi_p^k}{k} (B_k(\alpha) - \alpha^k) \right] \end{aligned}$$

which coincides with the LHS of the aimed identity. □

Thus, our proof of Theorem 5.1 is completed. □

Remark 5.3. We mention that the proof in [W3] also carries over in the case $p \mid m$; One needs only consider rational paths in $\pi_1^{\text{pro-}p}(V_1; \xi_m^a, \overrightarrow{01}) \hat{\otimes} \mathbb{Q}$ and in $\pi_1^{\text{pro-}p}(V_1, \overrightarrow{01}) \hat{\otimes} \mathbb{Q}$. The embedding of the latter $\pi_1 \hat{\otimes} \mathbb{Q}$ into $\mathbb{Q}_p\langle\langle X, Y \rangle\rangle$ extends to that of the former pro-p path space into $\mathbb{Q}_p\langle\langle X, Y \rangle\rangle$.

Remark 5.4. If $p \nmid m$, then for any $\alpha \in \mathbb{Z}_p$ there is $\gamma \in \pi_1^{\text{pro-}p}(V_1; \xi_m^a, \overrightarrow{01})$ such that $\rho_{z,\gamma} = \alpha(\chi_p - 1)$. Hence we have

$$\frac{(k-1)!}{\chi_p(\sigma)^k - 1} \left(\mathrm{Li}_k(\xi_m^{-a})_{\bar{\gamma}x^\alpha}(\sigma) + (-1)^k \mathrm{Li}_k(\xi_m^a)_{\gamma x^{-\alpha}}(\sigma) \right) = -\frac{B_k(\alpha)}{k}$$

as long as $\chi_p(\sigma)^k \neq 1$. A key observation here is the following: Taking $\alpha = 0$ we get values of the Riemann zeta function at negative integers (cf. [W2]), while taking $\alpha = \frac{a}{m} \in \mathbb{Q}^\times$ we get values of Hurwitz zeta function $\zeta(s, \frac{a}{m})$ at negative integers. If we choose γ from topological paths $\Gamma_{a/m} \in \pi(V_1(\mathbb{C}); \xi_m^a, \overrightarrow{01})$ (§3.1), then we get $-\frac{1}{k} B_k(\frac{a}{m})$ for every choice of rational prime p.

5.2. **Moment integrals of p-adic Hurwitz measure.** First we shall rewrite the formula in Theorem 5.1 in terms of measures $\varkappa_{1,p}(\overrightarrow{01}\overset{\gamma}{\dashrightarrow}\xi_m^a)$ and $\varkappa_{1,p}(\overrightarrow{01}\overset{\bar\gamma}{\dashrightarrow}\xi_m^{-a})$ after multiplied by m^{k-1}. Set $\alpha := \frac{a}{m} \in \mathbb{Q}$. By Proposition 4.4 we find that, for $\sigma \in G_F$,

$$\begin{aligned}
(5.7)\quad m^{k-1}\mathrm{Li}_k(\xi_m^{-a})_{\bar\gamma x^\alpha}(\sigma) &= \frac{m^{k-1}}{(k-1)!}\int_{\mathbb{Z}_p}(v+\alpha\chi_p(\sigma))^{k-1}d\left(\varkappa_{1,p}(\overrightarrow{01}\overset{\bar\gamma}{\dashrightarrow}\xi_m^{-a})(\sigma)\right)(v)\\
&= \frac{1}{(k-1)!}\int_{a\chi_p(\sigma)+m\mathbb{Z}_p} b^{k-1}d\left([m,a\chi_p(\sigma)]_*\varkappa_{1,p}(\xi_m^{-a})_{\bar\gamma}(\sigma)\right)(b)\\
&= \frac{1}{(k-1)!}\int_{\mathbb{Z}_p} b^{k-1}d\left([m,a\chi_p(\sigma)]_*\varkappa_{1,p}(\xi_m^{-a})_{\bar\gamma}(\sigma)\right)(b),
\end{aligned}$$

where the last equality follows as the measure $[m,a\chi_p(\sigma)]_*\varkappa_{1,p}(\xi_m^{-a})_{\bar\gamma}$ is supported on $a\chi_p(\sigma)+m\mathbb{Z}_p \subset \mathbb{Z}_p$. In the same way, we get that

$$\begin{aligned}
(5.8)\qquad m^{k-1}\mathrm{Li}_k(\xi_m^{a})_{\gamma x^{-\alpha}}(\sigma) &= \frac{m^{k-1}}{(k-1)!}\int_{\mathbb{Z}_p}(v-\alpha\chi_p(\sigma))^{k-1}d\varkappa_{1,p}(\overrightarrow{01}\overset{\gamma}{\dashrightarrow}\xi_m^{a})(\sigma)(v)\\
&= -(-1)^{k-1}\frac{m^{k-1}}{(k-1)!}\int_{\mathbb{Z}_p}(v+\alpha\chi_p(\sigma))^{k-1}d\left(\iota\cdot\varkappa_{1,p}(\overrightarrow{01}\dashrightarrow\xi_m^{a})_\gamma(\sigma)\right)(v)\\
&= \frac{(-1)^k}{(k-1)!}\int_{\mathbb{Z}_p} b^{k-1}d\Big([m,a\chi_p(\sigma)]_*\left(\iota\cdot\varkappa_{1,p}(\xi_m^{a})_\gamma(\sigma)\right)\Big)(b).
\end{aligned}$$

Now, we enter the situation of Theorem 1.1 and §3, that is, $a, m \in \mathbb{Z}$ ($m > 1$) are integers with $m \nmid a$, and set $\gamma := \Gamma_{a/m}$, $\alpha := a/m$.

Corollary 5.5. *For the adelic Hurwitz measure $\hat\zeta_{a,m} = [m,a\chi_p(\sigma)]_*\boldsymbol{\zeta}_{a/m} \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$, the p-adic image $\hat\zeta_{p,a.m}(\sigma) \in \mathbb{Z}_p[[\mathbb{Z}_p]]$ satisfies*

$$\int_{\mathbb{Z}_p} b^{k-1}d\hat\zeta_{p,a,m}(\sigma)(b) = \frac{m^{k-1}}{k}B_k\Big(\frac{a}{m}\Big)(1-\chi_p(\sigma)^k)\quad (\sigma\in G_{\mathbb{Q}(\mu_m)},\ k\geq 2).$$

Proof. Combining the above calculations (5.7) and (5.8), we obtain from Theorem 5.1:

$$\begin{aligned}
&\frac{m^{k-1}}{k}B_k(\alpha)(1-\chi_p(\sigma)^k) = (k-1)!\,m^{k-1}\left(\mathrm{Li}_k(\xi^{-a})_{\bar\gamma x^\alpha}(\sigma)+(-1)^k\mathrm{Li}_k(\xi^{a})_{\bar\gamma x^{-\alpha}}(\sigma)\right)\\
&= \int_{\mathbb{Z}_p} b^{k-1}d[m,a\chi_p(\sigma)]_*\left(\varkappa_{1,p}(\overrightarrow{01}\overset{\bar\gamma}{\dashrightarrow}\xi_m^{-a})(\sigma)+\iota\cdot\varkappa_{1,p}(\overrightarrow{01}\overset{\gamma}{\dashrightarrow}\xi_m^{a})(\sigma)\right)(b)\\
&= \int_{\mathbb{Z}_p} b^{k-1}d[m,a\chi_p(\sigma)]_*\boldsymbol{\zeta}_{p,a/m}(\sigma)(b),
\end{aligned}$$

where $\boldsymbol{\zeta}_{p,a/m}(\sigma)$ is the image of $\boldsymbol{\zeta}_{a/m}(\sigma)$ by the projection $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \to \mathbb{Z}_p[[\mathbb{Z}_p]]$. This concludes the proof of the corollary. $\square$

5.3. **Proof of Theorem 1.1.** Note first that the support of the measure $\hat\zeta_{p,a,m}(\sigma)$ is $a\chi_p(\sigma)+m\mathbb{Z}_p$. When $p \mid m$ and $p \nmid a$, it is included in $\mathbb{Z}_p^\times$ so that the above Corollary proves the case.

Remark 5.6. It is worth noting that we do not need to assume $0 < a < m$ for the construction of the measure $\hat\zeta_{a,m}$ and the integration property in the above case of $p \mid m$. This leads to Remark 1.3 of Introduction.

The case $p \nmid m$ was treated [W3] in the setting where pro-p path γ is taken suitably for a fixed p. In our present case, we are taking γ to be the topological path $\Gamma_{a/m} : \overrightarrow{01} \dashrightarrow \xi_m^a$ (cf. Remark 5.4). We also need the assumption $0 < a < m$ for the following

Lemma 5.7. *Given $m, a \in \mathbb{Z}$, $m > 1$ as in Theorem 1.1, suppose that a prime p does not divide m. Let $a_1, \delta \in \mathbb{Z}$ be integers such that $a = pa_1 + \delta m$ with $1 \leq a, a_1 < m$. Then,*
(i) $(\Gamma_{a/m})_{\langle -\delta \rangle, p*} = \Gamma_{a_1/m}$*;*
(ii) $(\bar{\Gamma}_{a/m})_{\langle \delta \rangle, p*} = \bar{\Gamma}_{a_1/m}$*;*
(iii) $\boldsymbol{\zeta}_{a/m}(\sigma) = [p, -\delta\chi(\sigma)]_* \boldsymbol{\zeta}_{a_1,m}(\sigma)$ *for* $\sigma \in G_{\mathbb{Q}(\mu_m)}$.

Proof. (i) results from a good compatibility of our topological paths $\Gamma_{a/m}$ introduced in §3.1 with the lifting along $V_r \twoheadrightarrow V_1$. Indeed,

$$(\Gamma_{a/m} \cdot x^{-\delta})_p = (\Gamma_{\frac{a}{m} - \delta})_p = \Gamma_{\frac{a}{pm} - \frac{\delta}{p}} = \Gamma_{a_1/m}$$

which derives (i). For (ii), suppose $1 \leq a, a_1 < m$. Then, noting that $\bar{\Gamma}_{a/m}$, $\bar{\Gamma}_{a_1/m}$ are homotopic to the complex conjugates of $\Gamma_{a/m}$, $\Gamma_{a_1/m}$ respectively (cf. Remark 3.2), we have

$$(\bar{\Gamma}_{a/m} \cdot x^{\delta})_p = (\Gamma_{-\frac{a}{m} + \delta})_p = \Gamma_{\frac{-a + m\delta}{mp}} = \Gamma_{-a_1/m} = \bar{\Gamma}_{a_1/m}.$$

This derives (ii). Finally, using (3.11), we see from (i) and (ii):

$$\begin{aligned} \varkappa_1(\Gamma_{a/m})(\sigma) &= [p, \delta\chi(\sigma)]_* \varkappa_1(\Gamma_{a_1/m}), \\ \varkappa_1(\bar{\Gamma}_{a/m})(\sigma) &= [p, -\delta\chi(\sigma)]_* \varkappa_1(\bar{\Gamma}_{a_1/m}), \end{aligned}$$

hence from (3.12) we find

$$\begin{aligned} [p, -\delta]_* \boldsymbol{\zeta}_{a_1,m}(\sigma) &= [p, -\delta]_* \left(\varkappa_1(\bar{\Gamma}_{a_1/m})(\sigma) + \iota \cdot \varkappa_1(\Gamma_{a_1/m})(\sigma) \right) \\ &= [p, -\delta]_* \varkappa_1(\bar{\Gamma}_{a_1/m})(\sigma) + \iota \cdot [p, \delta]_* \varkappa_1(\Gamma_{a_1/m})(\sigma) \\ &= \varkappa_1(\bar{\Gamma}_{a/m})(\sigma) + \iota \cdot \varkappa_1(\Gamma_{a/m})(\sigma) \\ &= \boldsymbol{\zeta}_{a/m}(\sigma). \end{aligned}$$

This settles the proof of (iii). □

Now, we compute the target integral of Theorem 1.1 in the case $p \nmid m$:

$$\begin{aligned} &\int_{\mathbb{Z}_p^\times} b^{k-1} d[m, a\chi_p(\sigma)]_* \boldsymbol{\zeta}_{p,a/m}(\sigma)(b) \\ &= \int_{\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)]_* \boldsymbol{\zeta}_{p,a/m}(\sigma)(b) - \int_{p\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)]_* \boldsymbol{\zeta}_{p,a/m}(\sigma)(b), \end{aligned}$$

where the first term is calculated as:

$$\int_{\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)]_* \boldsymbol{\zeta}_{p,a/m}(\sigma)(b) = \frac{m^{k-1}}{k} B_k \left(\frac{a}{m} \right) (1 - \chi_p(\sigma)^k) \tag{5.9}$$

by Corollary 5.5. For the second term, we observe:

$$\int_{p\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)]_* \boldsymbol{\zeta}_{p,a/m}(\sigma)(b) = \int_S (mv + a\chi_p(\sigma))^{k-1} d\boldsymbol{\zeta}_{p,a/m}(\sigma)(v)$$

with $S := \{ v \in \mathbb{Z}_p \mid mv + a\chi_p(\sigma) \in p\mathbb{Z}_p \}$. Since $p \nmid m$, we can choose integers $a_1, \delta \in \mathbb{Z}$ such that $a = a_1 p + m\delta$. We set $a_1 = \langle ap^{-1} \rangle$ to be the least positive one as introduced in Theorem 1.1. In this set up, the condition $mv + a\chi_p(\sigma) = m(v + \delta\chi_p(\sigma)) + pa_1\chi_p(\sigma) \in p\mathbb{Z}_p$

is equivalent to the condition $v + \delta\chi_p(\sigma) \in p\mathbb{Z}_p$; hence the space S is a coset form: $S = -\delta\chi_p(\sigma) + p\mathbb{Z}_p$. If $v \in S$ is written as $v = -\delta\chi_p(\sigma) + p\beta$ ($\beta \in \mathbb{Z}_p$), then $mv + a\chi_p(\sigma) = p(m\beta + a_1)$. Noting that Lemma 5.7 (iii) implies $\boldsymbol{\zeta}_{p,a/m}(\sigma) = [p, -\delta\chi(\sigma)]_* \boldsymbol{\zeta}_{p,a_1,m}(\sigma)$ for the p-adic images of measures by $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \to \mathbb{Z}_p[[\mathbb{Z}_p]]$, we obtain

$$\begin{aligned}
\int_S (mv + a\chi_p(\sigma))^{k-1} d\boldsymbol{\zeta}_{p,a/m}(\sigma)(v) &= p^{k-1} \int_{\mathbb{Z}_p} (m\beta + a_1\chi_p(\sigma))^{k-1} d\boldsymbol{\zeta}_{p,a_1,m}(\sigma)(\beta) \\
&= p^{k-1} \int_{\mathbb{Z}_p} b^{k-1} d[m, a_1\chi_p(\sigma)]_* \boldsymbol{\zeta}_{p,a_1,m}(\sigma)(b) \\
&= p^{k-1} \int_{\mathbb{Z}_p} b^{k-1} d\hat{\zeta}_{p,a_1,m}(\sigma)(b) \\
&= \frac{(pm)^{k-1}}{k} B_k\left(\frac{a_1}{m}\right)(1 - \chi_p(\sigma)^k),
\end{aligned}$$

where the last identity follows from Corollary 5.5. This, combined with (5.9), settles the remained case of Theorem 1.1. □

Appendix A. Cohen's p-adic Hurwitz zeta function

In this appendix, we shall relate the p-adic Hurwtiz zeta function $\zeta_p(s, x)$ introduced in H. Cohen's book [Co] to the p-adic Hurwtiz zeta function of Shiratani type ([Sh]) discussed in our main text. Let p be a prime, and let $q = p$ or $q = 4$ according as $p > 2, p = 2$ respectively. Set $C\mathbb{Z}_p := \mathbb{Q}_p \setminus \frac{p}{q}\mathbb{Z}_p$. Cohen's $\zeta_p(s, x)$ is defined first in [Co, §11.2.2] for $x \in C\mathbb{Z}_p$ and is then defined also for $x \in \mathbb{Z}_p$ in [Co, §11.2.4]. Our main goal is to give connection with $\zeta_p(s, \frac{a}{m})$ in the formulas (A.4) and (A.9).

The case $\zeta_p(s, x)$ for $x \in C\mathbb{Z}_p$:

In [Wa, Theorem 5.9], a p-adic meromorphic function $H_p(s, a, m)$ in s is introduced for a pair of integers a, m such that $0 < a < m$, $q \mid m$ and $p \nmid a$. It satisfies

$$\text{(A.1)} \qquad H_p(1 - k, a, m) = -\omega(a)^{-k} \frac{m^{k-1}}{k} B_k(\frac{a}{m}) \quad (k \in \mathbb{N}),$$

where $\omega : \mathbb{Z}_p^\times \to \mu_e$ ($e := |(\mathbb{Z}/q\mathbb{Z})^\times|$) is the p-adic Teichmüller character. Cohen extends ω to $\omega_v : \mathbb{Q}_p^\times \to p^\mathbb{Z} \cdot \mu_e$ by $\omega_v(up^n) = p^n\omega(u)$ ($u \in \mathbb{Z}_p^\times$, $n \in \mathbb{Z}$). Then, the interpolation property of $\zeta_p(s, x)$ for $x \in C\mathbb{Z}_p$ given in [Co, Theorem 11.2.9] reads

$$\zeta_p(1 - k, x) = -\omega_v(x)^{-k} \frac{B_k(x)}{k} \quad (k \in \mathbb{N}).$$

This specializes for $x = a/m \in \mathbb{Q} \cap C\mathbb{Z}_p$ ($p \nmid a$, $q \mid m$ without assuming $0 < a < m$) to

$$\text{(A.2)} \qquad \zeta_p(1 - k, \frac{a}{m}) = -\omega_v(m)^k \cdot \omega(a)^{-k} \frac{1}{k} B_k(\frac{a}{m}).$$

Restricting k to those positive integers in a same class in $\mathbb{Z}/e\mathbb{Z}$, we obtain a relation between special values of $\zeta_p(s, a/m)$ and $\{L_p^{[\beta]}(s; a, m)\}_{\beta \in \mathbb{Z}/e\mathbb{Z}}$ of Remark 1.4: For $k \equiv$

$\beta(\mathrm{mod}\ e)$, $k \geq 1$, writing $m = p^{v_p(m)} m_1$, we have

$$\zeta_p(1-k, \frac{a}{m}) = -\left(\frac{\omega_v(m)}{\omega(a)}\right)^k m^{1-k} L_p^{[\beta]}(1-k; a, m) \tag{A.3}$$
$$= -\left(\frac{\omega_v(m)}{\omega(a)^\beta}\right)\left(\frac{\omega_v(m)}{m}\right)^{k-1} L_p^{[\beta]}(1-k; a, m)$$
$$= -\left(\frac{\omega_v(m)}{\omega(a)^\beta}\right)\left(\frac{\omega(m_1)^{\beta-1}}{m_1^{k-1}}\right) L_p^{[\beta]}(1-k; a, m).$$

Hence, under the assumption $q \mid m$ and $p \nmid a$, for any $\beta \in \mathbb{Z}/e\mathbb{Z}$, it follows that

$$\zeta_p(s, \frac{a}{m}) = -\left(\frac{\omega_v(m)}{\omega(a)^\beta}\right)\left(\frac{\omega(m_1)^{\beta-1}}{m_1^{-s}}\right) L_p^{[\beta]}(s; a, m) \tag{A.4}$$

for s in the space $\beta + \frac{q}{p}\mathbb{Z}_p$ which is one of the forms $\mathbb{Z}_p$ $(p > 2)$, $2\mathbb{Z}_2$ or $1 + 2\mathbb{Z}_2$. Due to Remark 1.3, this formula holds true for all $a \in \mathbb{Z}$ with $m \nmid a$ and $p \nmid a$.

The case $\zeta_p(s, x)$ for $x \in \mathbb{Z}_p$:

In this case, let us first observe the following identity:

$$\zeta_p(1-k, x) = -\frac{1}{k} B_k(\tilde{\omega}^{-k}, x) \qquad (k \in \mathbb{Z}_{\geq 1}) \tag{A.5}$$

where $\tilde{\omega}$ is the Teichmüller character on $\mathbb{Z}_p$ (extended by 0 on $p\mathbb{Z}_p$) and $B_k(\chi, *)$ is the χ-Bernoulli polynomial defined in [Co, §9.4.1] .

Proof. Write $x = p^u \alpha \in \mathbb{Z}_p$ with $p \nmid \alpha$ and set $N = p^v = p^{u+1}$. We use [Co, Corollary 11.2.15] and the notations there. As $\omega_v(N) = p^v$, $\langle N \rangle = 1$, we have for $s = 1-k$:

$$p^v \cdot \zeta_p(1-k, x) = \sum_{\substack{0 \leq j < p^v \\ p \nmid j}} \zeta_p(1-k, \frac{x}{p^v} + \frac{j}{p^v}).$$

In RHS here, it follows from [Co, Theorem 11.2.9] that

$$\zeta_p(1-k, \frac{x}{p^v} + \frac{j}{p^v}) = -(p^{-v}\omega(j))^{-k} \frac{1}{k} B_k(\frac{x}{p^v} + \frac{j}{p^v}).$$

Hence

$$p^v \cdot \zeta_p(1-k, x) = -\frac{p^{kv}}{k} \cdot \sum_{j=0}^{p^v} \tilde{\omega}^{-k}(j) \cdot B_k(\frac{x}{p^v} + \frac{j}{p^v})$$
$$= -\frac{p^{kv}}{k} \cdot p^{v(1-k)} B_k(\tilde{\omega}^{-k}, x) \quad \text{(by [Co, Lemma 9.4.7]).}$$

This proves (A.5). □

Let $e = |(\mathbb{Z}/q\mathbb{Z})^\times|$. Then,

$$\zeta_p(1-k, x) = -\frac{1}{k}\left(B_k(x) - p^{k-1} B_k(\frac{x}{p})\right) \tag{A.6}$$

for $k \in \mathbb{Z}_{\geq 1}$ and $k \equiv 0 \bmod e$.

Proof. When $k \equiv 0 \bmod e$, we have $\tilde{\omega}^k(j) = 0, 1$ according as $p \mid j$, $p \nmid j$. Putting this into the basic identity of χ-Bernoulli polynomial ([Co, Proposition 9.4.5]), we find

$$B_k(\tilde{\omega}^{-k}, x) = p^{k-1}\sum_{j=0}^{p-1}\tilde{\omega}(j)^{-k}B_k(\frac{x+j}{p}) = p^{k-1}\sum_{j=1}^{p-1}B_k(\frac{x+j}{p}).$$

On the other hand, from the usual distribution formula of the Bernoulli polynomial: it follows that

$$B_k(x) = p^{k-1}\sum_{j=0}^{p-1}B_k(\frac{x}{p}+\frac{j}{p}).$$

By comparing these two identities, we obtain (A.6). □

Suppose $x = \frac{a}{m} \in \mathbb{Q}\cap\mathbb{Z}_p$ with $p \nmid m$. Observe that the above interpolation property of Cohen's $\zeta_p(s,x)$ (A.6) reads then

$$\text{(A.7)} \qquad \zeta_p(1-k, \frac{a}{m}) = -\frac{1}{k}\left(B_k(\frac{a}{m}) - p^{k-1}B_k(\frac{a}{mp})\right) \qquad (k \geq 1,\ k \equiv 0 (\mathrm{mod}\ e)).$$

It is not straightforward to find a connection from this to the interpolation property of Shiratani's $\zeta_p^{Sh}(s,a,m) = -L_p^{[0]}(s;a,m)$ (cf. Remark 1.4):

$$\text{(A.8)} \qquad \zeta_p^{Sh}(1-k;a,m) = -\frac{m^{k-1}}{k}\left(B_k(\frac{a}{m}) - p^{k-1}B_k(\frac{\langle ap^{-1}\rangle}{m})\right)$$

for all $k \in \mathbb{Z}_{>0}$, $k \equiv 0 \bmod e$, where a and $\langle ap^{-1}\rangle$ are supposed to be positive integer $< m$ such that $\langle ap^{-1}\rangle p \equiv a \mod m$. To connect (A.7) and (A.8), let r be the *unique* integer with $a + mr = \langle ap^{-1}\rangle p$ so that $\frac{a+mr}{mp} = \frac{\langle ap^{-1}\rangle}{m}$. Note that $r > 0$, due to the assumption $0 < a, \langle ap^{-1}\rangle < m$. Then, replacing a by $a + mr$ in (A.7), we find

$$\begin{aligned}\zeta_p(1-k, \frac{a+mr}{m}) - m^{1-k}\zeta_p^{Sh}(1-k;a,m) &= -\frac{1}{k}B_k(r+\frac{a}{m}) + \frac{1}{k}B_k(\frac{a}{m}) \\ &= -\sum_{v=0}^{r-1}(\frac{a+mv}{m})^{k-1}\end{aligned}$$

for all $k > 0$, $k \equiv 0 \bmod e$. We claim below that $p \nmid (a+mv)$ for all $v \in [0, r-1]$ so that the existence of p-adic analytic functions $(\frac{a}{m}+v)^s$ provides a connection between Cohen's $\zeta_p(s,x)$ with $x \in \mathbb{Z}_p$ and Shiratani's $\zeta_p^{Sh}(s;a,m) = -L_p^{[0]}(s;a,m)$, namely, it holds that

$$\text{(A.9)} \qquad \zeta_p(s, \frac{a+mr}{m}) = m^s \cdot \zeta_p^{Sh}(s;a,m) - \sum_{v=0}^{r-1}(\frac{a+mv}{m})^{-s}$$

for $s \in \frac{q}{p}\mathbb{Z}_p$, under the assumptions $p \nmid m$, $0 < a < m$, $p \mid (a+mr)$ and $0 < a+mr < pm$. The assertion (A.9) is thus reduced to the following elementary

Claim. Notations being as above, let r_0 be the least nonnegative integer such that $a + mr_0 \equiv 0 \mod p$. Then, $a + mr_0 = \langle ap^{-1}\rangle p$.

Proof. If $r_0 \geq p$, then $a + m(r_0 - p) \equiv a + mr_0 \equiv 0 \mod p$ which contradicts the minimality of r_0. Therefore $r_0 < p$. If $r_0 = p-1$, then writing $a + m(p-1) = xp$, we have $p(m-x) = m - a > 0$. Hence $m > x > 0$, i.e., $x = \langle ap^{-1}\rangle$. Assume that $r_0 < p-1$. If $mp \leq a + mr_0$, then since $a < m$, it follows that $0 \leq a + m(r_0 - p) < m(r_0 - p + 1)$ hence that $p-1 \le r_0$ contradicting the assumption. Thus $mp > a+mr_0$. Writing $a+mr_0 = xp$, we obtain $m > x$, that is, $x = \langle ap^{-1}\rangle$. $\square$

Example. Let $p = 11$, $a = 3$ and $m = 106$. Noting $106 \equiv 7 \mod 11$ and $3+7\cdot 9 = 6\cdot 11$, one finds $3+106\cdot 9 = 957 = 87\cdot 11$. Hence $\langle 3\cdot 11^{-1}\rangle = 87$. Now, the core sum in the above construction reads $\sum_v (\frac{a}{m}+v)^{k-1} = (\frac{3}{106})^{k-1} + (\frac{109}{106})^{k-1} + (\frac{215}{106})^{k-1} + (\frac{321}{106})^{k-1} + (\frac{427}{106})^{k-1} + (\frac{533}{106})^{k-1} + (\frac{639}{106})^{k-1} + (\frac{745}{106})^{k-1} + (\frac{851}{106})^{k-1}$. There do exist 11-adic analytic functions that interpolate $(\frac{3+106v}{106})^{k-1}$ at $s = 1-k$ ($k \equiv 0 \mod 10$) for $v = 0, 1, \ldots, 8$ respectively.

Question. It is unclear if $L_p^{[\beta]}(s; a, m)$ for $p \nmid m$, $\beta \not\equiv 0$ (e) can be expressed in terms of Cohen's $\zeta_p(s, x)$.

Appendix B. Path conventions

In this Appendix, we quickly summarize two conventions on étale paths mostly used in our papers. Just for simplicity, we call one system of conventions the traditional form ('t-form') and another system the electronic form ('e-form'). The present paper and most papers by the second author obey the t-form, whereas most papers by the first author and our previous common papers [NW1-3] obey the e-form. The purpose of this Appendix is to serve a dictionary to translate formulas between these two forms.

Let $\mathcal{C}$ be a Galois category, for example, that of the finite étale covers of an algebraic variety. We write $a, b, c, \ldots$ for general symbols playing roles of base points for $\pi_1(\mathcal{C})$ and $\omega_a, \omega_b, \omega_c, \ldots$ for the corresponding Galois functors $\mathcal{C} \to \mathit{Sets}$. The path space between two points a and b is by definition the set $\mathrm{Isom}(\omega_a, \omega_b)$ whose element is a compatible family of isomorphisms of fibre sets $\gamma_U : \omega_a(U) \xrightarrow{\sim} \omega_b(U)$ over $U \in \mathrm{Ob}(\mathcal{C})$. In t-form, an element γ of $\mathrm{Isom}(\omega_a, \omega_b)$ is called a (t-)path from a to b and written as $\gamma : a \dashrightarrow b$. In e-form, the same $\gamma \in \mathrm{Isom}(\omega_a, \omega_b)$ is called a(n e-)path <u>from b to a</u> and written as $\gamma : b \rightsquigarrow a$. Remind that, for each $U \in \mathrm{Ob}(\mathcal{C})$, $\gamma_U(s)$ is defined for elements $s \in \omega_a(U)$. [In e-form, we may imagine that the waving arrow $\gamma : b \rightsquigarrow a$ flows like an electronic current that conveys electron $s \in \omega_a(U)$ back into $\omega_b(U)$.] We shall use the notation

$$\pi_1(\mathcal{C}; b, a) := \mathrm{Isom}(\omega_a, \omega_b) \tag{B.1}$$

to designate the set of t-paths from a to b as well as the set of e-paths from b to a. Accordingly, if $\gamma_1 \in \mathrm{Isom}(\omega_a, \omega_b)$ and $\gamma_2 \in \mathrm{Isom}(\omega_b, \omega_c)$, then the composite $\gamma_2\gamma_1 \in \mathrm{Isom}(\omega_a, \omega_c)$ is defined. We have

$$[a \overset{\gamma_2\gamma_1}{\dashrightarrow} c] = [b \overset{\gamma_2}{\dashrightarrow} c] \cdot [a \overset{\gamma_1}{\dashrightarrow} b] \ \left(\text{viz. } [c \overset{\gamma_2\gamma_1}{\dashleftarrow} a] = [c \overset{\gamma_2}{\dashleftarrow} b] \cdot [b \overset{\gamma_1}{\dashleftarrow} a]\right), \tag{B.2}$$

$$[c \overset{\gamma_2\gamma_1}{\rightsquigarrow} a] = [c \overset{\gamma_2}{\rightsquigarrow} b] \cdot [b \overset{\gamma_1}{\rightsquigarrow} a]. \tag{B.3}$$

Next, let F be a subfield of $\mathbb{C}$ and let $\mathcal{C}$ be the Galois category of finite étale covers of an algebraic variety V over F. If a is an F-rational (tangential) points on V, then the sequence of finite sets $\{\omega_a(U)\}_{U \in \mathrm{Ob}(\mathcal{C})}$ have compatible actions by G_F, which defines the map $G_F \to \mathrm{Isom}(\omega_a, \omega_a) = \pi_1(V, a)$. For two such points a, b, we define the canonical left G_F-action on $\mathrm{Isom}(\omega_a, \omega_b)$ by $\gamma \mapsto \sigma\gamma\sigma^{-1}$ ($\sigma \in G_F$). Observe that, concerning Galois actions, no difference occurs between t-form and e-form.

Suppose now that $V = \mathbf{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$. Denote by x, y the standard loops based at $\overrightarrow{01}$ running around the punctures 0, 1 respectively with anticlockwise t-arrows $\dashrightarrow$, and let $\mathtt{x}$, $\mathtt{y}$ be those loops with anticlockwise e-arrows $\rightsquigarrow$. Then, $x = \mathtt{x}^{-1}$, $y = \mathtt{y}^{-1}$. Let z be a

F-rational (tangential) point on V. For a t-path $\gamma : \overrightarrow{01} \dashrightarrow z$ on $V \otimes F$, we define a Galois associator in t-form by

$$f_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1^{\text{ét}}(V, \overrightarrow{01}) \qquad (\sigma \in G_F) \tag{B.4}$$

as in §2 of the present paper, whereas, for an e-path $\delta : \overrightarrow{01} \rightsquigarrow z$ on $V \otimes F$, we define another Galois associator in e-form by

$$\mathfrak{f}_\sigma^\delta := \delta \cdot \sigma(\delta)^{-1} \in \pi_1^{\text{ét}}(V, \overrightarrow{01}) \qquad (\sigma \in G_F) \tag{B.5}$$

as in [NW1]. Therefore, assuming $\delta = \gamma^{-1}$, we find

$$f_\gamma(\sigma) = \mathfrak{f}_\sigma^\delta \qquad (\sigma \in G_F). \tag{B.6}$$

Given a prime ℓ, let $\pi_{\mathbb{Q}_\ell}$ be the pro-unipotent completion of the maximal pro-ℓ quotient of $\pi_1(V_{\overline{\mathbb{Q}}}, \overrightarrow{01})$. Consider the above $x, y, \mathtt{x}, \mathtt{y}$ as $\mathbb{Q}_\ell$-loops based at $\overrightarrow{01}$ on V, and regard $\gamma : \overrightarrow{01} \dashrightarrow z$ and $\delta : \overrightarrow{01} \rightsquigarrow z$ as $\mathbb{Q}_\ell$-paths on V. If $\delta = \gamma^{-1}$, then $(\gamma x^\alpha)^{-1} = \mathtt{x}^\alpha \delta$; hence it follows from (B.6) that

$$f_{\gamma x^\alpha}(\sigma) = \mathfrak{f}_\sigma^{\mathtt{x}^\alpha \delta} \qquad (\delta = \gamma^{-1},\ \sigma \in G_F,\ \alpha \in \mathbb{Q}_\ell). \tag{B.7}$$

Now, let us compare ℓ-adic Galois polylogarithms in t-form and e-form. Define generators X, Y (resp. $\bar{X}, \bar{Y}$) of $\mathrm{Lie}(\pi_{\mathbb{Q}_\ell})$ so that e^X, e^Y (resp. $e^{\bar{X}}, e^{\bar{Y}}$) are the ℓ-adic images of x, y (resp. $\mathtt{x}, \mathtt{y}$) in $\pi_{\mathbb{Q}_\ell}$. Then $X = -\bar{X}$, $Y = -\bar{Y}$. Let I_Y (resp. $I_{\bar{Y}}$) denote the ideal of $\mathrm{Lie}(\pi_{\mathbb{Q}_\ell})$ generated by those Lie words in X, Y (resp. $\bar{X}, \bar{Y}$) containing Y (resp. $\bar{Y}$) twice or more. Obviously we have $I_Y = I_{\bar{Y}} \subset \mathrm{Lie}(\pi_{\mathbb{Q}_\ell})$. In e-form, we have the Lie expansion

$$\log(\mathfrak{f}_\sigma^\delta)^{-1} \equiv \rho_{z,\delta}(\sigma)\bar{X} + \sum_{k=1}^{\infty} \ell i_k(z)_\delta(\sigma)(\mathrm{ad}\,\bar{X})^{k-1}(\bar{Y}) \mod I_{\bar{Y}} \tag{B.8}$$

extending [NW2, Definition 5.4] to any $\mathbb{Q}_\ell$-paths $\delta : \overrightarrow{01} \rightsquigarrow z$. Note that interpretation of $\rho_{z,\delta}$ as a Kummer 1-cocycle along power roots of z is basically available only when γ is a pro-ℓ path. On the side of t-form, one also has

$$\log(f_\gamma(\sigma)) \equiv \rho_{z,\gamma}(\sigma)X + \sum_{k=1}^{\infty} \ell i_k(z)_\gamma(\sigma)[..[Y, \underbrace{X], \ldots, X]}_{k-1} \mod I_Y \tag{B.9}$$

extending [W1, Definition 11.0.1] for any $\mathbb{Q}_\ell$-path $\gamma : \overrightarrow{01} \dashrightarrow z$. Comparing (B.8) and (B.9) under the situation (B.6), we see that the ρ_z and the ℓ-adic polylogarithms $\ell i_m(z)$ (written also as $\ell_m(z)$ in older papers) for $\gamma : \overrightarrow{01} \dashrightarrow z$ in t-form and for $\delta : \overrightarrow{01} \rightsquigarrow z$ in e-form coincide with each other as functions on G_F as long as $\delta = \gamma^{-1}$, that is,

$$\rho_{z,\gamma}(\sigma) = \rho_{z,\delta}(\sigma),\ \ell i_k(z)_\gamma(\sigma) = \ell i_k(z)_\delta(\sigma) \qquad (\sigma \in G_F,\ k \geq 1,\ \delta = \gamma^{-1}). \tag{B.10}$$

Next, embed $\mathrm{Lie}(\pi_{\mathbb{Q}_\ell})$ into the ring of non-commutative power series $\mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle = \mathbb{Q}_\ell\langle\langle \bar{X}, \bar{Y}\rangle\rangle$ and expand $f_\gamma(\sigma) = \mathfrak{f}_\sigma^\delta$ into series in X, Y or in $\bar{X}, \bar{Y}$. The coefficient at YX^{k-1} appearing in the former expansion is the ℓ-adic polylogarithm $\mathrm{Li}_k(z)_\gamma(\sigma)$ in §4.1 of this paper in t-form (with $\ell = p$), while the coefficient at $\bar{Y}\bar{X}^{k-1}$ in the latter expansion, which we denote by $\mathscr{L}i_k(z)_\delta(\sigma)$ in e-form, was discussed in [NW3, §6]. By definition, we have

$$\mathrm{Li}_k(z)_{\overrightarrow{01} \overset{\gamma}{\dashrightarrow} z}(\sigma) = (-1)^k \mathscr{L}i_k(z)_{\overrightarrow{01} \overset{\delta}{\rightsquigarrow} z}(\sigma) \qquad (\sigma \in G_F,\ k \geq 1,\ \delta = \gamma^{-1}). \tag{B.11}$$

Finally, we recall from [NW3, §6]. the function

$$\tilde{\boldsymbol{\chi}}_k^{z,\delta} : G_F \to \mathbb{Q}_\ell$$

associated to any $\mathbb{Q}_\ell$-path $\delta : \overrightarrow{01} \rightsquigarrow z$ for $k \geq 1$ by the equation:

$$\tilde{\boldsymbol{\chi}}_k^{z,\delta}(\sigma) = (-1)^{k+1}(k-1)! \sum_{i=1}^{k} \frac{\rho_{z,\delta}(\sigma)^{k-i}}{(k+1-i)!} \ell i_i(z)_\delta(\sigma). \tag{B.12}$$

It is related to the above $\mathscr{L}i_k(z)_{\overrightarrow{01}\overset{\delta}{\rightsquigarrow} z}(\sigma)$ by

$$-\frac{\tilde{\boldsymbol{\chi}}_k^{z,\delta}(\sigma)}{(k-1)!} = \mathscr{L}i_k(z)_{\overrightarrow{01}\overset{\delta}{\rightsquigarrow} z}(\sigma) \quad (\sigma \in G_F,\, k \geq 1). \tag{B.13}$$

When $\delta : \overrightarrow{01} \rightsquigarrow z$ is a pro-ℓ path, then $\tilde{\boldsymbol{\chi}}_k^{z,\delta}$ is the polylogarithmic character studied in [NW1] and is known to be valued in $\mathbb{Z}_\ell$ with explicit Kummer properties along a sequence of numbers.

For a path $\gamma : \overrightarrow{01} \dashrightarrow z$ in t-form, we employ the notation

$$\tilde{\boldsymbol{\chi}}_k(z)_\gamma(\sigma) := \tilde{\boldsymbol{\chi}}_k^{z,\delta}(\sigma) \qquad (\sigma \in G_F) \tag{B.14}$$

where $\delta = \gamma^{-1} : \overrightarrow{01} \rightsquigarrow z$ is the corresponding path in e-form. It follows then that

$$\frac{\tilde{\boldsymbol{\chi}}_k(z)_\gamma(\sigma)}{(k-1)!} = (-1)^{k-1} \mathrm{Li}_k(z)_{\overrightarrow{01}\overset{\gamma}{\dashrightarrow} z}(\sigma) \qquad (\sigma \in G_F) \tag{B.15}$$

for any $\mathbb{Q}_\ell$-path $\gamma : \overrightarrow{01} \dashrightarrow z$.

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