# On a two-parameter family of tropical Edwards curves

HIROAKI NAKAMURA AND RANI SASMITA TARMIDI

ABSTRACT. In this paper, a certain two-parameter family of plane-embeddings of Edwards elliptic curve $E_a : x^2 + y^2 = a^2(1 + x^2y^2)$ is introduced to provide explicitly computed tropical curves corresponding to degeneration in $a \to 1$. Applying the theta uniformization of $E_a$ with the method of ultradiscretization by Kajiwara-Kaneko-Nobe-Tsuda, we give a formula for the coordinate functions that traces the cycle part of the tropical elliptic curve. We also illustrate how one can recover the whole part of the tropical curve as a quotient of the Bruhat-Tits tree after Speyer's algebraic approach in smooth cases.

CONTENTS



## 1. Introduction

Tropical elliptic curves on the plane have called attentions of many authors from various viewpoints. They generally have a unique cycle whose length is the negative of the tropicalization of the $j$-invariant (cf. e.g., [V09], [KMM]). In [CS], Chan and Sturmfels studied symmetric cubics in two variables having honeycomb form tropicalizations, whereas Nobe ([N08]) closely observed a one-parameter family of tropical elliptic curves with cycles ranging over various polygons. In particular, Kajiwara-Kaneko-Nobe-Tsuda [KKNT] found a beautiful bridge from the theta functions of level 3 to the Hessian elliptic curves $E_\mu : x^3 + y^3 + 1 = 3\mu xy$ which enables one to uniformize the cycle part of the corresponding tropical curve explicitly by what are called the ultradiscrete theta functions. The purpose of this paper is to provide a simple variant of [KKNT]

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in the case of Edwards curves $E_a : x^2+y^2 = a^2(1+x^2y^2)$ where only classical Jacobian (viz. level 2) theta functions are enough to play the role for uniformization. Our treatment mostly follows the lines of arguments in [KKNT], while, since the direct tropicalization of $E_a$ is never faithful on the cycle, we introduce a variation of the plane-embedding of $E_a$ with certain two parameters. Our family then turns out to contain fairly rich plane elliptic curves (isomorphic to $E_a$) whose tropical cycles range over $n$-gons ($n = 4, 5, 7$) with explicit uniformization by ultradiscrete theta functions. We now illustrate our main results. Let

$$\epsilon = \epsilon(q) = \prod_{n=1}^{\infty}(1+q^n) \ \left(= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}} = 1+q+q^2+2q^3+\cdots\right)$$

be the Euler generating function counting the number of partitions of $n$ with distinct parts (which is the same as the number of partitions of $n$ with odd parts; see [A84, (1.2.5)]), and set

$$\bar{\epsilon} = \bar{\epsilon}(q) := \epsilon(-q) \ \left(= \prod_{n=1}^{\infty}\frac{1}{1+q^{2n-1}} = 1-q+q^2-2q^3+\cdots\right).$$

Let $K$ be a complete discrete valuation field of characteristic 0 with normalized valuation $v_K : K \twoheadrightarrow \mathbb{Z}\cup\{\infty\}$. Pick and fix an element $q \in K$ with $v_K(q) > 0$ and consider $\epsilon$, $\bar{\epsilon} \in K$ to be the convergent limits of the above generating functions respectively. Let $\mathbb{K}$ be the algebraic closure of $K$ which has a unique valuation $v_{\mathbb{K}} : \mathbb{K} \twoheadrightarrow \mathbb{Q}\cup\{\infty\}$ extending $\frac{1}{v_K(q)}v_K$ so that $v_{\mathbb{K}}(q) = 1$. For two parameters $r, s \in \mathbb{K}$ with $\epsilon r \neq \bar{\epsilon}s$, let us consider a polynomial

$$f_{r,s}(\mathtt{x},\mathtt{y}) = d_{12}(\mathtt{x}+\mathtt{y}) + d_{34}(\mathtt{x}^2+\mathtt{y}^2) + d_5\mathtt{x}\mathtt{y} + d_{67}(\mathtt{x}^2\mathtt{y}+\mathtt{y}^2\mathtt{x}) + d_8\mathtt{x}^2\mathtt{y}^2 \tag{1.1}$$

in two variables $\mathtt{x}, \mathtt{y}$ with

$$\begin{cases} d_{12} &= 2\epsilon\bar{\epsilon}(\epsilon^4-\bar{\epsilon}^4)(\bar{\epsilon}s-\epsilon r), \\ d_{34} &= (\epsilon^4-\bar{\epsilon}^4)(\bar{\epsilon}^2s^2-\epsilon^2r^2), \\ d_5 &= 8\epsilon\bar{\epsilon}(\epsilon r-\bar{\epsilon}s)(\bar{\epsilon}^3r-\epsilon^3s), \\ d_{67} &= 2(\epsilon r-\bar{\epsilon}s)\{(\bar{\epsilon}^4-\epsilon^4)rs+2\epsilon\bar{\epsilon}(\bar{\epsilon}^2r^2-\epsilon^2s^2)\}, \\ d_8 &= 2(\epsilon^2s^2-\bar{\epsilon}^2r^2)(\bar{\epsilon}^2s^2-\epsilon^2r^2). \end{cases} \tag{1.2}$$

We have then:

**Proposition 1.1.** *The equation $f_{r,s}(\mathtt{x},\mathtt{y}) = 0$ defines an elliptic curve over $\mathbb{K}$ birationally equivalent to the Edwards curve $E_a : x^2+y^2 = a^2(1+x^2y^2)$ with $a^2 = \dfrac{2\epsilon^2\bar{\epsilon}^2}{\epsilon^4+\bar{\epsilon}^4} \in K$. The $j$-invariant is given by:*

$$j(q^8) = \frac{1}{q^8} + 744 + 196884q^8 + \cdots$$

*with $j(q)$ the standard $q$-series for the $j$-invariant.*

By virtue of the known relation between $j$-invariant and tropical cycle length (cf. [V09], [KMM]), the above Proposition 1.1 implies that the tropicalization of the plane curve $f_{r,s}(\mathtt{x},\mathtt{y}) = 0$ has a unique cycle of length 8, if it is tropically smooth. Write $u_{12}, u_{34}, u_5, u_{67}, u_8 \in \mathbb{Q}\cup\{\infty\}$ for the values $v_{\mathbb{K}}(d_{12}), v_{\mathbb{K}}(d_{34}), v_{\mathbb{K}}(d_5), v_{\mathbb{K}}(d_{67}), v_{\mathbb{K}}(d_8)$ respectively. Our primary concern is the tropical curve $C(\mathrm{trop}(f_{r,s}))$ on the $XY$-plane which is by definition the graph obtained as the set of points $(X,Y) \in \mathbb{R}^2$ where the piecewise linear function $\mathrm{trop}(f_{r,s}) : \mathbb{R}^2 \to \mathbb{R}$ with

$$(X,Y) \mapsto \mathrm{trop}(f_{r,s})(X,Y) := \min\left\{\begin{matrix} u_{12}+X, u_{12}+Y, u_{34}+2X, u_{34}+2Y, u_5+X+Y, \\ u_{67}+2X+Y, u_{67}+2Y+X, u_8+2X+2Y \end{matrix}\right\}$$

is not differentiable. Our first result on the family $C(\mathrm{trop}(f_{r,s}))$ concerns an explicit parametrization of its cycle part (viz. the maximal subgraph with no end points) in terms of a pair of 'ultradiscrete' theta functions $\Theta^{\mathrm{odd}}(u)$, $\Theta^{even}(u)$ defined in the same spirit as [KKNT]:

**Theorem 1.2.** *Let $C(\mathrm{trop}(f_{r,s}))$ be the tropicalization of the plane curve $f_{r,s}(\mathtt{x},\mathtt{y})=0$ for $r,s\in\mathbb{K}$ ($\epsilon r\neq\bar{\epsilon}s$). Then, the cycle part of $C(\mathrm{trop}(f_{r,s}))$ has one-parameter expression $(-X(u),-Y(u))_{u\in\mathbb{R}}$ as follows:*

$$\begin{cases} X(u)= & Y(u-\frac{1}{2}),\\ Y(u)= & \max\left(\Theta^{\mathrm{odd}}(u),-1+\Theta^{even}(u)\right)\\ & -\max\left(-v_{\mathbb{K}}(r-s)+\Theta^{even}(u),-v_{\mathbb{K}}(r+s)+\Theta^{\mathrm{odd}}(u)\right), \end{cases}$$

*where* $\Theta^{\mathrm{odd}}(u):=-2(2\lfloor\frac{u}{2}\rfloor+1-u)^2$, $\Theta^{even}(u):=-2(2\lfloor\frac{u+1}{2}\rfloor-u)^2$.

As an immediate application of the above theorem, it follows that the shape of the cycle part of $C(\mathrm{trop}(f_{r,s}))$ relies only on the value of $v_{\mathbb{K}}(r+s)-v_{\mathbb{K}}(r-s)$. More concretely:

**Corollary 1.3.** *Under the same notations and assumptions as in Theorem 1.2, set*

$$\delta(=\delta_{r,s}):=v_{\mathbb{K}}(r+s)-v_{\mathbb{K}}(r-s).$$

*Then, we have the following assertions.*

(i) *The curve $C(\mathrm{trop}(f_{r,s}))$ has a pentagonal cycle of length 8 if and only if $2\leq\delta$.*
(ii) *The curve $C(\mathrm{trop}(f_{r,s}))$ has a heptagonal cycle of length 8 if and only if $1<\delta<2$.*
(iii) *The curve $C(\mathrm{trop}(f_{r,s}))$ has a square cycle of length $4(\delta+1)$ if and only if $-1<\delta\leq 1$. In particular, it has a square cycle of length 8 if and only if $\delta=1$.*
(iv) *If $\delta\leq -1$, then the locus of $(-X(u),-Y(u))_{u\in\mathbb{R}}$ degenerates to a connected union of two segments of length $\min(1,-\delta-1)$.*

Our next result concerns a criterion when $C(\mathrm{trop}(f_{r,s}))$ is tropically smooth. We obtain:

**Proposition 1.4.** *Notations being as in Corollary 1.3, the following assertions hold.*

(i) *If $2\leq\delta$, then $C(\mathrm{trop}(f_{r,s}))$ is never a smooth tropical curve.*
(ii) *If $1<\delta<2$, then $C(\mathrm{trop}(f_{r,s}))$ is always a smooth tropical curve.*
(iii) *If $\delta=1$, then the curve $C(\mathrm{trop}(f_{r,s}))$ can be either a smooth curve or a non-smooth curve according to the choice of $(r,s)$. It is smooth if and only if the principal coefficient of $r+s$ equals $-2$, i.e., $r+s$ is of the form $q^{v_{\mathbb{K}}(r+s)}(-2+\boldsymbol{\kappa})$ for some $\boldsymbol{\kappa}\in\mathbb{K}$ with $v_{\mathbb{K}}(\boldsymbol{\kappa})>0$.*
(iv) *If $\delta<1$, then $C(\mathrm{trop}(f_{r,s}))$ is never a smooth tropical curve.*

The contents of this paper are organized as follows. In Sect.2, we review the basic setup on the family of Edwards elliptic curves $E_a: x^2+y^2=a^2(1+x^2y^2)$ and their parametrization in terms of the classical Jacobian theta functions. We then introduce our above two-parameter family (1.1) of elliptic curves $f_{r,s}(\mathtt{x},\mathtt{y})=0$ as fractional transformations of $E_a$ at a specific localization $a\to 1$ given in Proposition 1.1. In Sect.3, we compute its ultradiscretization by following the method of [KKNT]. This yields the essential part of the formula in Theorem 1.2. Then, in Sect.4, we verify Theorem 1.2 in the general setting of the base field $\mathbb{K}$ and deduce Corollary 1.3. In Sect.5, we turn to investigate the smoothness criterion for $C(\mathrm{trop}(f_{r,s}))$ by looking closely at subdivisions of Newton polytopes, and prove Proposition 1.4. Sect.6 is devoted to illustrating examples of smooth tropical curves $C(\mathrm{trop}(f_{r,s}))$ to be given as the quotient of Speyer's subtree of the Bruhat-Tits tree corresponding to Tate uniformization $\mathbb{K}^\times/\langle q^8\rangle\cong E_{r,s}(\mathbb{K})$.

Throughout this paper, we shall write $\mathtt{i}:=\sqrt{-1}\in\mathbb{C}$.

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## 2. **Algebraic and analytic theory of Edwards curves**

In [E07], H.Edwards introduced the normal form of elliptic curves $E_a : x^2 + y^2 = a^2(1 + x^2y^2)$ and established the basic theory. Every complex elliptic curve is isomorphic to $E_a$ for some $a \in \mathbb{C} - \{0, \pm 1, \pm \mathtt{i}\}$ which has a simple symmetric addition law with regard to the origin $O = (0, a)$. The space $\mathbb{C} - \{0, \pm 1, \pm \mathtt{i}\}$ of the parameter $a$ is in fact the set of complex points of the modular curve $Y(4)$ which can be identified with a dense open subset of the degree-2 Fermat curve $X^2 + Y^2 = 1$ with $(X, Y) = (\frac{2a}{1+a^2}, \frac{1-a^2}{1+a^2})$ (cf. [M06, Chap. I, §10]). The algebraic theory mostly works without big changes over any field of characteristic different from 2.

**Remark 2.1.** The simplicity of addition law of Edwards curves has attracted cryptographic studies, e.g., [BL07]. See also [GS14] for advantages of tropicalization in view of efficiency of computations and of security against various network attacks.

One of the primary features elaborated in [E07] is a complex uniformization of the curve $E_a : x^2 + y^2 = a^2(1 + x^2y^2)$ by the upper half plane $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. We shall rephrase it in terms of standard theta functions:

$$(2.1)\quad \begin{cases} \theta_1(z|\tau) &= -\mathtt{i} \sum_{n\in\mathbb{Z}} (-1)^n q_\tau^{(\frac{1}{2}+n)^2} q_z^{2n+1} \left( = -\mathtt{i}(q_z - q_z^{-1}) \prod_{n=1}^{\infty} (1 - q_\tau^{2n})(1 - q_\tau^{2n} q_z^2)(1 - q_\tau^{2n} q_z^{-2}) \right), \\ \theta_2(z|\tau) &= \sum_{n\in\mathbb{Z}} q_\tau^{(\frac{1}{2}+n)^2} q_z^{2n+1} \left( = q_\tau^{1/4}(q_z + q_z^{-1}) \prod_{n=1}^{\infty} (1 - q_\tau^{2n})(1 + q_\tau^{2n} q_z^2)(1 + q_\tau^{2n} q_z^{-2}) \right), \\ \theta_3(z|\tau) &= \sum_{n\in\mathbb{Z}} q_\tau^{n^2} q_z^{2n} \left( = \prod_{n=1}^{\infty} (1 - q_\tau^{2n})(1 + q_\tau^{2n-1} q_z^2)(1 + q_\tau^{2n-1} q_z^{-2}) \right), \\ \theta_4(z|\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q_\tau^{n^2} q_z^{2n} \left( = \prod_{n=1}^{\infty} (1 - q_\tau^{2n})(1 - q_\tau^{2n-1} q_z^2)(1 - q_\tau^{2n-1} q_z^{-2}) \right), \end{cases}$$

where $q_\tau = \exp(\pi \mathtt{i} \tau)$, $q_z = \exp(\pi \mathtt{i} z)$ for $\tau \in \mathfrak{H}$, $z \in \mathbb{C}$.

**Proposition 2.2** ([E07] Theorem 15.1)**.** *For any fixed $\tau \in \mathfrak{H}$, set*

$$a = a(\tau) := \frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)}.$$

*and let*

$$x(z) := \frac{\theta_1(z|2\tau)}{\theta_4(z|2\tau)}, \qquad y(z) := \frac{\theta_2(z|2\tau)}{\theta_3(z|2\tau)}.$$

*Then we have*

$$x(z)^2 + y(z)^2 = a^2(1 + x(z)^2 y(z)^2)$$

*for all $z \in \mathbb{C}$. The mapping of the complex $z$-plane $\mathbb{C}_z$ to the complex points $(x(z), y(z))$ of $E_a$ gives a uniformization of the elliptic curve : $\mathbb{C}_z \twoheadrightarrow \mathbb{C}_z/(2\mathbb{Z} + 2\tau\mathbb{Z}) \overset{\sim}{\to} E_a(\mathbb{C})$.* □

Our motivating idea is to variate the equation of $E_a$ by fractional substitutions of the form

$$(2.2)\quad x = \frac{r\mathtt{x} + \alpha}{s\mathtt{x} + \beta}, \quad y = \frac{r\mathtt{y} + \alpha}{s\mathtt{y} + \beta}$$

for some constants $r, s, \alpha, \beta$ ($\alpha s \neq \beta r$) to obtain nicer equations in regards of tropicalization. Let

$$f_{r,s}^{\alpha,\beta}(\mathtt{x},\mathtt{y}) = d_0 + d_{12}(\mathtt{x}+\mathtt{y}) + d_{34}(\mathtt{x}^2+\mathtt{y}^2) + d_5\mathtt{x}\mathtt{y} + d_{67}(\mathtt{x}^2\mathtt{y}+\mathtt{y}^2\mathtt{x}) + d_8\mathtt{x}^2\mathtt{y}^2 \tag{2.3}$$

be the numerator of the rational function $x^2+y^2-a^2(1+x^2y^2)$ in $\mathtt{x},\mathtt{y}$ obtained by the variable change from $(x,y)$ to $(\mathtt{x},\mathtt{y})$ after (2.2).

**Lemma 2.3.** *The constant term $d_0$ of $f_{r,s}^{\alpha,\beta}(\mathtt{x},\mathtt{y})$ is given by*

$$d_0 = 2\alpha^2\beta^2 - a^2(\alpha^4+\beta^4).$$

In order to obtain effective tropical curves, following an idea of [KKNT], we shall change the focus of ultradiscretization process from the standard limit $\tau\to\mathtt{i}\infty$ along the imaginary axis to the limit $\tau\to 0$ along the semicircle emanating from $\frac{1}{4}$. We will implement this idea by, roughly speaking, replacing $q$-expansions of relevant analytic functions in $q_\tau$ by those in

$$q := \exp(\pi\mathtt{i}\frac{4\tau-1}{4\tau}). \tag{2.4}$$

**Lemma 2.4.**

$$2\epsilon(q)^2\bar\epsilon(q)^2 - \left(\frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)}\right)^2(\epsilon(q)^4+\bar\epsilon(q)^4) = 0.$$

*Proof.* Applying the Landen type transformations (cf. [L10, (1.8.5-6) & Ex.2 of Chap.1]), we find

$$\left(\frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)}\right)^2 = \frac{\theta_3(0|\tau)^2-\theta_4(0|\tau)^2}{\theta_3(0|\tau)^2+\theta_4(0|\tau)^2} = \frac{2\theta_2(0|4\tau)\theta_3(0|4\tau)}{\theta_3(0|4\tau)^2+\theta_2(0|4\tau)^2}.$$

Combining this with the theta transformation (cf. [F1916, p.482 (5)]) with $4\tau\to\frac{4\tau-1}{4\tau}$ in the form:

$$\frac{\theta_2(0|4\tau)}{\theta_3(0|4\tau)} = \frac{\theta_3(0|\frac{4\tau-1}{4\tau})}{\theta_4(0|\frac{4\tau-1}{4\tau})} = \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1+q^{2n-1})^2}{(1-q^{2n})(1-q^{2n-1})^2} = \frac{\epsilon(q)^2}{\bar\epsilon(q)^2},$$

we obtain

$$\left(\frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)}\right)^2 = \frac{2}{(\epsilon(q)/\bar\epsilon(q))^2+(\bar\epsilon(q)/\epsilon(q))^2} = \frac{2\epsilon(q)^2\bar\epsilon(q)^2}{\epsilon(q)^4+\bar\epsilon(q)^4} \tag{2.5}$$

for our $q$ given in (2.4). This proves the assertion. □

In the sequel, we set the parameters $\alpha=\bar\epsilon(q)$ and $\beta=\epsilon(q)$ in the fractional substitution (2.2), so that the resulting polynomial (2.3) has no constant term $d_0$ by virtue of Lemmas 2.3-2.4. Accordingly, we restrict ourselves to focusing on the two-parameter family

$$f_{r,s}(\mathtt{x},\mathtt{y}) := f_{r,s}^{\bar\epsilon(q),\epsilon(q)}(\mathtt{x},\mathtt{y}) = d_{12}(\mathtt{x}+\mathtt{y}) + d_{34}(\mathtt{x}^2+\mathtt{y}^2) + d_5\mathtt{x}\mathtt{y} + d_{67}(\mathtt{x}^2\mathtt{y}+\mathtt{y}^2\mathtt{x}) + d_8\mathtt{x}^2\mathtt{y}^2 \tag{2.6}$$

with $r,s$ given as Laurent power series in $q^{1/N}$ ($N\in\mathbb{N}$) convergent around (possibly with poles at) $q=0$. Note that $\epsilon r\neq\bar\epsilon s$ is assumed as the non-degeneracy condition for the substitutions (2.2). The coefficients $d_{12}, d_{34}, d_5, d_{67}, d_8$ are derived in the form (1.2) by a simple computation (using, say, Maple [M]).

*Proof of Proposition 1.1.* By construction, the plane curve $f_{r,s}(\mathtt{x},\mathtt{y})$ is birationally equivalent to the Edwards curve $E_a: x^2+y^2=a^2(1+x^2y^2)$ with $a^2=\frac{2\epsilon^2\bar\epsilon^2}{\epsilon^4+\bar\epsilon^4}$. In [E07], the $j$-invariant of $E_a$ is known to be

$$j(E_a) = \frac{1728}{108}\cdot\frac{(a^8+14a^4+1)^3}{a^4(a^4-1)^4}.$$

If $a = a(\tau)$ is given as in Proposition 2.2 in terms of theta-zero values, then it follows by simple calculation that $j(E_a) = j(\tau) = j(\frac{4\tau-1}{\tau})$ where the latter equality is due to the $\mathrm{PSL}_2(\mathbb{Z})$-invariance of $j$-function. Thus

$$j(E_a) = j\left(\exp(2\pi\mathtt{i}\frac{4\tau-1}{\tau})\right)$$

with $j(*)$ in RHS being the standard Fourier series of the $j$-function $j(z) = \frac{1}{z} + 744 + 196884z + \cdots$ in $z = e^{2\pi\mathtt{i}\tau}$. Since $\exp(2\pi\mathtt{i}\frac{4\tau-1}{\tau})) = q^8$ for $q$ given in (2.4), the assertion follows. □

## 3. **Ultra-discretization**

The power series $\epsilon(q), \bar{\epsilon}(q)$ converge on the Poincaré disk $|q| < 1$. In this section, we assume the two parameters $r(q), s(q)$ are contained in $\mathbb{C}\{q^{1/N}\}[\frac{1}{q}]$, the fractional field of the convergent power series ring in $q^{1/N}$ for some large integer $N \in \mathbb{N}$. We shall pursue the ultradiscretization of the points $(x, y)$ with

$$x = x(z) = \frac{\theta_1(z|2\tau)}{\theta_4(z|2\tau)}, \qquad y = y(z) = \frac{\theta_2(z|2\tau)}{\theta_3(z|2\tau)}$$

on the Edwards curve

$$x^2 + y^2 = \left(\frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)}\right)^2 (1 + x^2y^2)$$

at the limit

$$q = \exp(\pi\mathtt{i}\frac{4\tau-1}{4\tau}) \to 0. \tag{3.1}$$

Our method mostly follows the argument given in [KKNT] for the Hessian elliptic curves. Introduce an adjustment constant $\theta \in \mathbb{R}_{>0}$ for the above limit in the form

$$\frac{4\tau-1}{4\tau} = \frac{\mathtt{i}\theta}{\varepsilon} \to \mathtt{i}\infty \quad (\varepsilon \to 0). \tag{3.2}$$

Let us first observe the behaviors of $x(z), y(z)$ with $z = u + \mathtt{i}v$ $(u, v \in \mathbb{R})$ under the Fourier expansion (2.1) in $q$. Note first that the theta transformation [F1916, p.482 (5)] yields

$$x(z) = \frac{\theta_1(z|2\tau)}{\theta_4(z|2\tau)} = -\mathtt{i}\frac{\theta_1(\frac{-z}{2\tau}|\frac{2\tau-1}{2\tau})}{\theta_2(\frac{-z}{2\tau}|\frac{2\tau-1}{2\tau})}, \quad y(z) = \frac{\theta_2(z|2\tau)}{\theta_3(z|2\tau)} = \frac{\theta_3(\frac{-z}{2\tau}|\frac{2\tau-1}{2\tau})}{\theta_4(\frac{-z}{2\tau}|\frac{2\tau-1}{2\tau})}, \tag{3.3}$$

and our above setting (3.1)-(3.2) implies

$$\exp(\pi\mathtt{i}\frac{2\tau-1}{2\tau}) = -q^2 = \exp\left[\pi\mathtt{i}(-1 + \frac{2\theta\mathtt{i}}{\varepsilon})\right], \quad \frac{u+\mathtt{i}v}{2\tau} = 2(u+\mathtt{i}v)(1 - \frac{\mathtt{i}\theta}{\varepsilon}), \tag{3.4}$$

which enables us to evaluate the Fourier expansion of $x(z), y(z)$. Let us first look at the numerator of $y(z)$ as follows:

$$\begin{aligned}\theta_3(\frac{-z}{2\tau}|\frac{2\tau-1}{2\tau}) &= \sum_{n\in\mathbb{Z}} \exp\left[n^2\pi\mathtt{i}(-1 + \frac{2\theta\mathtt{i}}{\varepsilon}) + 2\pi\mathtt{i}n(-u - v\mathtt{i})2(1 - \frac{\mathtt{i}\theta}{\varepsilon})\right] \\ &= \sum_{n\in\mathbb{Z}} \exp\left[-\pi\left(\frac{2\theta}{\varepsilon}n^2 + 4n(\frac{\theta}{\varepsilon}u - v) + \mathtt{i}(n^2 + 4n(u + \frac{\theta}{\varepsilon}v))\right)\right].\end{aligned}$$

In the ultradiscretization process, the imaginary exponents should be annihilated. This determines the imaginary part $v$ of $z$ to be equal to $-u\varepsilon/\theta$:

$$z = u - \mathtt{i}\varepsilon\frac{u}{\theta}. \tag{3.5}$$

Since $\exp(-n^2\pi\mathtt{i}) = (-1)^n$, this implies

$$\theta_3(\frac{-z}{2\tau}|\frac{2\tau-1}{2\tau}) \sim \sum_{n\in\mathbb{Z}}(-1)^n \exp\left[-\frac{2\pi\theta}{\varepsilon}\left((n+u)^2-u^2\right)\right]$$

as $\varepsilon \to 0$. After similar calculations for the denominator of $y(z)$ and the numerator and denominator of $x(z)$, we summarize evaluations as follows:

$$\text{(3.6)} \qquad \begin{cases} x(u-\mathtt{i}\varepsilon\frac{u}{\theta}) & \sim -\dfrac{\sum_{n\in\mathbb{Z}}(-1)^n\exp\left[-\frac{2\pi\theta}{\varepsilon}(n+u+\frac{1}{2})^2\right]}{\sum_{n\in\mathbb{Z}}\exp\left[-\frac{2\pi\theta}{\varepsilon}(n+u+\frac{1}{2})^2\right]} \\ & \qquad = \dfrac{\sum_{n\in\mathbb{Z}}(-1)^n\exp\left[-\frac{2\pi\theta}{\varepsilon}(n+u-\frac{1}{2})^2\right]}{\sum_{n\in\mathbb{Z}}\exp\left[-\frac{2\pi\theta}{\varepsilon}(n+u-\frac{1}{2})^2\right]}, \\ y(u-\mathtt{i}\varepsilon\frac{u}{\theta}) & \sim \dfrac{\sum_{n\in\mathbb{Z}}(-1)^n\exp\left[-\frac{2\pi\theta}{\varepsilon}(n+u)^2\right]}{\sum_{n\in\mathbb{Z}}\exp\left[-\frac{2\pi\theta}{\varepsilon}(n+u)^2\right]}. \end{cases}$$

Here we note that the factors $\exp(\frac{2\pi\theta}{\varepsilon}(u^2+\frac{1}{4}))$ (resp. $\exp(\frac{2\pi\theta}{\varepsilon}u^2)$) from $\theta_1, \theta_4$ (resp. from $\theta_2, \theta_3$) are cancelled out in the numerator and denominator in the above right hand sides' expressions (and that $\exp(-\pi\mathtt{i}(n+\frac{1}{2})^2) = -\mathtt{i}$ and $-(-1)^n = (-1)^{n+1}$ for the former expressions from $\theta_1, \theta_2$). We next convert the above evaluations (3.6) of $x, y$ to those of $\mathtt{x}, \mathtt{y}$ under the substitutions (2.2) with $\alpha = \bar{\epsilon}(q)$ and $\beta = \epsilon(q)$ via

$$\text{(3.7)} \qquad \mathtt{x} = \frac{\epsilon x - \bar{\epsilon}}{-sx+r}, \quad \mathtt{y} = \frac{\epsilon y - \bar{\epsilon}}{-sy+r}.$$

Write $\epsilon(q) = \sum_{k=0}^{\infty} a_k q^k$ so that $\bar{\epsilon}(q) = \sum_{k=0}^{\infty}(-1)^k a_k q^k$ (where we know $a_k > 0$ for all $k \geq 0$) and suppose that the two parameters $r, s \in \mathbb{C}\{q^{1/N}\}[\frac{1}{q}]$ are given in the form

$$\text{(3.8)} \qquad r = \sum_{k=0}^{\infty} r_k q^{a+\frac{k}{N}} = q^a(r_0 + r_1 q^{1/N} + \cdots), \quad s = \sum_{k=0}^{\infty} s_k q^{a+\frac{k}{N}} = q^a(s_0 + s_1 q^{1/N} + \cdots)$$

with $a \in \frac{1}{N}\mathbb{Z}$, $|r_0| + |s_0| \neq 0$. Then, we obtain:

$$\text{(3.9)} \qquad \begin{cases} \mathtt{x}(u-\mathtt{i}\varepsilon\frac{u}{\theta}) \sim \dfrac{P(u-\frac{1}{2},\theta,\varepsilon)\left(\sum_{k:odd\geq 1} 2a_k q^k\right) - Q(u-\frac{1}{2},\theta,\varepsilon)\left(\sum_{k:even\geq 0} 2a_k q^k\right)}{P(u-\frac{1}{2},\theta,\varepsilon)\left(\sum_k (r_k - s_k) q^{a+\frac{k}{N}}\right) + Q(u-\frac{1}{2},\theta,\varepsilon)\left(\sum_k (r_k + s_k) q^{a+\frac{k}{N}}\right)}, \\ \mathtt{y}(u-\mathtt{i}\varepsilon\frac{u}{\theta}) \sim \dfrac{P(u,\theta,\varepsilon)\left(\sum_{k:odd\geq 1} 2a_k q^k\right) - Q(u,\theta,\varepsilon)\left(\sum_{k:even\geq 0} 2a_k q^k\right)}{P(u,\theta,\varepsilon)\left(\sum_k (r_k - s_k) q^{a+\frac{k}{N}}\right) + Q(u,\theta,\varepsilon)\left(\sum_k (r_k + s_k) q^{a+\frac{k}{N}}\right)}, \end{cases}$$

where $P(u,\theta,\varepsilon) := \sum_{n:even} \exp[-\frac{2\pi\theta}{\varepsilon}(n+u)^2]$, $Q(u,\theta,\varepsilon) := \sum_{n:odd} \exp[-\frac{2\pi\theta}{\varepsilon}(n+u)^2]$. At this stage, we are led to define the ultradiscrete theta functions $\Theta^{even}$ and $\Theta^{\mathrm{odd}}$ as those limits of the quantities $P(u,\theta,\varepsilon)$ and $Q(u,\theta,\varepsilon)$ respectively:

**Definition 3.1.** For $u \in \mathbb{R}$, define

$$\Theta^{even}(u) := \lim_{\varepsilon\to 0+} \varepsilon \log P(u,\theta,\varepsilon) = 2\pi\theta \max_{n:even}(-(n+u)^2) = -2\pi\theta\left[2\lfloor\frac{u+1}{2}\rfloor - u\right]^2,$$

$$\Theta^{\mathrm{odd}}(u) := \lim_{\varepsilon\to 0+} \varepsilon \log Q(u,\theta,\varepsilon) = 2\pi\theta \max_{n:odd}(-(n+u)^2) = -2\pi\theta\left[2\lfloor\frac{u}{2}\rfloor + 1 - u\right]^2.$$

Note here that the distance from any $u \in \mathbb{R}$ to the nearest even (resp. odd) integer can be written as $|2\lfloor \frac{u+1}{2} \rfloor - u|$ (resp. $|2\lfloor \frac{u}{2} \rfloor + 1 - u|$). It is easy to see that these functions $\Theta^{even}, \Theta^{odd}$ are even functions of period 2 on $\mathbb{R}$ and satisfy $\Theta^{even}(\pm u \pm 1) = \Theta^{odd}(\pm u)$. With the above definition of $\Theta^{even}, \Theta^{odd}$ together with (3.9), we conclude that the ultradiscrete limit $(X(u), Y(u))$ of the point $(\mathtt{x}(u - \mathtt{i}\varepsilon u/\theta), \mathtt{y}(u - \mathtt{i}\varepsilon u/\theta))$ is given by:

$$
\begin{cases} X(u) & = \lim_{\varepsilon \to 0+} \varepsilon \log \mathtt{x}(u - \mathtt{i}\varepsilon \frac{u}{\theta}) = Y(u - \frac{1}{2}), \\ Y(u) & = \lim_{\varepsilon \to 0+} \varepsilon \log \mathtt{y}(u - \mathtt{i}\varepsilon \frac{u}{\theta}) \\ & = \max\left(\Theta^{\mathrm{odd}}(u), -1 + \Theta^{even}(u)\right) \\ & \quad -\max\left(-v_{\mathbb{K}}(r-s) + \Theta^{\mathrm{even}}(u), -v_{\mathbb{K}}(r+s) + \Theta^{\mathrm{odd}}(u)\right). \end{cases} \tag{3.10}
$$

**Remark 3.2.** The minus sign problem for ultradiscrete limits can be avoided in our case. See [KNT08, Remark 2.1] (cf. also [KL06]).

Now let us normalize our adjustment constant $\theta$ introduced above in (3.2) by comparing our ultradiscrete limit with the non-archimedean amoeba studied in Einsiedler-Kapranov-Lind [EKL06] and in Speyer [S14]. In fact, our fundamental quantities $\epsilon(q), \bar{\epsilon}(q)$ and the two parameters $r(q), s(q)$ are considered as elements of the field of convergent Laurent series $\mathbb{C}\{q^{1/N}\}[\frac{1}{q}]$ for taking analytic limits $q \to 0$, however we are able to enhance this analytic procedure to a more general algebraic process via the valuation theory: Consider $\mathbb{C}\{q^{1/N}\}[\frac{1}{q}]$ as a subfield of the standard Puiseux power series $\mathbb{C}\{\{q\}\}$ which has the standard valuation $v : \mathbb{C}\{\{q\}\} \to \mathbb{Q} \cup \{\infty\}$ with $v(q) = 1$ and has the non-archimedean norm $||a|| := e^{-v(a)}$ $(a \in \mathbb{C}\{\{q\}\})$. According to [EKL06], the non-archimedean amoeba $\mathcal{A}(S) \in \mathbb{R}^2$ of a subscheme $S \subset (\mathbb{C}\{\{q\}\}^\times)^2$ is by definition the closure of the image of $S$ by the map $\mathrm{Log} : (\mathbb{C}\{\{q\}\}^\times)^2 \to \mathbb{R}^2$ $((\mathtt{x}, \mathtt{y}) \mapsto (\log ||\mathtt{x}||, \log ||\mathtt{y}||))$, and the tropical variety $\mathrm{Trop}(S) \subset \mathbb{R}^2$ is the closure of the image of $X$ by the map $\mathrm{val} : (\mathbb{C}\{\{q\}\}^\times)^2 \to \mathbb{R}^2$ defined by $(\mathtt{x}, \mathtt{y}) \mapsto (v(\mathtt{x}), v(\mathtt{y}))$; this is equivalent to saying that

$$
\mathrm{Trop}(S) = -\mathcal{A}(S). \tag{3.11}
$$

As given in (3.1)-(3.2), our ultradiscrete limits are taken with respect to the base scaling $q = \exp(-\pi\theta/\varepsilon)$ $(\varepsilon \to 0)$, in particular $\varepsilon \log q \to -\pi\theta$. Comparing this with $v(q) = 1$, $\log ||q|| = -1$ in the non-archimedean metric of $\mathbb{C}\{\{q\}\}$, we are led to normalize our above adjustment constant $\theta$ as follows:

$$
\theta := 1/\pi. \tag{3.12}
$$

This enables us to identify the plane curve $\{(X(u), Y(u)) \mid u \in \mathbb{R}\} \subset \mathbb{R}^2$ to lie on the cycle part of the non-archimedean amoeba $\mathcal{A}(f_{r,s}(\mathtt{x}, \mathtt{y}) = 0)$, viz., the cycle part of the closure of

$$
\mathrm{Log}\left(\left\{(||\mathtt{x}||, ||\mathtt{y}||) \,\middle|\, f_{r,s}(\mathtt{x}, \mathtt{y}) = 0, \mathtt{xy} \neq 0,\ (\mathtt{x}, \mathtt{y}) \in \mathbb{C}\{\{q\}\}^2\right\}\right) \subset \mathbb{Q}^2. \tag{3.13}
$$

## 4. **Proofs of Theorem 1.2 and Corollary 1.3**

Now we convert the analytic theory of Edwards curve to algebraic theory over the complete discrete valuation field $K$ of characteristic 0 with the algebraic closure $\mathbb{K}$ as set up in Introduction. The Edwards curve

$$
E_a : x^2 + y^2 = a^2(1 + x^2y^2)
$$

around the neighborhood of $q = \exp(\pi \mathtt{i} \frac{4\tau - 1}{4\tau}) \to 0$ as in (2.4) hints us how to convert necessary identities into the algebraic situation. As shown in (2.5), the parameter $a$ should be given by

$$a^2 = \frac{2\epsilon(q)^2 \bar{\epsilon}(q)^2}{\epsilon(q)^4 + \bar{\epsilon}(q)^4} \in K$$

through which we define the curve $E_a$ over $K$. We next convert the analytic uniformization of $E_a$ (Proposition 2.2) by the complex $z$-plane to the algebraic uniformization by $\mathbb{K}^\times$ in Tate form. We first convert the Jacobian theta functions (2.1) into the reduced algebraic series $\bar{\theta}_1(t, -q^2), \ldots, \bar{\theta}_4(t, -q^2)$ by substituting $q_\tau \mapsto -q^2$, $q_z \mapsto t$ and by dropping the fragment factors $-\mathtt{i}(-q^2)^{1/4}$, $(-q^2)^{1/4}$ from $\theta_1, \theta_2$ respectively:

$$\begin{cases} \bar{\theta}_1(t, -q^2) & = t \sum_{n \in \mathbb{Z}} (-1)^n q^{2n^2 + 2n} t^{2n}, \\ \bar{\theta}_2(t, -q^2) & = t \sum_{n \in \mathbb{Z}} q^{2n^2 + 2n} t^{2n}, \\ \bar{\theta}_3(t, -q^2) & = \sum_{n \in \mathbb{Z}} (-1)^n q^{2n^2} t^{2n}, \\ \bar{\theta}_4(t, -q^2) & = \sum_{n \in \mathbb{Z}} q^{2n^2} t^{2n}. \end{cases} \tag{4.1}$$

It turns out from Fricke's transformation (3.3) that those fragment factors from $\theta_1, \theta_2$ cancel each other and that the algebraic uniformization is given by

$$x(t) = -\frac{\bar{\theta}_1(t, -q^2)}{\bar{\theta}_2(t, -q^2)}, \quad y(t) = \frac{\bar{\theta}_3(t, -q^2)}{\bar{\theta}_4(t, -q^2)} \qquad (t \in \mathbb{K}^\times). \tag{4.2}$$

Noting the identities

$$\begin{cases} \bar{\theta}_1(t, -q^2) & = t\bar{\theta}_3(tq, -q^2), \\ \bar{\theta}_2(t, -q^2) & = t\bar{\theta}_4(tq, -q^2), \\ \bar{\theta}_3(t, -q^2) & = -tq\bar{\theta}_1(tq, -q^2), \\ \bar{\theta}_4(t, -q^2) & = tq\bar{\theta}_2(tq, -q^2); \end{cases} \text{ and } \begin{cases} x(t) & = x(-t), \\ y(t) & = y(-t), \\ x(tq) & = y(t), \\ y(tq) & = -x(t). \end{cases} \tag{4.3}$$

we see that the Tate uniformization map $\mathbb{K}^\times \to E_a(\mathbb{K})$ by $t \mapsto (x(t), y(t))$ gives rise to

$$E_a(\mathbb{K}) = \mathbb{K}^\times / \langle \pm q^{4\mathbb{Z}} \rangle.$$

We then apply the fractional substitutions in the same way as (2.2)

$$x = \frac{r\mathtt{x} + \bar{\epsilon}(q)}{s\mathtt{x} + \epsilon(q)}, \quad y = \frac{r\mathtt{y} + \bar{\epsilon}(q)}{s\mathtt{y} + \epsilon(q)}. \tag{4.4}$$

for given constants $r, s \in \mathbb{K}$ with $\bar{\epsilon}(q)s \neq \epsilon(q)r$ to obtain the curve (2.6)

$$f_{r,s}(\mathtt{x}, \mathtt{y}) = d_{12}(\mathtt{x} + \mathtt{y}) + d_{34}(\mathtt{x}^2 + \mathtt{y}^2) + d_5\mathtt{x}\mathtt{y} + d_{67}(\mathtt{x}^2\mathtt{y} + \mathtt{y}^2\mathtt{x}) + d_8\mathtt{x}^2\mathtt{y}^2 = 0 \tag{4.5}$$

with coefficients $d_{12}, d_{34}, d_5, d_{67}, d_8$ given as in Introduction (1.2). To obtain the tropical curve $C(\mathrm{trop}(f_{r,s}))$ is thus reduced to evaluating the points $(\mathtt{x}, \mathtt{y})$ in $(\mathbb{K}^\times)^2$ by the valuation map $\mathrm{val} : (\mathbb{K}^\times)^2 \to \mathbb{Q}^2$ that applies $v_{\mathbb{K}} : \mathbb{K} \to \mathbb{Q} \cup \{\infty\}$ to each component $\mathtt{x}, \mathtt{y}$ of (4.4) through:

$$\mathtt{x} = \frac{\epsilon(q)x - \bar{\epsilon}(q)}{-sx + r}, \quad \mathtt{y} = \frac{\epsilon(q)y - \bar{\epsilon}(q)}{-sy + r}. \tag{4.6}$$

In order to prove Theorem 1.2, it suffices to get partial information on (the cycle part of) $C(\mathrm{trop}(f_{r,s}))$ by performing a procedure parallel to (3.9). We first observe from (4.2) and (4.6) that

$$\begin{aligned}\mathtt{y}(t) &= \frac{(\epsilon-\bar\epsilon)(\bar\theta_4+\bar\theta_3)-(\epsilon+\bar\epsilon)(\bar\theta_4-\bar\theta_3)}{(r-s)(\bar\theta_4+\bar\theta_3)+(r+s)(\bar\theta_4-\bar\theta_3)}\\ &= \frac{(\epsilon-\bar\epsilon)(\sum_{n:even}t^{2n}q^{2n^2})-(\epsilon+\bar\epsilon)(\sum_{n:odd}t^{2n}q^{2n^2})}{(r-s)(\sum_{n:even}t^{2n}q^{2n^2})+(r+s)(\sum_{n:odd}t^{2n}q^{2n^2})}.\end{aligned} \tag{4.7}$$

Recall that the valuation $v_{\mathbb{K}}$ of $\mathbb{K}$ is normalized as $v_{\mathbb{K}}(q)=1$. To introduce a variable $u$ for the valuation $v_{\mathbb{K}}(t)$, let us take into account the ultradiscrete limits discussed in §3 where the variable $u$ is normalized as

$$t=q_z=\exp(\pi\mathtt{i}\frac{-u-v\mathtt{i}}{2\tau})=\exp\left(-2\pi\left(\frac{\theta}{\varepsilon}+\frac{\varepsilon}{\theta}\right)u\right),\quad \lim_{\varepsilon\to0}\varepsilon\log(t)=-2\pi\theta u,$$

whereas

$$q=\exp\left(\pi\mathtt{i}\frac{4\tau-1}{4\tau}\right)=\exp(-\frac{\pi\theta}{\varepsilon}),\quad \lim_{\varepsilon\to0}\varepsilon\log(q)=-\pi\theta.$$

Thus, let us set

$$v_{\mathbb{K}}(t)\big(=2v_{\mathbb{K}}(q)u\big)=2u \qquad (u\in\mathbb{Q}). \tag{4.8}$$

We shall first evaluate $v_{\mathbb{K}}(\mathtt{y}(t))$ when the numerator and denominator have a unique term with distinguished valuation respectively. First note that $v_{\mathbb{K}}(\epsilon-\bar\epsilon)=1$, $v_{\mathbb{K}}(\epsilon+\bar\epsilon)=0$. Since $v_{\mathbb{K}}(t^{2n}q^{2n^2})=2(n+u)^2-2u^2$, we find that

$$v_{\mathbb{K}}(\sum_{n:even}t^{2n}q^{2n^2})\ge 2(2\lfloor\frac{u+1}{2}\rfloor-u)^2-2u^2=-\Theta^{\mathrm{even}}(u)-2u^2$$

with equality held for $u\in\mathbb{Q}\setminus(1+2\mathbb{Z})$ and that

$$v_{\mathbb{K}}(\sum_{n:odd}t^{2n}q^{2n^2})\ge 2(2\lfloor\frac{u}{2}\rfloor+1-u)^2-2u^2=-\Theta^{\mathrm{odd}}(u)-2u^2$$

with equality held for $u\in\mathbb{Q}\setminus 2\mathbb{Z}$. Therefore,

$$\begin{aligned}v_{\mathbb{K}}(\mathtt{y}(t))&=\min\Big(1-\Theta^{even}(u),-\Theta^{\mathrm{odd}}(u)\Big)\\ &\quad-\min\Big(v_{\mathbb{K}}(r-s)-\Theta^{\mathrm{even}}(u),v_{\mathbb{K}}(r+s)-\Theta^{\mathrm{odd}}(u)\Big)\end{aligned} \tag{4.9}$$

for $u=\frac{1}{2}v_{\mathbb{K}}(t)\in\mathbb{Q}$ not belonging to the exceptional set

$$\Xi_{r,s}:=\mathbb{Z}\cup\left\{u\in\mathbb{Q}\;\middle|\;\begin{array}{c}1-\Theta^{even}(u)=-\Theta^{\mathrm{odd}}(u),\\ v_{\mathbb{K}}(r-s)-\Theta^{\mathrm{even}}(u)=v_{\mathbb{K}}(r+s)-\Theta^{\mathrm{odd}}(u)\end{array}\right\}. \tag{4.10}$$

As for $v_{\mathbb{K}}(\mathtt{x}(t))$, noting that (4.3) and (4.6) imply

$$\mathtt{x}(t)=\mathtt{y}(tq^{-1}) \tag{4.11}$$

and that $v_{\mathbb{K}}(tq^{-1})=2u-1=2(u-\frac{1}{2})$, we easily derive from (4.9) the formula:

$$\begin{aligned}v_{\mathbb{K}}(\mathtt{x}(t))&=\min\Big(1-\Theta^{even}(u-\frac{1}{2}),-\Theta^{\mathrm{odd}}(u-\frac{1}{2})\Big)\\ &\quad-\min\Big(v_{\mathbb{K}}(r-s)-\Theta^{\mathrm{even}}(u-\frac{1}{2}),v_{\mathbb{K}}(r+s)-\Theta^{\mathrm{odd}}(u-\frac{1}{2})\Big)\end{aligned} \tag{4.12}$$

for $u = \frac{1}{2}v_{\mathbb{K}}(t)$ such that $u - \frac{1}{2} \notin \Xi_{r,s}$. Thus, $C(\mathrm{trop}(f_{r,s})) \subset \mathbb{R}^2$ contains the particular set

$$\left\{ \begin{array}{l} \mathrm{val}(\mathtt{x}(t), \mathtt{y}(t)) \in \mathbb{Q}^2 \\ \left(= (v_{\mathbb{K}}(\mathtt{x}(t)), v_{\mathbb{K}}(\mathtt{y}(t)))\right) \end{array} \;\middle|\; \frac{1}{2}v_{\mathbb{K}}(t) \notin \Xi_{r,s} \cup \left(\frac{1}{2} + \Xi_{r,s}\right), \quad t \in \mathbb{K}^\times \right\} \tag{4.13}$$

whose coordinates are explicitly known by (4.12) and (4.9). It is then not difficult to see that the closure of the above set in the Euclidean plane $\mathbb{R}^2$ is equal to the locus of the points $(-X(u), -Y(u))$ with $u \in \mathbb{R}$ given in Theorem 1.2 (iii), after noting the general equality $-\max(A, B) = \min(-A, -B)$. The proof of Theorem 1.2 is completed. □

*Proof of Corollary 1.3.* Since the cycle part of $C(\mathrm{trop}(f_{r,s}))$ is parametrized as $(-Y(u-\frac{1}{2}), -Y(u))$ for $u \in \mathbb{R}$ by Theorem 1.2, each side of the cycle can be captured by the shape of the piecewise linear graph

$$Z = Y(u) = \max\left(\Theta^{\mathrm{odd}}(u), -1 + \Theta^{even}(u)\right) - \max\left(\delta + \Theta^{even}(u), \Theta^{\mathrm{odd}}(u)\right) - v_{\mathbb{K}}(r+s),$$

and its translation $Z = Y(u - \frac{1}{2})$ on (u,Z)-plane. Setting

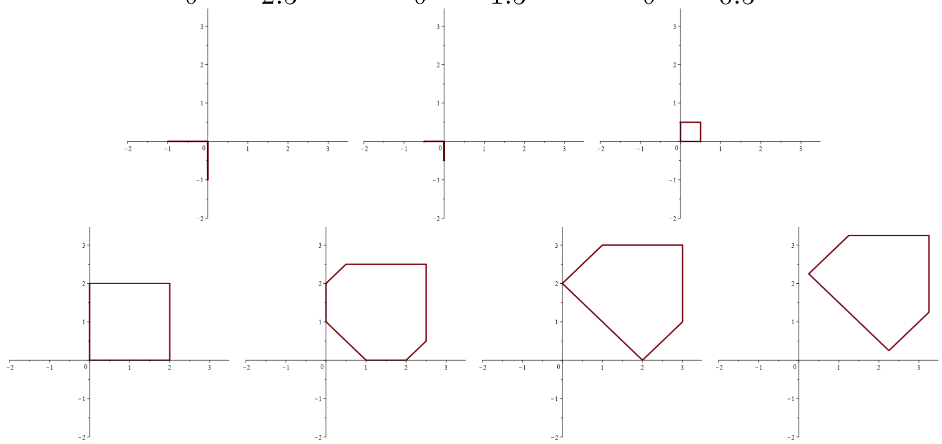
$$Y_\delta(u) := \max\left(\Theta^{\mathrm{odd}}(u), -1 + \Theta^{even}(u)\right) - \max\left(\delta + \Theta^{even}(u), \Theta^{\mathrm{odd}}(u)\right),$$

so that

$$\begin{pmatrix} -Y(u-\frac{1}{2}) \\ -Y(u) \end{pmatrix} = \begin{pmatrix} -Y_\delta(u-\frac{1}{2}) \\ -Y_\delta(u)) \end{pmatrix} + v_{\mathbb{K}}(r+s)\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we find that the shape of the cycle part of $C(\mathrm{trop}(f_{r,s}))$ depends only on $\delta$. The rest is not difficult after tracing the move of the cycle $(-Y_\delta(u - \frac{1}{2}), -Y_\delta(u))_{u\in\mathbb{R}}$ on $XY$-plane under the parameter $\delta$ varied in $\mathbb{R}$ (summarized in the following figures for $\delta = -2.5, -1.5, -0.5, 1, 1.5, 2, 2.25$).

$\delta = -2.5$ $\delta = -1.5$ $\delta = -0.5$



$\delta = 1$ $\delta = 1.5$ $\delta = 2$ $\delta = 2.25$

□

## 5. **Proof of Proposition 1.4**

In [T], the second named author studied the condition for a truncated symmetric cubic to have smooth tropicalization. Let

$$f(\mathtt{x}, \mathtt{y}) = d_{12}(\mathtt{x} + \mathtt{y}) + d_{34}(\mathtt{x}^2 + \mathtt{y}^2) + d_5\mathtt{x}\mathtt{y} + d_{67}(\mathtt{x}^2\mathtt{y} + \mathtt{y}^2\mathtt{x}) + d_8\mathtt{x}^2\mathtt{y}^2$$

and let $u_{12}, u_{34}, u_5, u_{67}, u_8$ be valuations of $d_{12}, d_{34}, d_5, d_{67}, d_8$ respectively. Then, we have

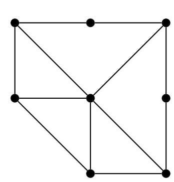
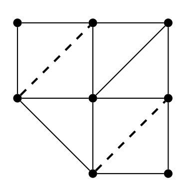
**Proposition 5.1** ([T]). *Suppose that the tropical elliptic curve trop(f) has a polygonal cycle $P$ and let $(u_{12}, u_{34}, u_5, u_{67}, u_8)$ be the associated parameter whose entries are valuations of the coefficients $d_{12}, d_{34}, d_5, d_{67}, d_8$ respectively. Then, trop(f) is tropically smooth if and only if $(P; u_{12}, u_{34}, u_5, u_{67}, u_8)$ satisfies one of the conditions listed in Table 1:*

TABLE 1. Cases of smooth $trop(f)$

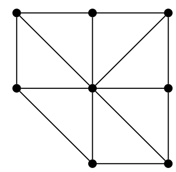
| $P$ | $(u_{12}, u_{34}, u_5, u_{67}, u_8)$ |
|---|---|
| [i] *Triangle* | $\begin{cases} -u_{34} + 2u_{67} - u_8 < 0, \\ u_{12} - u_5 - u_{67} + u_8 < 0, \\ -2u_{12} + 3u_5 - u_8 < 0. \end{cases}$ |
| [ii] *Square* | $\begin{cases} -u_5 + 2u_{67} - u_8 < 0, \\ -u_{12} + 2u_5 - u_{67} < 0, \\ u_{12} - u_{34} - u_5 + u_{67} < 0. \end{cases}$ |
| [iii] *Pentagon* | $\begin{cases} u_5 - 2u_{67} + u_8 < 0, \\ -u_{12} + u_5 + u_{67} - u_8 < 0, \\ u_{12} - u_{34} - u_5 + u_{67} < 0. \end{cases}$ |
| [iv] *Hexagon* | $\begin{cases} -u_5 + 2u_{67} - u_8 < 0, \\ -u_{34} + u_5 < 0, \\ -u_{12} + u_{34} + u_5 - u_{67} < 0. \end{cases}$ |
| [v] *Heptagon* | $\begin{cases} u_5 - 2u_{67} + u_8 < 0, \\ -u_{34} + 2u_{67} - u_8 < 0, \\ -u_{12} + u_{34} + u_5 - u_{67} < 0. \end{cases}$ |

*Proof.* This follows from a close look at the "subdivision" associated to trop($f$). For more details, we refer the reader to [T, 5.4]. □

*Proof of Proposition 1.4.* Now, let us prove Proposition 1.4 by combining Corollary 1.3 and the above Proposition 5.1. Note that, for our two-parameter family $f_{r,s}$, only cycles of squares, heptagons or pentagons can occur as observed in Corollary 1.3 . We discuss case by case according to the value of $\delta = \delta_{r,s}$. (i) Suppose that $P$ is a pentagon, i.e., $2 \leq \delta$. Then, the possible subdivisions producing a pentagon are the following two types:

 ,  ,

where dashed lines may or may not exist. Since a smooth tropical curve corresponds to a subdivision by triangles of area $\frac{1}{2}$, the smooth case occurs only from the latter type. Both of these subdivisions produce pentagons as dual graphs consisting of three right angles and two obtuse angles, however their shapes differ from each other in that the former has separated obtuse angles (like the baseball homeplate) while the latter has adjacent obtuse angles. On the other hand, the explicit parametrization of the cycle given in Theorem 1.2 claims that only the former type of pentagon occurs in our family, all of which turn our to correspond to non-smooth tropical curves. (ii) Suppose next that $P$ is a heptagon, i.e., $1 < \delta < 2$. In this case, the corresponding subdivision is pictured as follows.



This produces a smooth tropical curve. (iii) Suppose that $P$ is a square, i.e., $-1 < \delta \leq 1$. By the result of [V09]-[KMM] and Proposition 1.1, the smooth tropical curve has a cycle of length 8, hence by Corollary 1.3, it can occur only when $\delta = 1$. Then, after dividing $r, s$ by $q^{v_{\mathbb{K}}(r+s)}$, without loss of generality, we may assume that the parameters $r, s$ are of the form:

$$r = 1 + r_0 q + \sum_{k \geq 1} r_k q^{1+\frac{k}{N}},$$
$$s = -1 + s_0 q + \sum_{k \geq 1} s_k q^{1+\frac{k}{N}}$$

with $r_0 + s_0 \neq 0$ for some integer $N > 0$. By simple computations, we find $u_{12} = 1$, $u_5 = 0$ and $u_{67} = 1$ hold independently of the choice of $N$. Then, the smoothness condition (for the square cycle case) in Proposition 5.1 (the third inequality of Table 1 [ii]) implies

$$2 < u_{34} = 1 + v_{\mathbb{K}}(\bar{\epsilon}^2 s^2 - \epsilon^2 r^2) = v_{\mathbb{K}}\Big((-4 - 2s_0 - 2r_0)q^2 + O(q^{2+\frac{1}{N}})\Big),$$

hence $r_0 + s_0 = -2$. Conversely, if $r_0 + s_0 = -2$, then

$$v_{\mathbb{K}}(\epsilon^2 s^2 - \bar{\epsilon}^2 r^2) = v_{\mathbb{K}}\Big((4 - 2s_0 - 2r_0)q + O(q^{1+\frac{1}{N}})\Big) = 1,$$

so that $u_8 = u_{34} > 2$. This satisfies the smoothness condition given in Proposition 5.1 [ii]. □

## 6. **Examples**

By virtue of Speyer's work [S14, Sect. 7], the structure of $C(\mathrm{trop}(f_{r,s}))$ as a metric graph can be viewed as the projection from a certain subtree of the Bruhat-Tits tree, if the divisors (zeros and poles) of the elliptic functions $\mathtt{x}$, $\mathtt{y}$ are known on the curve

$$E_{r,s} : f_{r,s}(\mathtt{x}, \mathtt{y}) = d_{12}(\mathtt{x} + \mathtt{y}) + d_{34}(\mathtt{x}^2 + \mathtt{y}^2) + d_5 \mathtt{x}\mathtt{y} + d_{67}(\mathtt{x}^2\mathtt{y} + \mathtt{y}^2\mathtt{x}) + d_8 \mathtt{x}^2\mathtt{y}^2 = 0 \tag{6.1}$$

with coefficients $d_{12}, d_{34}, d_5, d_{67}, d_8$ given as in (1.2). We investigate those divisors in view of the theta uniformization $\wp\colon \mathbb{K}^\times \twoheadrightarrow \mathbb{K}^\times/\langle \pm q^{4\mathbb{Z}} \rangle = E_{r,s}(\mathbb{K})$ determined by the pair $(\mathtt{x}(t), \mathtt{y}(t))$ of functions in $t \in \mathbb{K}^\times$ (given by (4.7) and (4.11)) as in §4, and illustrate $C(\mathrm{trop}(f_{r,s}))$ in some special cases of parameters $r, s \in \mathbb{K}$. In the following examples, we content ourselves with observing smooth cases, while we hope to look into details of (subtle) phenomena appearing in non-smooth cases in a future separate article.

6.1. **Bruhat-Tits tree.** Let $E$ be the elliptic curve over $\mathbb{K}$, the smooth completion of the affine curve defined by $f_{r,s}(\mathtt{x}, \mathtt{y}) = 0$. Noting that $\mathbb{K}$ is assumed to be an algebraically closed field of characteristic 0, we may compose the above theta parametrization with the square power map to fit in the usual form of Tate uniformization

$$\mathbb{K}^\times \overset{\wp}{\twoheadrightarrow} \mathbb{K}^\times/\langle \pm q^4 \rangle = E_{r,s}(\mathbb{K}) \xrightarrow[t \mapsto t^2]{\sim} \mathbb{K}^\times/\langle q^8 \rangle. \tag{6.2}$$

Denote by $Z_{\mathtt{x}}$ (resp. $Z_{\mathtt{y}}$) the set of zeros in $t \in \mathbb{K}^\times$ of the function $\mathtt{x} \circ \wp$ (resp. $\mathtt{y} \circ \wp$)) and by $P_{\mathtt{x}}$ (resp. $P_{\mathtt{y}}$) the set of poles of $\mathtt{x} \circ \wp$ (resp. $\mathtt{y} \circ \wp$), and set

$$Z := Z_{\mathtt{x}} \cup Z_{\mathtt{y}}, \quad P := P_{\mathtt{x}} \cup P_{\mathtt{y}}.$$

Note that from (4.11) we have

$$Z_{\mathtt{x}} = qZ_{\mathtt{y}}, \quad P_{\mathtt{x}} = qP_{\mathtt{y}}. \tag{6.3}$$

We also see from the equation (6.1) that both $\mathtt{x}$ and $\mathtt{y}$ are rational functions of degree 2 on $E_{r,s}$, hence that each of the sets $Z_{\mathtt{x}}$, $P_{\mathtt{x}}$, $Z_{\mathtt{y}}$, $P_{\mathtt{y}}$ has the image of cardinality at most 2 under the projection (6.2).

Basic tools devised in [S14] are the Bruhat-Tits $\mathbb{Q}$-tree $BT(\mathbb{K})$ and its completion $\overline{BT}(\mathbb{K})$ with ends in $\mathbb{P}^1(\mathbb{K})$:

$$\overline{BT}(\mathbb{K}) = BT(\mathbb{K}) \cup \mathbb{P}^1(\mathbb{K}).$$

The group $\mathrm{GL}_2(\mathbb{K})$ acts naturally on $\overline{BT}(\mathbb{K})$ so that the multiplication by $\xi \in \mathbb{K}^\times$ on $\mathbb{P}^1(\mathbb{K}) = \mathbb{K} \cup \{\infty\}$ is extended to the action of $\left(\begin{smallmatrix} \xi & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{K})$. We consider $Z \cup P$ as an infinite subset of $\mathbb{P}^1(\mathbb{K})$ (necessarily stable under the action of $\langle \pm q^4 \rangle \subset \mathbb{K}^\times$) and its spanning tree

$$\Gamma_{r,s} := \bigcup_{z,z' \in Z \cup P} [z, z'] \quad (\subset BT(\mathbb{K})) \tag{6.4}$$

which has, for every point $z \in Z \cup P$ $(\subset \mathbb{K}^\times \subset \mathbb{P}^1(\mathbb{K}))$, a semi-infinite path to the 'end' $z$. It turns out that $\Gamma_{r,s}$ contains an infinite central road $[0, \infty]$. The metric structure on the internal edges of $\Gamma_{r,s}$ is determined by the rule that, for every 4 points $w, x, y, z \in P \cup Z$, the length of the internal edge $[w, x] \cap [y, z]$ is given by $|v_{\mathbb{K}}(c(w, x : y, z))|$, the valuation of the cross ratio $c(w, x : y, z)$ defined by

$$c(w, x : y, z) = \frac{(w - y)(x - z)}{(w - z)(x - y)} \tag{6.5}$$

(cf. [S14, Lemma 4.2]). Now, in regards of the above Tate uniformization (6.2), $-1 \in \mathbb{K}^\times$ acts on $\Gamma_{r,s}$ by switching two sides of the central line so that the quotient tree $\overline{\Gamma}_{r,s} := \Gamma_{r,s}/\langle \pm 1 \rangle$ (that corresponds to the projection image of $\Gamma_{r,s}$ by $\mathbb{K}^\times \to \mathbb{K}^\times$ by $t \mapsto t^2$) projects onto the dense image in $C(\mathrm{trop}(f_{r,s}))$.

**Lemma 6.1.** *Suppose that $C(\mathrm{trop}(f_{r,s}))$ is tropically smooth. Let $\mathbb{R}\overline{\Gamma}_{r,s}$ is the $\mathbb{R}$-tree naturally extended from the $\mathbb{Q}$-tree $\overline{\Gamma}_{r,s}$. Then the tropical curve $C(\mathrm{trop}(f_{r,s}))$ is isometric to $\mathbb{R}\overline{\Gamma}_{r,s}/\langle q^8 \rangle$.*

*Proof.* This follows easily from the argument in [S14, Sect. 7]. The smoothness assumption is used to apply [J20, Theorem 5.7] to see the fully faithfulness of the tropicalization map. □

6.2. **The set $Z$ of zeros of $\mathtt{x} \circ \wp, \mathtt{y} \circ \wp$.** The zeros of the function

$$\mathtt{y}(t) = \frac{\epsilon y - \bar{\epsilon}}{-sy + r} = \frac{\epsilon(q)\bar{\theta}_3(t, -q^2) - \bar{\epsilon}(q)\bar{\theta}_4(t, -q^2)}{-s(q)\bar{\theta}_3(t, -q^2) + r(q)\bar{\theta}_4(t, -q^2)} \tag{6.6}$$

(independent of the choice of $r, s$) are obtained from the equation $\bar{\epsilon}/\epsilon = \bar{\theta}_3(t, -q^2)/\bar{\theta}_4(t, -q^2)$ in $t$, which is equivalent to

$$\prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 + q^{2n-1}} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n-2}t^2)(1 - q^{4n-2}t^{-2})}{(1 + q^{4n-2}t^2)(1 + q^{4n-2}t^{-2})}.$$

It follows that $Z_{\mathtt{y}} = \{\pm q^{\pm\frac{1}{2}+4n} \mid n \in \mathbb{Z}\}$ so that $Z_{\mathtt{y}}$ has the image of cardinality 2 under the projection (6.2). By (6.3), we conclude

$$Z = Z_{\mathtt{y}} \cup qZ_{\mathtt{y}} = \{\pm q^{\pm\frac{e}{2}+4n} \mid n \in \mathbb{Z},\, e = \pm 1, 3\}. \tag{6.7}$$

6.3. **A smooth square case.** We first consider the case

$$\begin{cases} r(q) & = 1 - 3q, \\ s(q) & = -1 + q. \end{cases} \tag{6.8}$$

To investigate the set of poles of $\mathtt{y}(t)$, set

$$\Delta := -s(q)\bar{\theta}_3(t, -q^2) + r(q)\bar{\theta}_4(t, -q^2),$$

and compare it with a product of functions of the theta form

$$\Theta_{\xi,a} := \prod_{\substack{n\in\mathbb{Z} \\ 8n+a>0}} (1 + \xi q^{8n+a} t^2) \prod_{\substack{n\in\mathbb{Z} \\ 8n+a<0}} (1 + \xi^{-1} q^{-8n-a} t^{-2}) \quad (a \in \mathbb{Q}/8\mathbb{Z},\ \xi \in \mathbb{K}^\times).$$

Since $\mathtt{y}(t)$ has degree 2, we may assume $\Delta \sim \Theta_{\xi,a} \cdot \Theta_{\eta,b}$ for some $a, b \in \mathbb{Q}$, $\xi, \eta \in \mathbb{K}^\times$ (where $\sim$ means up to $\mathbb{K}^\times$) so that $P_{\mathtt{y}} = \{\pm\sqrt{-\xi^{-1}} q^{-\frac{a}{2}+4n}, \pm\sqrt{-\eta^{-1}} q^{-\frac{b}{2}+4n} \mid n \in \mathbb{Z}\}$. Comparing the logarithmic derivative

$$\begin{aligned} \frac{d}{dt}\log(\Delta) =& \frac{-2t^4+2}{t^3} q^3 + \frac{-4t^4+4}{t^3} q^4 + \frac{-8t^4+8}{t^3} q^5 + \frac{-2t^8 - 16t^6 + 16t^2 + 2}{t^5} q^6 \\ &+ \frac{-8t^8 - 32t^6 + 32t^2 + 8}{t^5} q^7 + \frac{-20t^8 - 64t^6 + 64t^2 + 20}{t^5} q^8 + \cdots \end{aligned}$$

with that of $\Theta_{\xi,a} \cdot \Theta_{\eta,b}$ successively from lower degree terms in $q$, we find

$$\begin{cases} a = 3,\ \xi = -(1 + 2q + 3q^2 + 10q^3 + 15q^4 + 38q^5 + 51q^6 + 162q^7 + \cdots), \\ b = 5,\ \eta = -(1 - 2q + q^2 - 6q^3 + 14q^4 - 28q^5 + 84q^6 - 232q^7 + \cdots), \end{cases}$$

where we only know $\xi, \eta$ as approximate values of $q$-expansions. By (6.3), we conclude

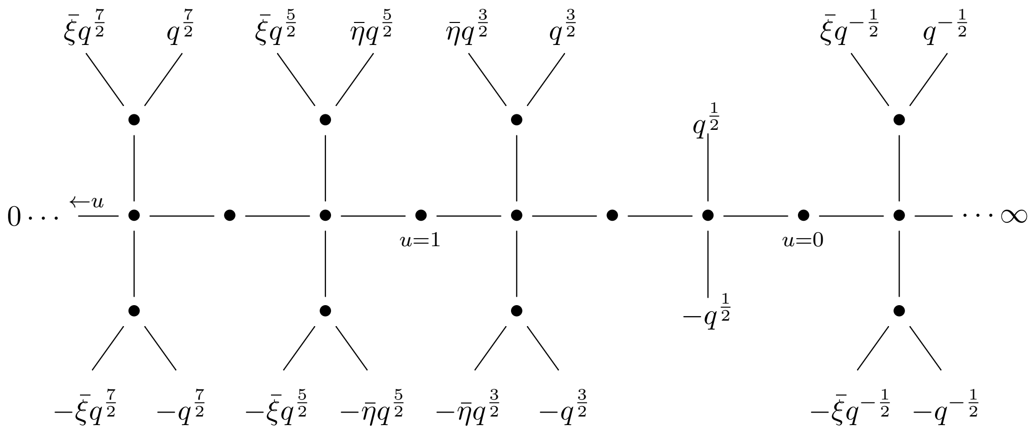
$$P = P_{\mathtt{y}} \cup qP_{\mathtt{y}} = \{\pm\bar{\xi} q^{\frac{5}{2}+4n}, \pm\bar{\eta} q^{\frac{3}{2}+4n}, \pm\bar{\xi} q^{\frac{7}{2}+4n}, \pm\bar{\eta} q^{\frac{5}{2}+4n} \mid n \in \mathbb{Z}\} \tag{6.9}$$

with

$$\begin{cases} \bar{\xi} := \sqrt{-\xi^{-1}} = 1 - q - 3q^3 + 4q^4 - 10q^5 + \frac{55}{2}q^6 - \frac{153}{2}q^7 + \cdots, \\ \bar{\eta} := \sqrt{-\eta^{-1}} = 1 + q + q^2 + 4q^3 + 3q^4 + 12q^5 + \frac{5}{2}q^6 + \frac{109}{2}q^7 + \cdots. \end{cases}$$

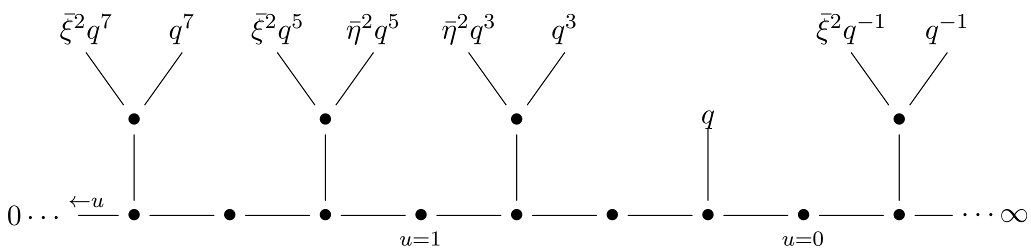
We compute various cross ratios (6.5) and their valuations in $v_{\mathbb{K}}$. For example, we have: the length of $[0, \bar{\xi}q^{-\frac{1}{2}}] \cap [\bar{\eta}q^{\frac{5}{2}}, \infty] = v_{\mathbb{K}}(c(0, \bar{\xi}q^{-\frac{1}{2}} : \bar{\eta}q^{\frac{5}{2}}, \infty)) = 3$; the length of $[\bar{\xi}q^{-\frac{1}{2}}, \bar{\xi}q^{\frac{5}{2}}] \cap [\bar{\eta}q^{\frac{5}{2}}, \infty] = v_{\mathbb{K}}(c(\bar{\xi}q^{-\frac{1}{2}}, \bar{\xi}q^{\frac{5}{2}} : \bar{\eta}q^{\frac{5}{2}}, \infty)) = 4$ etc. Eventually, we find the shape of $\Gamma_{r,s}$ as in the following picture, where each of the vertical (resp. horizontal) internal edges between two adjacent $\bullet$'s has length one (resp. $\frac{1}{2}$).

(6.10) $\Gamma_{r,s}$ :




Noting that $-1 \in \mathbb{K}^\times$ acts by switching the upper and lower sides of the central line, we see that the quotient of the above tree by $\langle \pm 1 \rangle$ (that corresponds to the projection $\mathbb{K}^\times \to \mathbb{K}^\times$ by $t \mapsto t^2$) forms the following tree:

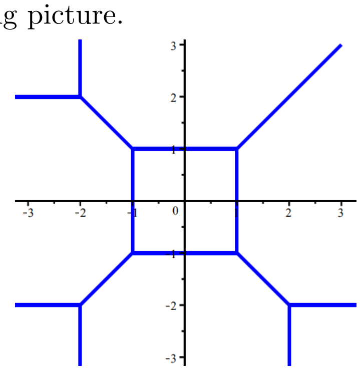
(6.11) $\overline{\Gamma}_{r,s}$ $(\cong \Gamma_{r,s}/\langle \pm 1 \rangle)$ :




where the central $u$-line acquires the double metric of the original one while the other edges keeps the original lengths: Consequently each of the internal edges between two adjacent $\bullet$'s has length one. Thus, after the Tate uniformization

$$\mathbb{K}^\times \xrightarrow{\wp} \mathbb{K}^\times/\langle \pm q^4 \rangle = E_{r,s}(\mathbb{K}) \underset{t \mapsto t^2}{\xrightarrow{\sim}} \mathbb{K}^\times/\langle q^8 \rangle,$$

we find the tropical curve $C(\mathrm{trop}(f_{r,s}))$ is isometric to the quotient of the above tree modulo $\langle q^8 \rangle$ as shown in the following picture.



6.4. **A heptagon case.** We next consider the case

$$\begin{cases} r(q) &= 1+q^{\frac{3}{2}}, \\ s(q) &= -1+q^{\frac{3}{2}}. \end{cases} \tag{6.12}$$

We begin by investigating the set of poles of $\mathtt{y}(t)$ by setting

$$\Delta := -s(q)\bar{\theta}_3(t,-q^2)+r(q)\bar{\theta}_4(t,-q^2),$$

and compare it with a product of theta functions of the form $\Theta_{\xi,a}\Theta_{\eta,b}$. By analogous consideration to the above square case, in this heptagon case, we are led to finding $a=\frac{7}{2}, b=\frac{9}{2}$ and comparison of the form $\Delta \sim \Theta_{\xi,\frac{7}{2}} \cdot \Theta_{\eta,\frac{9}{2}}$ for some $\xi, \eta \in \mathbb{K}^\times$ so that

$$P_{\mathtt{y}} = \{\pm\sqrt{-\xi^{-1}}q^{-\frac{7}{4}+4n}, \pm\sqrt{-\eta^{-1}}q^{-\frac{9}{4}+4n} \mid n \in \mathbb{Z}\}.$$

Comparing the logarithmic derivative

$$\begin{aligned} \frac{d}{dt}\log(\Delta) =& \frac{2t^4-2}{t^3}q^{\frac{7}{2}} + \frac{-2t^8+2}{t^5}q^7 + \frac{4t^8-4}{t^5}q^8 + \frac{2t^{12}+2t^8-2t^4-2}{t^7}q^{\frac{21}{2}} \\ &+ \frac{-6t^{12}-2t^8+2t^4+6}{t^7}q^{\frac{23}{2}} - \frac{2(t^4-1)(t^4+1)^3}{t^9}q^{14} + \cdots \end{aligned}$$

with that of $\Theta_{\xi,a} \cdot \Theta_{\eta,b}$ successively from lower degree terms in $q$, we find

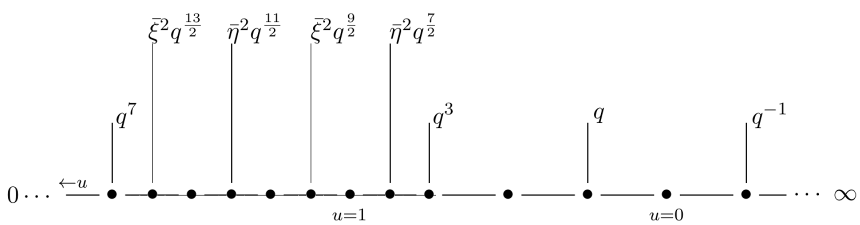
$$\begin{cases} a=\frac{7}{2},\ \xi = -(1+q+q^2+2q^3+2q^4+5q^5+42q^6+131q^7+\cdots), \\ b=\frac{9}{2},\ \eta = 1+q+2q^2+5q^3+14q^4+42q^5+132q^6+428q^7+\cdots. \end{cases}$$

We may skip computing cross ratios since internal edges disappear in $\Gamma_{r,s}$ in this case. Consequently $\overline{\Gamma}_{r,s}$ has external rays from the central line

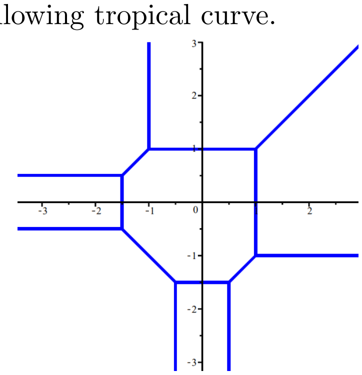
(i) for zeros at $\{q^{-1+8\mathbb{Z}}, q^{1+8\mathbb{Z}}, q^{3+8\mathbb{Z}}\}$, and

(ii) for poles at $\{\bar{\eta}^2 q^{\frac{7}{2}+8\mathbb{Z}}, \bar{\xi}^2 q^{\frac{9}{2}+8\mathbb{Z}}, \bar{\eta}^2 q^{\frac{11}{2}+8\mathbb{Z}}, \bar{\xi}^2 q^{\frac{13}{2}+8\mathbb{Z}}\}$,

where $\bar{\xi} := \sqrt{-\xi^{-1}}$, $\bar{\eta} := \sqrt{-\eta^{-1}}$. It follows then that $\mathbb{R}\overline{\Gamma}_{r,s}$ looks like the following tree:




whose quotient modulo $\langle q^8 \rangle$ projects onto the tropical curve $C(\mathrm{trop}(f_{r,s}))$. We observe that it is indeed isometric to the following tropical curve.



## References


[A84] G.E.Andrews, *The theory of partitions*, Cambridge UP, 1984.

[BL07] D.J.Bernstin, T.Lange, *Faster addition and doubling on elliptic curves*, Lecture Notes in Computer Science, **4833** (2007), 29–50.

[CS] M.Chan, B.Sturmfels, *Elliptic curves in honeycomb form*, in "Algebraic and combinatorial aspects of tropical geometry" , Contemp. Math., **589** (2013), 87–107.

[E07] H.M.Edwards, *A normal form for elliptic curves*, Bull. Amer. Math. Soc. **44** (2007), 393–422.

[EKL06] M.Einsiedler, M.Kapranov, D.Lind, *Non-archimedean amoebas and tropical varieties*, J. reine angew. Math. **601** (2006), 139–157.

[F1916] R.Fricke, *Die Elliptischen Funktionen und ihre anwndungen*, Erster Teil, Teubner 1916.

[GS14] D.Grigoriev, V.Shpilrain, *Tropical cryptography*, Communications in Algebra, **42** (2014), 2624–2632.

[J20] P.Jell, *Constructing smooth and fully faithful tropicalizations for Mumford curves*, Selecta Math. (2020) 26:60.

[KL06] A.Kasman, S.Lafortune, *When is negativity not a problem for the ultradiscrete limit?*, J. Math. Phys. **47** (2006), 103510.

[KKNT] K.Kajiwara, M.Kaneko, A.Nobe, T.Tsuda, *Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve*, Kyushu J. Math. **63** (2009) 315–338.

[KNT08] K.Kajiwara, A.Nobe, T.Tsuda, *Ultradiscretization of solvable one-dimensional chaotic maps*, J. Phys. A: Math. Theore. **41** (2008) 395202.

[KMM] E.Katz, H.Markwig, T.Markwig, *The tropical $j$-invariant*, LMS J. Comput. Math. **12** (2009), 275–294.

[L10] D.F.Lawden, *Elliptic Functions and Applications*, Springer 2010.

[M] Maplesoft, a division of Waterloo Maple Inc.., 2019. Maple, Waterloo, Ontario.

[M06] D.Mumford, *Tata Lectures on Theta, I*, Birkhäuser 2006.

[N08] A.Nobe, *Ultradiscrete QRT maps and tropical elliptic curves*, J. Phys. A **41** (2008), no. 12, 125205 (12 pp).

[S14] D.E.Speyer, *Parameterizing tropical curves I: Curves of genus zero and one*, Algebra and Number theory, **8:4** (2014).

[T] R.S.Tarmidi, *Note on smoothness condition on tropicall elliptic curves of symmetric truncated cubic forms*, Math. J. Okayama Univ. (to appear).

[V09] M.D.Vigeland, *The group law on a tropical, elliptic curve*, Math. Scand. **104** (2009), 188–204.



Hiroaki Nakamura: Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

*Email address*: nakamura@math.sci.osaka-u.ac.jp

Rani Sasmita Tarmidi: Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

*Email address*: u644627d@ecs.osaka-u.ac.jp, ranitarmidi@yahoo.com