# The automorphism groups of the profinite braid groups

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Abstract. In this paper we determine the automorphism groups of the profinite braid groups with four or more strings in terms of the profinite Grothendieck-Teichmüller group.

Contents



1. **Introduction.** Let $B_n$ be the Artin braid group with $n(\geq 2)$ strings defined by generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and relations:

- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (i = 1, \dots, n-1)$,
- $\sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| \geq 2)$.

In [DG], J. L. Dyer and E. K. Grossman studied the automorphism group $\mathrm{Aut}(B_n)$ and showed $\mathrm{Out}(B_n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. In this paper, we study the continuous automorphisms of the profinite completion $\widehat{B}_n$ of $B_n$. We prove

**Theorem A.** *Let $n \geq 4$. There exists a natural isomorphism*

$$\mathrm{Out}(\widehat{B}_n) \cong \widehat{\mathrm{GT}} \times (1 + n(n-1)\widehat{\mathbb{Z}})^\times,$$

*where $\widehat{\mathrm{GT}}$ is the profinite Grothendieck-Teichmüller group introduced by V. Drinfeld* [Dr], *Y. Ihara* [I90]-[I95] *and $(1 + n(n-1)\widehat{\mathbb{Z}})^\times$ is the kernel of the natural projection $\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/n(n-1)\mathbb{Z})^\times$.*

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It is well known that the center $\widehat{C}_n$ of $\widehat{B}_n$ is (topologically) generated by $\zeta_n := (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ and is isomorphic to $\widehat{\mathbb{Z}}$. Write

$$\widehat{\mathcal{B}}_n \ := \ \widehat{B}_n/\widehat{C}_n.$$

Since $\widehat{C}_n$ is a characteristic subgroup of $\widehat{B}_n$, there is induced the natural homomorphism $\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{\mathcal{B}}_n)$. The key fact for the proof of Theorem A is the following isomorphism theorem.

**Theorem B** (Theorem 4.3)**.** *Let $n \geq 4$. Then, it holds that* $\widehat{\mathrm{GT}} \overset{\sim}{\to} \mathrm{Out}(\widehat{\mathcal{B}}_n)$.

Our proofs of Theorems A and B rely on preceding works by many authors on the Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$ and the profinite completion $\widehat{\Gamma}_{0,n}$ of the mapping class group $\Gamma_{0,n}$ of the sphere with $n$ marked points (cf. [I95], [LS1], [LS2], [C12]). The permutation of labels defines a natural inclusion of the symmetric group of degree $n$: $\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\widehat{\Gamma}_{0,n})$, whose image commutes with the standard action of $\widehat{\mathrm{GT}}$ on $\widehat{\Gamma}_{0,n}$ ([I95]). D.Harbater and L.Schneps [HS] remarkably showed that when $n \geq 5$, $\widehat{\mathrm{GT}}$ is characterized as a "special" subgroup of the centralizer of $\mathfrak{S}_n$ in $\mathrm{Out}(\widehat{\Gamma}_{0,n})$. In a recent work [HMM], this result has been improved by showing that the focused centralizer is indeed *full* as large as possible in $\mathrm{Out}(\widehat{\Gamma}_{0,n})$. In particular,

**Theorem 1.1** (Hoshi-Minamide-Mochizuki [HMM] Corollary C)**.** There is a natural isomorphism of profinite groups

$$\widehat{\mathrm{GT}} \times \mathfrak{S}_{n+1} \ \overset{\sim}{\to} \ \mathrm{Out}(\widehat{\Gamma}_{0,n+1})$$

for every integer $n \geq 4$. □

Theorems A and B will be derived by translating the ingredient of Theorem 1.1 for $\mathrm{Out}(\widehat{\Gamma}_{0,n+1})$ into the language of $\mathrm{Out}(\widehat{\mathcal{B}}_n)$ and $\mathrm{Out}(\widehat{B}_n)$. Arguments given by Dyer-Grossman [DG] for discrete braid groups generically guide us also in profinite context. However, for the case $n = 4$, we elaborate a different treatment in Section 3 due to the existence of non-standard surjections $B_4 \twoheadrightarrow \mathfrak{S}_4$ found in E. Artin's classic [A47]. Our argument in Section 3 looks at the "Cardano-Ferrari" homomorphism $B_4 \twoheadrightarrow B_3$ which has close relations with the universal monodromy representation in once-punctured elliptic curves. Noting that $\mathcal{B}_4$ is isomorphic to the mapping class group $\Gamma_{1,2}$ of a topological torus with two marked points, we obtain from Theorem B the following remarkable

**Corollary C.** *There is a natural isomorphism* $\widehat{\mathrm{GT}} \overset{\sim}{\to} \mathrm{Out}(\widehat{\Gamma}_{1,2})$. □

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2. **Generalities on braid groups.** We begin with recalling basic facts on braid groups (cf. e.g., [KT]). Let $n \geq 3$ be an integer. The pure braid group $P_n$ is the kernel of the epimorphism

$$\begin{aligned} \varpi_n : B_n &\twoheadrightarrow \mathfrak{S}_n \\ \sigma_i &\mapsto (i, i+1) \qquad (i = 1, \dots, n-1). \end{aligned}$$

The center $C_n$ of $P_n$ coincides with the center of $B_n$ which is a free cyclic group generated by

$$\zeta_n := (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n.$$

Write $\mathcal{P}_n := P_n / C_n$ and $\mathcal{B}_n := B_n / C_n$. The above $\varpi_n$ factors through $\pi_n : \mathcal{B}_n \twoheadrightarrow \mathfrak{S}_n$ and there arise the following exact sequences of finitely generated groups:

$$1 \longrightarrow \mathcal{P}_n \longrightarrow \mathcal{B}_n \xrightarrow{\pi_n} \mathfrak{S}_n \longrightarrow 1, \tag{2.1}$$

$$1 \longrightarrow C_n \longrightarrow B_n \longrightarrow \mathcal{B}_n \longrightarrow 1, \tag{2.2}$$

$$1 \longrightarrow C_n \longrightarrow P_n \longrightarrow \mathcal{P}_n \longrightarrow 1. \tag{2.3}$$

We introduce the *mapping class group of the $n$-times punctured sphere* $\Gamma_{0,[n]}$ to be the group generated by $\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_{n-1}$ with the relations

- $\bar{\sigma}_i \bar{\sigma}_{i+1} \bar{\sigma}_i = \bar{\sigma}_{i+1} \bar{\sigma}_i \bar{\sigma}_{i+1} \quad (i = 1, \dots, n-1)$,
- $\bar{\sigma}_i \bar{\sigma}_j = \bar{\sigma}_j \bar{\sigma}_i \quad (|i - j| \geq 2)$,
- $\bar{\sigma}_1 \cdots \bar{\sigma}_{n-2} \bar{\sigma}_{n-1}^2 \bar{\sigma}_{n-2} \cdots \bar{\sigma}_1 = 1$,
- $(\bar{\sigma}_1 \bar{\sigma}_2 \cdots \bar{\sigma}_{n-1})^n = 1$.

Observe that there is a natural epimorphism

$$\begin{aligned} \Psi_n : B_n &\twoheadrightarrow \Gamma_{0,[n]} \\ \sigma_i &\mapsto \bar{\sigma}_i \qquad (i = 1, \dots, n-1) \end{aligned} \tag{2.4}$$

which factors through $\mathcal{B}_n = B_n / C_n$. We also write $\Gamma_{0,n}$ for the *pure mapping class group of the $n$-times punctured sphere* which is by definition the kernel of the epimorphism

$$\begin{aligned} \gamma_n : \Gamma_{0,[n]} &\twoheadrightarrow \mathfrak{S}_n \\ \bar{\sigma}_i &\mapsto (i, i+1) \qquad (i = 1, \dots, n-1) \end{aligned} \tag{2.5}$$

fitting in the exact sequence

$$1 \longrightarrow \Gamma_{0,n} \longrightarrow \Gamma_{0,[n]} \longrightarrow \mathfrak{S}_n \longrightarrow 1. \tag{2.6}$$

In this paper, besides the above epimorphism $\Psi_n$ (2.4), another shifted morphism

$$\begin{aligned} \Phi_n : B_n &\to \Gamma_{0,[n+1]} \\ \sigma_i &\mapsto \bar{\sigma}_i \qquad (i = 1, \dots, n-1) \end{aligned} \tag{2.7}$$

plays an important role, whose kernel is known to coincide with $C_n$ ([FM, Section 9.2-3]). The homomorphism $\Phi_n$ induces the following commutative diagram of groups

$$
\begin{tikzcd}
1 \arrow[r] & \mathcal{P}_n \arrow[r] \arrow[d, "\wr"] & \mathcal{B}_n \arrow[r, "\pi_n"] \arrow[d, hook] & \mathfrak{S}_n \arrow[r] \arrow[d, hook, "\iota_n"] & 1 \\
1 \arrow[r] & \Gamma_{0,n+1} \arrow[r] & \Gamma_{0,[n+1]} \arrow[r, "\gamma_{n+1}"] & \mathfrak{S}_{n+1} \arrow[r] & 1,
\end{tikzcd}
\tag{2.8}
$$

where the horizontal sequences are exact; the left-hand (resp. middle; right-hand) vertical arrow is the isomorphism (resp. the injection; the natural injection which trivially extends each permutation of $\{1,2,\ldots,n\}$ to that of $\{1,2,\ldots,n+1\}$) induced from $\Phi_n$.

It is well known that the profinite completion functor preserves the (injectivity of the) kernel part of the exact sequences (2.1)-(2.3) and (2.6) respectively. If $Z(G)$ denotes the center of a profinite group $G$, then

$$
\begin{cases} Z(\mathcal{P}_n) = Z(\mathcal{B}_n) = Z(\widehat{\mathcal{P}}_n) = Z(\widehat{\mathcal{B}}_n) = \{1\}, \\ \widehat{C}_n = Z(\widehat{P}_n) = Z(\widehat{B}_n) \ (\cong \widehat{\mathbb{Z}}). \end{cases}
\tag{2.9}
$$

hold (cf. e.g., [N94, Section 1.2-1.3]).

**Definition 2.1.** Let $n \geq 3$ be an integer. We shall write $(*)$ for the commutative diagram of profinite groups

$$
\begin{tikzcd}
1 \arrow[r] & \widehat{\mathcal{P}}_n \arrow[r] \arrow[d, "\wr"] & \widehat{\mathcal{B}}_n \arrow[r, "\widehat{\pi}_n"] \arrow[d, hook] & \mathfrak{S}_n \arrow[r] \arrow[d, hook, "\iota_n"] & 1 \\
1 \arrow[r] & \widehat{\Gamma}_{0,n+1} \arrow[r] & \widehat{\Gamma}_{0,[n+1]} \arrow[r, "\widehat{\gamma}_{n+1}"] & \mathfrak{S}_{n+1} \arrow[r] & 1
\end{tikzcd}
\tag{$*$}
$$

which is obtained as the profinite completion of (2.8). Note that the horizontal sequences are exact as remarked as above.

**Proposition 2.2.** Suppose that $n \neq 4$, $n \geq 3$. Then every epimorphism $\widehat{\mathcal{B}}_n \twoheadrightarrow \mathfrak{S}_n$ has kernel $\widehat{\mathcal{P}}_n$. In particular, $\widehat{\mathcal{P}}_n$ is a characteristic subgroup of $\widehat{\mathcal{B}}_n$.

*Proof.* E.Artin ([A47, Theorem 1]) classified all surjective homomorphisms $B_n \twoheadrightarrow \mathfrak{S}_n$ up to equivalence by conjugation in $\mathfrak{S}_n$: When $n \neq 4, 6$, there is a unique equivalence class and when $n = 6$ there are two classes mutually equivalent by a nontrivial outer automorphism of $\mathfrak{S}_6$. This proves the assertion for discrete braid groups. Lemma 2.3 below with the residual finiteness of $\mathcal{B}_n$ settles the assertion for the profinite braid groups. □

**Lemma 2.3.** Let $G$ be a residually finite group, $N$ a normal subgroup of $G$ with finite quotient $Q := G/N$. Suppose that every epimorphism $G \twoheadrightarrow Q$ has the same kernel $N$. Then, every epimorphism $\widehat{G} \twoheadrightarrow Q$ has the same kernel $\widehat{N}$.

*Proof.* Note first that, by one-to-one correspondence between the finite index subgroups of $G$ and the open subgroups of $\widehat{G}$, the image of the monomorphism $\widehat{N} \to \widehat{G}$ coincides with the closure of $N$ in $\widehat{G}$. Let $p : \widehat{G} \twoheadrightarrow Q$ be a given epimorphism. Then, by [RZ, Proposition 3.2.2 (a)], the closure of $H := \ker(p) \cap G$ in $\widehat{G}$ coincides with $\ker(p)$. Consider the composite:

$$\varphi : G \twoheadrightarrow G/H \overset{\sim}{\to} \widehat{G}/\ker(p) \overset{\sim}{\to} Q,$$

where the first arrow is the projection, the second arrow is the isomorphism induced from the associated morphism $G \hookrightarrow \widehat{G}$ ([RZ, Proposition 3.2.2 (d)]) and the third arrow is the isomorphism induced by $p$. From the assumption, $\varphi$ has the kernel $N$, i.e., $N = \ker(\varphi) = H$. Thus, $\ker(p)$ coincides with $\widehat{N}$. $\square$

3. **Special case $\widehat{\mathcal{B}}_4$.** The main aim of this section is to provide a proof of the following

**Proposition 3.1.** $\widehat{\mathcal{P}}_4$ is a characteristic subgroup of $\widehat{\mathcal{B}}_4$.

In the proof of [DG, Theorem 11] claiming that $\mathcal{P}_n$ is characteristic in $\mathcal{B}_n$ for $n \geq 3$, we find an inaccurate argument for the case $n = 4$: By E. Artin's classic work ([A47, Theorem 1]), each surjective homomorphism $B_4 \twoheadrightarrow \mathfrak{S}_4$ is equivalent to one of the following $\epsilon_1, \epsilon_2, \epsilon_3$ up to change of labels in $\{1, 2, 3, 4\}$:

$$\begin{aligned}
\epsilon_1 : B_4 \twoheadrightarrow \mathfrak{S}_4 \qquad & (\sigma_1 \mapsto (12),\ \sigma_2 \mapsto (23),\ \sigma_3 \mapsto (34));\\
\epsilon_2 : B_4 \twoheadrightarrow \mathfrak{S}_4 \qquad & (\sigma_1 \mapsto (1234),\ \sigma_2 \mapsto (2134),\ \sigma_3 \mapsto (1234));\\
\epsilon_3 : B_4 \twoheadrightarrow \mathfrak{S}_4 \qquad & (\sigma_1 \mapsto (1234),\ \sigma_2 \mapsto (2134),\ \sigma_3 \mapsto (4321)).
\end{aligned}$$

Among them, $\ker(\epsilon_1) = P_4$, while neither $\ker(\epsilon_2)$ or $\ker(\epsilon_3)$ equals to $P_4$, for $\sigma_1^2 \in P_4$ has non-trivial images in $\mathfrak{S}_4$: $\epsilon_2(\sigma_1^2) = \epsilon_3(\sigma_1^2) = (13)(24)$.

Let $\bar{\epsilon}_1 : \mathcal{B}_4 \twoheadrightarrow \mathfrak{S}_4$ be the induced map. Given an arbitrary automorphism $\phi \in \mathrm{Aut}(\mathcal{B}_4)$, consider the composite

$$\epsilon_\phi : B_4 \to B_4/C_4 = \mathcal{B}_4 \underset{\phi}{\overset{\sim}{\to}} \mathcal{B}_4 \xrightarrow{\bar{\epsilon}_1} \mathfrak{S}_4.$$

Dyer-Grossman [DG, p.1159] discusses that $\epsilon_\phi$ cannot be equivalent to $\epsilon_2$, for $(\bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_3)^2$ has order exactly two in $\mathcal{B}_4$ hence does not belong to $\mathcal{P}_4$ (torsion-free), while $\epsilon_2((\sigma_1\sigma_2\sigma_3)^2) = 1$. If moreover one knew $\epsilon_\phi \not\sim \epsilon_3$, then one could get $\epsilon_\phi \sim \epsilon_1$ and hence $\phi(\mathcal{P}_4) = \mathcal{P}_4$ so as to conclude Proposition 3.1. However, in [DG], apparently omitted is a discussion about $\epsilon_3$ as the existence of $\epsilon_3$ is already missed in their citation of Artin's theorem in [DG, Theorem 2]. Since $\epsilon_3((\sigma_1\sigma_2\sigma_3)^2) = (12)(34) \neq 1$, a simple replacement of the above argument for $\epsilon_\phi \not\sim \epsilon_2$ does not work to eliminate another possibility $\epsilon_\phi \sim \epsilon_3$.

The fact that $\mathcal{P}_4$ is a characteristic subgroup of $\mathcal{B}_4$ has followed in a different approach by topologists (see, e.g., [Ko, Theorem 3]) by using finer analysis of the mapping class group action on the complex of curves $C(S)$ on a topological surface $S$. However, a profinite variant of $C(S)$ to derive Proposition 3.1 still remains unsettled even to this day. Below, we give an alternative argument looking closely at a family of characteristic subgroups of $\mathcal{B}_4$. We argue in the profinite context, however, our discussion works also for the discrete case in the obvious interpretation. Our main targets arise from the following epimorphisms $b_{43} : \widehat{\mathcal{B}}_4 \twoheadrightarrow \widehat{\mathcal{B}}_3$ and $s_{43} : \mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3$ defined by

$$(3.1) \qquad b_{43} : \widehat{\mathcal{B}}_4 \twoheadrightarrow \widehat{\mathcal{B}}_3 : \begin{cases} \bar{\sigma}_1, \bar{\sigma}_3 & \mapsto \bar{\sigma}_1, \\ \quad \bar{\sigma}_2 & \mapsto \bar{\sigma}_2; \end{cases} \quad s_{43} : \mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3 : \begin{cases} (12), (34) & \mapsto (12), \\ \quad (23) & \mapsto (23), \end{cases}$$

and the composition

$$(3.2) \qquad \mathfrak{P} := \widehat{\pi}_3 \circ b_{43} \, (= s_{43} \circ \widehat{\pi}_4) : \widehat{\mathcal{B}}_4 \twoheadrightarrow \mathfrak{S}_3$$

where $\widehat{\pi}_n : \widehat{\mathcal{B}}_n \twoheadrightarrow \mathfrak{S}_n$ is as in the previous section. The kernel of $s_{43}$ is what is called the Klein four group

$$V_4 \ := \ \ker(s_{43}) \ = \ \{id, (12)(34), (13)(24), (14)(23)\} \subset \mathfrak{S}_4.$$

Denote by $p_{43} : \widehat{\mathcal{P}}_4 \to \widehat{\mathcal{P}}_3$ the restriction of $b_{43} : \widehat{\mathcal{B}}_4 \to \widehat{\mathcal{B}}_3$ and write $\widehat{\Pi}_{0,4} := \ker(p_{43})$. We note that $p_{43}$ is not the same as the usual homomorphism obtained by forgetting one strand of pure 4-braids. These maps fit into the following commutative diagram of horizontal and vertical exact sequences:

$$(3.3) \qquad \begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\ 1 \arrow[r] & \widehat{\Pi}_{0,4} \arrow[r] \arrow[d] & \ker(b_{43}) \arrow[r] \arrow[d] & V_4 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \widehat{\mathcal{P}}_4 \arrow[r] \arrow[d, "p_{43}"] & \widehat{\mathcal{B}}_4 \arrow[r, "\widehat{\pi}_4"] \arrow[d, "b_{43}"'] \arrow[dr, "\mathfrak{P}"'] & \mathfrak{S}_4 \arrow[r] \arrow[d, "s_{43}"] & 1 \\ 1 \arrow[r] & \widehat{\mathcal{P}}_3 \arrow[r] \arrow[d] & \widehat{\mathcal{B}}_3 \arrow[r, "\widehat{\pi}_3"'] \arrow[d] & \mathfrak{S}_3 \arrow[r] \arrow[d] & 1 \\ & 1 & 1 & 1 & . \end{tikzcd}$$

Concerning the two sequences of subgroups $\widehat{\mathcal{B}}_4 \supset \ker(b_{43}) \supset \widehat{\Pi}_{0,4}$ and $\widehat{\mathcal{B}}_4 \supset \ker(\mathfrak{P}) \supset \widehat{\mathcal{P}}_4$, we shall prove

**Proposition 3.2.** (i) $\widehat{\Pi}_{0,4}$ is a characteristic subgroup of $\ker(b_{43})$.
(ii) $\ker(\mathfrak{P})$ is a characteristic subgroup of $\widehat{\mathcal{B}}_4$.
(iii) $\ker(b_{43})$ is a characteristic subgroup of $\widehat{\mathcal{B}}_4$.
(iv) $\widehat{\Pi}_{0,4}$ is a characteristic subgroup of $\widehat{\mathcal{B}}_4$.
(v) $\widehat{\mathcal{P}}_4$ is a characteristic subgroup of $\widehat{\mathcal{B}}_4$.

Proposition 3.1 is obtained as (v) of the above Proposition. Here is a simple immediate consequence of it:

**Corollary 3.3.** $\widehat{P}_n$ is a charactersitic subgroup of $\widehat{B}_n$ for every $n \geq 3$.

*Proof.* Proposition 2.2 and Proposition 3.1 show that $\widehat{\mathcal{P}}_n$ is a characteristic subgroup of $\widehat{\mathcal{B}}_n$ for every $n \geq 3$. Assertion follows from this and the fact that $\widehat{P}_n$ is the inverse image of $\widehat{\mathcal{P}}_n$ by the projection $\widehat{B}_n \twoheadrightarrow \widehat{\mathcal{B}}_n$ whose kernel is the center $\widehat{C}_n$ of $\widehat{B}_n$. □

For the proof of Proposition 3.2, note first that (iv) follows from (i) and (iii). We will apply (iv) for the proof of (v). Assertion (ii) will be used to prove (iii). In fact, (ii) follows from a stronger assertion that every epimorphism $\widehat{\mathcal{B}}_4 \twoheadrightarrow \mathfrak{S}_3$ has the same kernel as $\ker(\mathfrak{P})$. In fact, it is not difficult to see that every (discrete group) homomorphism $B_4 \twoheadrightarrow \mathfrak{S}_3$ is conjugate to the standard one $B_4 \twoheadrightarrow B_3 \twoheadrightarrow \mathfrak{S}_3$ (cf. e.g., [Lin, Theorem 3.19 (a)]). Since $\mathcal{B}_4$ is residually finite, the profinite version follows from Lemma 2.3. To complete the proof of Proposition 3.2, it remains to prove (i), (iii) and (v).

*Proof of Proposition* 3.2 (i): Let us begin with geometric interpretation of $\widehat{\Pi}_{0,4} \subset \ker(b_{43})$ which has been well studied by topologists (see, e.g., [ASWY, Section 2.1], [KS, Section 3]). One may regard the standard lift $\beta_{43} : \widehat{B}_4 \twoheadrightarrow \widehat{B}_3$ of $b_{43} : \widehat{\mathcal{B}}_4 \to \widehat{\mathcal{B}}_3$ (given by $\sigma_1, \sigma_2, \sigma_3 \mapsto \sigma_1, \sigma_2, \sigma_1$ respectively) as the $\pi_1^{\text{ét}}$-transform of the "Cardano-Ferrari mapping $\mathfrak{F}_0 : (\mathbb{A}^4 \setminus D)_0 \to (\mathbb{A}^3 \setminus D)_0$" assigning to a monic quartic (with no multiple zeros) its cubic resolvent (in the notations of [N13, Section 5.4]). The kernel of $\beta_{43}$ is isomorphic to the free profinite group $\widehat{F}_2$ of rank 2. In fact, after Mordell transformation, the homomorphism $\beta_{43} = \pi_1^{\text{ét}}(\mathfrak{F}_0)$ turns to interpret the monodromy of the universal family of the (affine part of) elliptic curves

$$\begin{array}{c} \mathcal{E} \setminus \{O\} = \{Y^2 = 4X^3 - g_2X - g_3\} \\ \Big\downarrow \\ \mathfrak{M}_{1,1}^{\omega} = \{(g_2, g_3) \mid \Delta \neq 0\}. \end{array}$$

Let $\sqrt{\zeta_4} := (\sigma_1\sigma_2\sigma_3)^2$ so that $\beta_{43}(\sqrt{\zeta_4}) = \zeta_3 \in B_3$. Then, the reduced sequence

$$(3.4) \qquad 1 \longrightarrow F_2 \longrightarrow B_4/\langle\sqrt{\zeta_4}\rangle \longrightarrow B_3/\langle\zeta_3\rangle = \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow 1$$

fits in the orbifold quotient of the complex model of elliptic curve family over the upper half plane. Taking into account that $\sqrt{\zeta_4}$ (mod $\langle\zeta_4\rangle$) acts on each elliptic curve $E : Y^2 = 4X^3 - g_2X - g_3$ by the switching $\pm Y$ involution, we see that $\ker(b_{43})$ can be regarded as the fundamental group of an orbicurve $\mathbb{P}^1_{\infty,2,2,2}$ obtained as the $X$-line from $(E\setminus\{O\})/\{\pm 1\}$; it turns out to be isomorphic to the profinite free product of three copies of $\mathbb{Z}/2\mathbb{Z}$:

$$\ker(b_{43}) = \pi_1^{\text{ét}}(\mathbb{P}^1_{\infty,2,2,2}/\mathbb{C}) = (\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z}) \tag{3.5}$$

which may also be regarded as the profinite completion of discrete free product $(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})$ ([RZ, Section 9.1]). The normal subgroup $\widehat{\Pi}_{0,4}$ of $\ker(b_{43})$ corresponds to the fundamental group of the Galois cover of $\mathbb{P}^1_{\infty,2,2,2}$ with group $V_4$ given in the Lattés cover diagram:

$$\begin{tikzcd} E\setminus E[2] \arrow[r] \arrow[d] & \mathbb{P}^1 - \{e_0,e_1,e_2,e_3\} \arrow[d] \\ E\setminus\{O\} \arrow[r] & \mathbb{P}^1_{\infty,2,2,2} \end{tikzcd} \tag{3.6}$$

where the left vertical arrow is the isogeny of punctured elliptic curves by multiplication by 2, and horizontal arrows correspond to the $\{\pm 1\}$-quotients. From this we obtain a cartesian diagram of profinite groups:

$$\begin{tikzcd} \ker(b_{43}) = (\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z}) \arrow[r, two heads] & (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \\ \widehat{\Pi}_{0,4} \arrow[u, hook] \arrow[r, two heads] & (\mathbb{Z}/2\mathbb{Z}), \arrow[u, hook, "\text{diagonal map}"'] \end{tikzcd} \tag{3.7}$$

where the upper horizontal arrow is the abelianization map. Moreover, according to Herfort-Ribes ([HR, Theorem 2 (i)]), the torsion elements of $(\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z})$ form exactly the three conjugacy classes of order two which, therefore, must be preserved as a set under $\mathrm{Aut}(\ker(b_{43}))$. This characterizes the diagonal image of $(\mathbb{Z}/2\mathbb{Z})$ in the right hand side of (3.7). Thus we conclude that $\widehat{\Pi}_{0,4}$ is characteristic in $\ker(b_{43})$ as the pull-back image of $(\mathbb{Z}/2\mathbb{Z}) \overset{diag.}{\hookrightarrow} (\mathbb{Z}/2\mathbb{Z})^3$ along the abelianization of $\ker(b_{43})$. $\square$

*Proof of Proposition* 3.2 (iii): To prove (iii), pick any $\phi \in \mathrm{Aut}(\widehat{\mathcal{B}}_4)$. We first show that $\phi(\ker(b_{43})) \subset \ker(b_{43})$. As $\ker(\mathfrak{P})$ is characteristic in $\widehat{\mathcal{B}}_4$ as shown in (ii), it follows that $\phi(\ker(b_{43})) \subset \ker(\mathfrak{P})$. Hence $b_{43}$ maps $\phi(\ker(b_{43}))$ onto a subgroup of $\widehat{\mathcal{P}}_3(\overset{\sim}{\to} \widehat{\Gamma}_{0,4} \cong \widehat{F}_2)$. But $\phi(\ker(b_{43}))$ is isomorphic to $\ker(b_{43})$ which is a topologically finitely generated closed normal subgroup of $\widehat{\mathcal{B}}_4$. Since $\widehat{F}_2$ has no nontrivial non-free finitely generated normal subgroups ([LvD, Corollary 3.14]) and since $\ker(b_{43}) \cong (\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z}) \amalg (\mathbb{Z}/2\mathbb{Z})$ has finite abelianization $(\mathbb{Z}/2\mathbb{Z})^3$, the image $\phi(\ker(b_{43}))$ must be annihilated by $b_{43}$, i.e., $\phi(\ker(b_{43})) \subset \ker(b_{43})$. We can argue in the same way after replacing $\phi$ by $\phi^{-1}$ to obtain $\phi^{-1}(\ker(b_{43})) \subset \ker(b_{43})$. Combining both inclusions implies $\phi(\ker(b_{43})) = \ker(b_{43})$. $\square$

*Proof of Proposition* 3.2 (v): Let us write $[*]^{\mathrm{ab}}$ for the abelianization of $[*]$. Since we already know '(iv): $\widehat{\Pi}_{0,4}$ is characteristic in $\widehat{\mathcal{B}}_4$' from (i)-(iii), for proving $\widehat{\mathcal{P}}_4$ characteristic in $\widehat{\mathcal{B}}_4$, it suffices to show the assertion that $\widehat{\mathcal{P}}_4$ is the kernel of the conjugate representation $\rho : \widehat{\mathcal{B}}_4 \to \mathrm{Aut}(\widehat{\Pi}_{0,4}^{\mathrm{ab}})$. First we note that $\rho$ factors through $\bar{\rho} : \widehat{\mathcal{B}}_4/\widehat{\mathcal{P}}_4 \cong \mathfrak{S}_4 \to \mathrm{Aut}(\widehat{\Pi}_{0,4}^{\mathrm{ab}})$. This follows from the observation that $p_{43}^{\mathrm{ab}}$ injects $\widehat{\Pi}_{0,4}^{\mathrm{ab}}$ into $\widehat{\mathcal{P}}_4^{\mathrm{ab}}$: Indeed, writing $\{\bar{x}_{ij}\}$ for the image of the standard generator system $\{x_{ij} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1} \mid 1 \le i < j \le n\}$ of $\mathcal{P}_n$, we find

$$p_{43}^{\mathrm{ab}} : \widehat{\mathcal{P}}_4^{\mathrm{ab}} \to \widehat{\mathcal{P}}_3^{\mathrm{ab}} : \begin{cases} \bar{x}_{12}, \bar{x}_{34} & \mapsto \bar{x}_{12}, \\ \bar{x}_{13}, \bar{x}_{24} & \mapsto \bar{x}_{13}, \\ \bar{x}_{14}, \bar{x}_{23} & \mapsto \bar{x}_{23}. \end{cases} \tag{3.8}$$

Taking into account the single relation $\bar{x}_{12} + \bar{x}_{13} + \bar{x}_{14} + \bar{x}_{23} + \bar{x}_{24} + \bar{x}_{34} = 0$ for $\widehat{\mathcal{P}}_4^{\mathrm{ab}}$ (respectively, $\bar{x}_{12} + \bar{x}_{13} + \bar{x}_{14} = 0$ for $\widehat{\mathcal{P}}_3^{\mathrm{ab}}$), we easily see from the description (3.8) of $p_{43}^{\mathrm{ab}} : \widehat{\mathbb{Z}}^5 \twoheadrightarrow \widehat{\mathbb{Z}}^2$ that $\ker(p_{43}^{\mathrm{ab}})$ is isomorphic to $\widehat{\mathbb{Z}}^3$ (torsion-free) into which $\widehat{\Pi}_{0,4}^{\mathrm{ab}}$ must inject. Then, to complete proof of the assertion, it suffices to see faithfulness of $\bar{\rho} : \widehat{\mathcal{B}}_4/\widehat{\mathcal{P}}_4 \cong \mathfrak{S}_4 \to \mathrm{Aut}(\widehat{\Pi}_{0,4}^{\mathrm{ab}})$. This is easily seen from the general fact that the action of $\mathcal{B}_n/\mathcal{P}_n = \mathfrak{S}_n$ on the $\bar{x}_{ij} \in \mathcal{P}_n^{\mathrm{ab}}$ is given by the natural action on indices, once declared $\bar{x}_{ij} = \bar{x}_{ji}$. The action of $\mathfrak{S}_4$ on $\widehat{\Pi}_{0,4}^{\mathrm{ab}}$ turns out to be the standard permutation representation $\widehat{\mathbb{Z}}^4$ modulo the diagonal line, which is faithful. □

4. **Proofs of Theorems A and B.** By virtue of Propositions 2.2 and 3.1, we know that $\widehat{\mathcal{P}}_n$ is a characteristic subgroup of $\widehat{\mathcal{B}}_n$ for $n \ge 3$. The following proposition follows immediately from this together with the well-known fact that $\mathrm{Out}(\mathfrak{S}_n) = \{1\}$ in the case $n \ne 6$. However, the case $n = 6$ requires a special care, since $\mathrm{Out}(\mathfrak{S}_6) \cong \mathbb{Z}/2\mathbb{Z}$. Theorem 1.1 (Hoshi-Minamide-Mochizuki) allows us to give a uniform proof working for all $n \ge 4$.

**Proposition 4.1.** Regard $\mathfrak{S}_n$ as the quotient of $\widehat{\mathcal{B}}_n$ and of $\widehat{\Gamma}_{0,[n]}$ by $\varpi_n : B_n \to \mathfrak{S}_n$ in Section 2.
(i) Every automorphism of $\widehat{\mathcal{B}}_n$ induces an inner automorphism of $\mathfrak{S}_n$ for $n \ge 3$.
(ii) $\widehat{\Gamma}_{0,n}$ is a characteristic subgroup of $\widehat{\Gamma}_{0,[n]}$ in the profinite completion of (2.6), and every automorphism of $\widehat{\Gamma}_{0,[n]}$ induces an inner automorphism of $\mathfrak{S}_n$ for $n \ge 5$.

*Proof.* (i) As $\mathrm{Out}(\mathfrak{S}_3) = \{1\}$, the assertion is trivial when $n = 3$. Suppose $n \ge 4$ and pick any $\phi \in \mathrm{Aut}(\widehat{\mathcal{B}}_n)$. Then, it follows from Propositions 2.2 and 3.1, that $\phi$ induces $(\phi_P, \phi_{\mathfrak{S}}) \in \mathrm{Aut}(\widehat{\mathcal{P}}_n) \times \mathrm{Aut}(\mathfrak{S}_n)$, Moreover $\phi_P$ induces $\phi_\Gamma \in \mathrm{Aut}(\widehat{\Gamma}_{0,n+1})$ via the natural isomorphism $\widehat{\mathcal{P}}_n \xrightarrow{\sim} \widehat{\Gamma}_{0,n+1}$ given by $\Phi_n$ of Section 2. Let $\bar{\phi}_\Gamma \in \mathrm{Out}(\widehat{\Gamma}_{0,n+1})$ be the outer class of $\phi_\Gamma$, and let $(\phi_0, \phi_1) \in \widehat{\mathrm{GT}} \times \mathfrak{S}_{n+1}$ be the image of $\bar{\phi}_\Gamma$ under the isomorphism

$\mathrm{Out}(\widehat{\Gamma}_{0,n+1}) \xrightarrow{\sim} \widehat{\mathrm{GT}} \times \mathfrak{S}_{n+1}$ of Theorem 1.1. Then we have the commutative diagram

$$\begin{tikzcd} \mathfrak{S}_n \arrow[r, "\chi_n"] \arrow[d, "\phi_{\mathfrak{S}}", "\wr"'] & \mathrm{Out}(\widehat{\Gamma}_{0,n+1}) \arrow[r, "\sim"] \arrow[d, "\mathrm{Inn}(\bar{\phi}_\Gamma)", "\wr"'] & \widehat{\mathrm{GT}} \times \mathfrak{S}_{n+1} \arrow[d, "\mathrm{Inn}(\phi_0{,}\phi_1)", "\wr"'] \\ \mathfrak{S}_n \arrow[r, "\chi_n"] & \mathrm{Out}(\widehat{\Gamma}_{0,n+1}) \arrow[r, "\sim"] & \widehat{\mathrm{GT}} \times \mathfrak{S}_{n+1} \end{tikzcd} \tag{4.1}$$

where $\chi_n : \mathfrak{S}_n \to \mathrm{Out}(\widehat{\mathcal{P}}_n) = \mathrm{Out}(\widehat{\Gamma}_{0,n+1})$ is the natural isomorphism regarding the commutative diagram $(*)$ in Definition 2.1. Since $\chi_n$ factors through $\iota_n : \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$, the above (4.1) makes the diagram

$$\begin{tikzcd} \mathfrak{S}_n \arrow[r, "\iota_n"] \arrow[d, "\phi_{\mathfrak{S}}"', "\wr"] & \mathfrak{S}_{n+1} \arrow[d, "\mathrm{Inn}(\phi_1)", "\wr"'] \\ \mathfrak{S}_n \arrow[r, "\iota_n"] & \mathfrak{S}_{n+1} \end{tikzcd}$$

commutative, hence $\phi_1$ normalizes (hence lies in) the image of $\iota_n$. From this follows that $\phi_{\mathfrak{S}}$ is an inner automorphism of $\mathfrak{S}_n$.

(ii): Recall from Section 2 that there is a surjection sequence $\widehat{B}_n \twoheadrightarrow \widehat{\mathcal{B}}_n \twoheadrightarrow \widehat{\Gamma}_{0,[n]} \twoheadrightarrow \mathfrak{S}_n$. By Proposition 2.2, every epimorphism from $\widehat{\mathcal{B}}_n$ to $\mathfrak{S}_n$ has kernel $\widehat{\mathcal{P}}_n$ for $n \geq 5$. This makes $\widehat{\Gamma}_{0,n}$ to be a characteristic subgroup of $\widehat{\Gamma}_{0,[n]}$ as the pull-back of $\widehat{\mathcal{P}}_n \subset \widehat{\mathcal{B}}_n$. For the rest, we can argue in exactly a similar (and simpler) way to the case (i) with employing $\chi'_n : \mathfrak{S}_n \to \mathrm{Out}(\widehat{\Gamma}_{0,n}) \cong \widehat{\mathrm{GT}} \times \mathfrak{S}_n$ for the role of $\chi_n$ in (i). We leave the rest of detail to the reader. $\square$

For the proof of Theorem B, we prepare a simple lemma of group theory. Let

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an exact sequence of profinite groups with $\rho : G \to \mathrm{Out}(\Delta)$ the associated outer representation. Let $Z_{\mathrm{Out}(\Delta)}(\mathrm{Im}(\rho))$ denote the centralizer of the image $\rho(G)$ in $\mathrm{Out}(\Delta)$. Assume that $\Delta$ and $G$ are topologically finitely generated so that $\mathrm{Aut}(\Delta)$, $\mathrm{Aut}(G)$ are profinite groups. Write $\mathrm{Aut}_G(\Pi)$ (resp. $\mathrm{Inn}_\Pi(\Delta)$) for the group of automorphisms of $\Pi$ which preserve $\Delta \subset \Pi$ and induce the identity automorphism of $G$ (resp. for the group of inner automorphisms of $\Pi$ by the elements of $\Delta$). Then,

**Lemma 4.2.** Notations being as above, we have the following assertions.
(i) Suppose $Z(\Delta) = \{1\}$. Then the restriction map $\mathrm{Aut}_G(\Pi) \to \mathrm{Aut}(\Delta)$ induces an isomorphism

$$\mathrm{Aut}_G(\Pi)/\mathrm{Inn}_\Pi(\Delta) \xrightarrow{\sim} Z_{\mathrm{Out}(\Delta)}(\mathrm{Im}(\rho)).$$

(ii) Suppose $Z(G) = \{1\}$ and that $\Delta$ is a characteristic subgroup of $\Pi$. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \mathrm{Aut}_G(\Pi)/\mathrm{Inn}_\Pi(\Delta) \xrightarrow{\jmath} \mathrm{Out}(\Pi) \xrightarrow{\varpi} \mathrm{Out}(G).$$

*Proof.* Assertion (i) follows immediately from [N94, Corollary 1.5.7]. We consider (ii). First, observing $\mathrm{Aut}_G(\Pi) \cap \mathrm{Inn}(\Pi) = \mathrm{Inn}_\Pi(\Delta)$ under the assumption $Z(G) = \{1\}$, we obtain the monomorphism

$$\jmath : \mathrm{Aut}_G(\Pi)/\mathrm{Inn}_\Pi(\Delta) \hookrightarrow \mathrm{Aut}(\Pi)/\mathrm{Inn}(\Pi) = \mathrm{Out}(\Pi)$$

from the natural injection $\mathrm{Aut}_G(\Pi) \hookrightarrow \mathrm{Aut}(\Pi)$. Next, since $\Delta$ is a characteristic subgroup of $\Pi$, there exists a natural homomorphism $\varpi : \mathrm{Out}(\Pi) \to \mathrm{Out}(G)$ with $\varpi \circ \jmath = 1$. Then, immediately from the surjectivity $\mathrm{Inn}(\Pi) \twoheadrightarrow \mathrm{Inn}(G)$ follows that $\mathrm{Im}(\jmath) = \ker(\varpi)$, which completes the proof of (ii). □

We now obtain Theorem B:

**Theorem 4.3.** (i) Let $n \geq 4$ be an integer. Then the composite

$$\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{\mathcal{B}}_n)$$

of the natural homomorphisms $\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{B}_n) \to \mathrm{Out}(\widehat{\mathcal{B}}_n)$ is an isomorphism.
(ii) Let $n \geq 5$. Then, the natural homomorphism

$$\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{\Gamma}_{0,[n]})$$

induced from $\widehat{\Psi}_n : \widehat{B}_n \twoheadrightarrow \widehat{\Gamma}_{0,[n]}$ (2.4) is an isomorphism.

*Proof.* First, we note that $\mathfrak{S}_n$ and $\widehat{\mathcal{P}}_n$ are center-free (2.9), and that $\widehat{\mathcal{P}}_n$ is a characteristic subgroup of $\widehat{\mathcal{B}}_n$ (Propositions 2.2 and 3.1). Consider the upper exact sequence

$$1 \longrightarrow \widehat{\mathcal{P}}_n \longrightarrow \widehat{\mathcal{B}}_n \longrightarrow \mathfrak{S}_n \longrightarrow 1$$

of $(*)$ in Definition 2.1, and write $\varphi_n : \mathfrak{S}_n \to \mathrm{Out}(\widehat{\mathcal{P}}_n)$ for the associated outer representation. Let us apply Lemma 4.2 to the above exact sequence. By virtue of Proposition 4.1 (i), the homomorphism $\varpi : \mathrm{Out}(\widehat{\mathcal{B}}_n) \to \mathrm{Out}(\mathfrak{S}_n)$ of Lemma 4.2 (ii) turns out trivial, so $\jmath$ in loc. cit. together with Lemma 4.2 (i) gives an isomorphism

$$Z_{\mathrm{Out}(\widehat{\mathcal{P}}_n)}(\varphi_n(\mathfrak{S}_n)) \xrightarrow{\sim} \mathrm{Out}(\widehat{\mathcal{B}}_n).$$

Then observe that the natural isomorphism $\widehat{\mathcal{P}}_n \xrightarrow{\sim} \widehat{\Gamma}_{0,n+1}$ in $(*)$ induces an isomorphism

$$Z_{\mathrm{Out}(\widehat{\mathcal{P}}_n)}(\varphi_n(\mathfrak{S}_n)) \xrightarrow{\sim} Z_{\mathrm{Out}(\widehat{\Gamma}_{0,n+1})}(\chi_n(\mathfrak{S}_n)),$$

where $\chi_n : \mathfrak{S}_n \to \mathrm{Out}(\widehat{\Gamma}_{0,n+1})$ is as in (4.1). But since $\iota_n(\mathfrak{S}_n)$ has trivial centralizer in $\mathfrak{S}_{n+1}$, Theorem 1.1 implies

$$\widehat{\mathrm{GT}} \xrightarrow{\sim} Z_{\mathrm{Out}(\widehat{\Gamma}_{0,n+1})}(\chi_n(\mathfrak{S}_n)).$$

It is easy to see that the composite of the above three displayed isomorphisms coincides with $\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{\mathcal{B}}_n)$ of the assertion. This completes the proof of (i).

(ii) Let $n \geq 5$. After Proposition 4.1 (ii), the argument goes in a similar (and simpler) way to the case (i) with applying Lemma 4.2 to the profinite completion of (2.6):

$$1 \longrightarrow \widehat{\Gamma}_{0,n} \longrightarrow \widehat{\Gamma}_{0,[n]} \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

We leave the rest of detail to the reader. □

Now, to prove Theorem A, let us follow an argument in [DG] (Theorem 20) to look closely at the short exact sequence

$$1 \longrightarrow \widehat{C}_n \longrightarrow \widehat{B}_n \longrightarrow \widehat{\mathcal{B}}_n \longrightarrow 1 \tag{4.2}$$

obtained as the profinite completion of (2.2). Since $\widehat{C}_n$ is characteristic in $\widehat{B}_n$, this yields two natural homomorphisms

$$\mathfrak{p}_0 : \mathrm{Aut}(\widehat{B}_n) \to \mathrm{Aut}(\widehat{C}_n), \quad \mathfrak{p}_1 : \mathrm{Aut}(\widehat{B}_n) \to \mathrm{Aut}(\widehat{\mathcal{B}}_n). \tag{4.3}$$

Recalling $\widehat{C}_n = \langle \zeta_n \rangle \cong \widehat{\mathbb{Z}}$, we now *canonically* identify $\mathrm{Aut}(\widehat{C}_n) = \widehat{\mathbb{Z}}^\times$.

**Definition 4.4.** For $n > 1$, define the subgroup $Z_n \subset \widehat{\mathbb{Z}}^\times$ by

$$Z_n := \left(1 + n(n-1)\widehat{\mathbb{Z}}\right)^\times = \ker\left(\widehat{\mathbb{Z}}^\times \twoheadrightarrow (\widehat{\mathbb{Z}}/n(n-1)\widehat{\mathbb{Z}})^\times\right).$$

It is clear that each $\nu \in Z_n$ has a unique element $e \in \widehat{\mathbb{Z}}$ such that

$$\nu = 1 + n(n-1)e.$$

(But note that this form of $\nu$ is not always in $\widehat{\mathbb{Z}}^\times$ for arbitrary $e \in \widehat{\mathbb{Z}}$.)

The next key lemma enables us to identify $\ker(\mathfrak{p}_1)$ with $Z_n$:

**Lemma 4.5.** There is an isomorphism

$$\phi : Z_n \xrightarrow{\sim} \ker(\mathfrak{p}_1) \subset \mathrm{Aut}(\widehat{B}_n)$$

which assigns to every $\nu \in Z_n$ an automorphism $\phi_\nu \in \mathrm{Aut}(\widehat{B}_n)$ determined by

$$\phi_\nu(\sigma_i) = \sigma_i \zeta_n^e \qquad (\nu = 1 + n(n-1)e,\ i = 1, \dots, n-1).$$

*Proof.* Given any $\nu \in Z_n$, let $e \in \widehat{\mathbb{Z}}$ be the unique element with $\nu = 1 + n(n-1)e$. By using this $e \in \widehat{\mathbb{Z}}$, we define $\phi_\nu \in \ker(\mathfrak{p}_1)$ as follows: First, set $\phi_\nu(\sigma_i) := \sigma_i \zeta_n^e$ for all $i = 1, \dots, n-1$. Since $\zeta_n$ lies in the center of $\widehat{B}_n$, it is easy to see that $\phi_\nu$ preserves the Artin's braid relations. Therefore, $\phi_\nu$ extends to an endomorphism of $\widehat{B}_n$ written by the same symbol $\phi_\nu$. One computes then

$$\phi_\nu(\zeta_n) = \phi_\nu((\sigma_1 \cdots \sigma_{n-1})^n) = (\sigma_1 \cdots \sigma_{n-1})^n \cdot \zeta_n^{n(n-1)e} = \zeta_n^{1+n(n-1)e} = \zeta_n^\nu. \tag{4.4}$$

From this, for $\nu_j = 1 + n(n-1)e_j \in Z_n$ $(j = 1, 2)$, it follows that $\phi_{\nu_1} \circ \phi_{\nu_2}(\sigma_i) = \sigma_i \zeta_n^{e_1 + \nu_1 e_2} = \phi_{\nu_1\nu_2}(\sigma_i)$ holds for every $i = 1, \dots, n-1$. Noting then that $\phi_1 = id$ and that $Z_n$ forms a multiplicative group, we see that $\phi_\nu$ $(\nu \in Z_n)$ belongs to $\mathrm{Aut}(\widehat{B}_n)$ and hence that the mapping $\phi : Z_n \to \mathrm{Aut}(\widehat{B}_n)$ defined by $\nu \mapsto \phi_\nu$ forms a group homomorphism. One verifies immediately that $\phi$ is injective and $Z_n \cong \mathrm{Im}(\phi) \subset \ker(\mathfrak{p}_1)$. To see $\mathrm{Im}(\phi) = \ker(\mathfrak{p}_1)$, pick any $\alpha \in \ker(\mathfrak{p}_1)$ and set $\nu := \mathfrak{p}_0(\alpha) \in \widehat{\mathbb{Z}}^\times$. Then, $\alpha(\zeta_n) = \zeta_n^\nu$ and there exist $e_i \in \widehat{\mathbb{Z}}$ $(i = 1, \ldots, n-1)$ such that $\alpha(\sigma_i) = \sigma_i\, \zeta_n^{e_i}$. It is easy to see from the braid relation that all $e_i$ are the same constant $e \in \widehat{\mathbb{Z}}$. But then, since $\zeta_n = (\sigma_1 \cdots \sigma_{n-1})^n$, we find $\nu = 1 + n(n-1)e$ which belongs to $Z_n$ and that $\alpha = \phi_\nu$. $\square$

Theorem A is obtained from Theorem 4.3 (i) together with the last part of the following

**Theorem 4.6.** Let $n \geq 4$ be an integer.

(i) There exists an exact sequence

$$1 \longrightarrow Z_n \xrightarrow{\phi} \mathrm{Aut}(\widehat{B}_n) \xrightarrow{\mathfrak{p}_1} \mathrm{Aut}(\widehat{\mathcal{B}}_n) \longrightarrow 1.$$

(ii) $\mathrm{Inn}(\widehat{B}_n) \cap \phi(Z_n) = \{1\}$.

(iii) The exact sequence (i) provides a split central extension, i.e.,

$$\mathrm{Aut}(\widehat{B}_n) \cong \mathrm{Aut}(\widehat{\mathcal{B}}_n) \times Z_n,$$

and gives rise to $\mathrm{Out}(\widehat{B}_n) \cong \mathrm{Out}(\widehat{\mathcal{B}}_n) \times Z_n$.

*Proof.* (i) It suffices to show $\mathfrak{p}_1$ is surjective. Note that $\mathrm{Inn}(\widehat{B}_n)$ is mapped onto $\mathrm{Inn}(\widehat{\mathcal{B}}_n)$. On the other hand, there is a well-known action $\iota_n : \widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{B}_n)$ in the form

$$\begin{cases} \sigma_1 & \mapsto \ \sigma_1^\lambda, \\ \sigma_i & \mapsto \ f(\sigma_i, \zeta_i)\sigma_i^\lambda f(\zeta_i, \sigma_i) \quad (i = 1, \ldots, n-1) \end{cases} \tag{4.5}$$

with $(\lambda, f) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]$ the standard parameter for the elements of $\widehat{\mathrm{GT}}$ ([Dr], [I90], [I95]). Let $\bar{\iota}_n : \widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{\mathcal{B}}_n)$ be the induced action. By virtue of Theorem 4.3 (i), $\widehat{\mathrm{GT}} \cong \mathrm{Out}(\widehat{\mathcal{B}}_n)$, hence $\mathrm{Aut}(\widehat{\mathcal{B}}_n) = \bar{\iota}_n(\widehat{\mathrm{GT}}) \cdot \mathrm{Inn}(\widehat{\mathcal{B}}_n)$. From this follows that $\mathfrak{p}_1$ maps $\iota_n(\widehat{\mathrm{GT}}) \cdot \mathrm{Inn}(\widehat{B}_n)(\subset \mathrm{Aut}(\widehat{B}_n))$ onto $\mathrm{Aut}(\widehat{\mathcal{B}}_n)$.

(ii) This is a consequence of Lemma 4.2 (ii) applied to (4.2). Here is an alternative direct proof: Recall that the abelianization $\widehat{B}_n^{\mathrm{ab}}$ of $\widehat{B}_n$ is isomorphic to $\widehat{\mathbb{Z}}$. Each inner automorphism acts trivially on $\widehat{B}_n^{\mathrm{ab}}$, while $\phi_\nu \in \phi(Z_n)$ $(\nu \in Z_n)$ acts on it by

$$(\sigma_i)^{\mathrm{ab}} \mapsto (\sigma_i \cdot \zeta_n^e)^{\mathrm{ab}} = (\sigma_i^{\mathrm{ab}})^{1+n(n-1)e} \qquad (i = 1, \ldots, n-1)$$

which is nontrivial unless $e = 0$. This concludes the assertion.

(iii) It follows from (ii) that $\mathfrak{p}_1$ induces $\mathrm{Inn}(\widehat{B}_n) \xrightarrow{\sim} \mathrm{Inn}(\widehat{\mathcal{B}}_n)$. Since $\iota_n(\widehat{\mathrm{GT}}) \xrightarrow{\sim} \bar{\iota}_n(\widehat{\mathrm{GT}})$, we find from Theorem 4.3 (i) that $\mathfrak{p}_1$ restricts to the isomorphism

$$\iota_n(\widehat{\mathrm{GT}}) \cdot \mathrm{Inn}(\widehat{B}_n) \xrightarrow{\sim} \mathrm{Aut}(\widehat{\mathcal{B}}_n), \tag{4.6}$$

i.e., $\iota_n(\widehat{\mathrm{GT}}) \cdot \mathrm{Inn}(\widehat{B}_n)$ gives a complementary factor of $\phi(Z_n)$ in $\mathrm{Aut}(\widehat{B}_n)$. To see that the exact sequence (i) gives a central extension, it suffices to show that both $\mathrm{Inn}(\widehat{B}_n)$ and $\iota_n(\widehat{\mathrm{GT}})$ commutes with $\phi(Z_n)$. The commutativity of $\mathrm{Inn}(\widehat{B}_n)$ and $\phi(Z_n)$ follows immediately from the definition of $\phi_\nu$ ($\nu \in Z_n$) in Lemma 4.5. The commutativity of $\iota_n(\widehat{\mathrm{GT}})$ and $\phi(Z_n)$ also follows from direct computation by using the above description of the $\widehat{\mathrm{GT}}$-action on $\widehat{B}_n$. Indeed, given $(\lambda, f) \in \widehat{\mathrm{GT}}$, noting that $\zeta_n$ lies in the center of $\widehat{B}_n$, and $f$ lies in the commutator subgroup of $\widehat{F}_2$, we have $f(\sigma_i, \zeta_i) = f(\sigma_i\zeta_n^e, \zeta_i)$ ($i = 1, \dots, n-1$). Since $(\lambda, f) \in \widehat{\mathrm{GT}}$ is known to act on $\zeta_n$ by $\zeta_n \mapsto \zeta_n^\lambda$ under the action (4.5), one computes:

$$\begin{aligned}(\lambda, f) \circ \phi_\nu(\sigma_i) &= (\lambda, f)(\sigma_i\zeta_n^e) = f(\sigma_i, \zeta_i)\sigma_i^\lambda f(\zeta_i, \sigma_i)\zeta_n^{\lambda e}, \\ &= f(\sigma_i, \zeta_i)(\sigma_i\zeta^e)^\lambda f(\zeta_i, \sigma_i) = \phi_\nu(f(\sigma_i, \zeta_n)\sigma_i^\lambda f(\zeta_i, \sigma_i)) \\ &= \phi_\nu \circ (\lambda, f)(\sigma_i).\end{aligned}$$

for every $i = 1, \dots, n-1$ (we understand $\zeta_1 = 1$ when $i = 1$). Thus we settle the first assertion $\mathrm{Aut}(\widehat{B}_n) \cong \mathrm{Aut}(\widehat{\mathcal{B}}_n) \times Z_n$ after identifying $Z_n \cong \phi(Z_n) \subset \mathrm{Aut}(\widehat{B}_n)$ and $\mathrm{Aut}(\widehat{\mathcal{B}}_n) \cong \iota_n(\widehat{\mathrm{GT}}) \cdot \mathrm{Inn}(\widehat{B}_n) \subset \mathrm{Aut}(\widehat{B}_n)$ via (4.6). The second assertion is then just a consequence of it. □

In our above discussion for the proof of Theorem A, important roles have been played by the pair of two maps (4.3), which was motivated from the profinite Wells exact sequence (cf. [N94, Section 1.5], [JL]) associated to the short exact sequence (4.2) in the form:

$$0 \longrightarrow Z^1_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n) \longrightarrow \mathrm{Aut}(\widehat{B}_n, \widehat{C}_n) \xrightarrow{\ \mathfrak{p}\ } \mathscr{C} \xrightarrow{\ q\ } H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n). \tag{4.7}$$

Since $\widehat{B}_n$ in (4.2) is a central extension and $\mathcal{B}_n^{\mathrm{ab}} \cong \mathbb{Z}/n(n-1)\mathbb{Z}$, we easily see that $Z^1_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n) = \{0\}$, $\mathrm{Aut}(\widehat{B}_n, \widehat{C}_n) = \mathrm{Aut}(\widehat{B}_n)$, and find the group of "compatible pairs" $\mathscr{C}$ to be $\mathrm{Aut}(\widehat{\mathcal{B}}_n) \times \mathrm{Aut}(\widehat{C}_n)$. Thus, the exact sequence (4.7) is reduced to

$$0 \longrightarrow \mathrm{Aut}(\widehat{B}_n) \xrightarrow{\mathfrak{p}=(\mathfrak{p}_1,\mathfrak{p}_0)} \mathrm{Aut}(\widehat{\mathcal{B}}_n) \times \mathrm{Aut}(\widehat{C}_n) \xrightarrow{\ q\ } H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n), \tag{4.8}$$

where $q$ is called the Wells pointed map (generally not a homomorphism).

The above sequence (4.8) is simply useful, for example, to see that *the exact sequence of* Theorem 4.6 (i) *provides a central extension*, reproving the core part of Theorem 4.6 (iii) without use of the explicit $\widehat{\mathrm{GT}}$-action (4.5): Indeed, according to (4.4), the image $\mathfrak{p}(\phi_\nu) = (\mathfrak{p}_1(\phi_\nu), \mathfrak{p}_0(\phi_\nu)) = (id, \nu)$ for every $\nu \in Z_n$ is easily seen to lie in the center of $\mathrm{Aut}(\widehat{\mathcal{B}}_n) \times \mathrm{Aut}(\widehat{C}_n)$. Besides this simple observation, it is a natural question to measure the size of the image of $\mathrm{Aut}(\widehat{B}_n)$ by the injection $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_0)$ into $\mathrm{Aut}(\widehat{\mathcal{B}}_n) \times \mathrm{Aut}(\widehat{C}_n)$. Now, recalling $\widehat{\mathrm{GT}} \subset \{(\lambda, f) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2\}$, $\mathrm{Aut}(\widehat{\mathcal{B}}_n) = \widehat{\mathrm{GT}} \cdot \mathrm{Inn}(\mathcal{B}_n)$ and $\mathrm{Aut}(\widehat{C}_n) = \widehat{\mathbb{Z}}^\times$, we define two characters

$$\lambda : \mathrm{Aut}(\widehat{\mathcal{B}}_n) \to \widehat{\mathbb{Z}}^\times \quad \text{and} \quad \nu : \mathrm{Aut}(\widehat{C}_n) \to \widehat{\mathbb{Z}}^\times \tag{4.9}$$

in the obvious way. One finds:

**Proposition 4.7.** Notations being as above, we have

$$\mathrm{Im}(\mathfrak{p}) = \{(\alpha,\beta) \in \mathscr{C} \mid \lambda(\alpha) \equiv \nu(\beta) \mod n(n-1)\}.$$

In particular, $\mathscr{C}/\mathrm{Im}(\mathfrak{p}) \cong (\mathbb{Z}/n(n-1)\mathbb{Z})^\times$.

*Proof.* Let $Z_n \subset \mathrm{Aut}(\widehat{C}_n) = \widehat{\mathbb{Z}}^\times$ be as above, and define $A_n \subset \mathrm{Aut}(\mathcal{B}_n)$ to be $\lambda^{-1}(Z_n)$. It is not difficult to see $A_n \times Z_n \subset \mathrm{Im}(\mathfrak{p})$. The assertion is derived from the observation that the image of $\mathrm{Im}(\mathfrak{p})$ in the quotient group $\mathscr{C}/(A_n \times Z_n) \cong (\mathbb{Z}/n(n-1)\mathbb{Z})^\times \times (\mathbb{Z}/n(n-1)\mathbb{Z})^\times$ forms the diagonal subgroup. This follows from the well-known fact that the restriction of the action of $(\lambda, f) \in \widehat{\mathrm{GT}}$ on $\widehat{B}_n$ to $\widehat{C}_n = \langle \zeta_n \rangle$ is given by $\zeta_n \mapsto \zeta_n^\lambda$, which completes the proof. $\square$

Before closing the paper, let us add some remark on the Wells map $q : \mathscr{C} \to H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n)$. Let $[\mu] \in H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n)$ be the class of factor sets associated to the central extension (4.2). For each pair $(\alpha, \nu) \in \mathscr{C} = \mathrm{Aut}(\widehat{\mathcal{B}}_n) \times \mathrm{Aut}(\widehat{C}_n)$, we denote by $[\mu]^{(\alpha,\nu)} \in H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n)$ the class of a central extension obtained by twisting (4.2) by $(\alpha, \nu)$. Then, one finds:

$$q(\alpha,\nu) = [\mu] - [\mu]^{(\alpha,\nu)}. \tag{4.10}$$

This means that $\mathrm{Im}(\mathfrak{p}) \subset \mathscr{C}$ can be characterized as the stabilizer of the twisting action of $\mathscr{C}$ on $[\mu]$. Concerning the precise position and size of $[\mu] \in H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n)$, we remark the following

**Proposition 4.8.** Let $n \geq 4$. The cohomology group $H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n)$ is isomorphic to $\mathbb{Z}/n(n-1)\mathbb{Z}$, and is generated by the class $[\mu]$.

*Proof.* According to V.Arnold [A68], $H^2(B_n, \mathbb{Z}) = \{0\}$ and $H^3(B_n, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Applying this to the long exact sequence associated with $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/r\mathbb{Z} \to 0$ ($r \in \mathbb{N}$), we obtain $H^2(B_n, \mathbb{Z}/r\mathbb{Z}) \cong \{0\}$, $\cong \mathbb{Z}/2\mathbb{Z}$ according to whether $r$ is odd or even respectively. For a positive integer $N$, (part of) the five term exact sequence for the central extension $1 \to C_n \to B_n \to \mathcal{B}_n \to 1$ reads

$$\begin{aligned} H^1(B_n, \mathbb{Z}/N\mathbb{Z}) &\xrightarrow{\mathrm{res}_N} H^1(C_n, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\mathrm{tg}_N} H^2(\mathcal{B}_n, \mathbb{Z}/N\mathbb{Z}) \\ &\xrightarrow{\mathrm{inf}_N} H^2(B_n, \mathbb{Z}/N\mathbb{Z}), \end{aligned} \tag{4.11}$$

where $\mathrm{res}_N$, $\mathrm{tg}_N$ and $\mathrm{inf}_N$ are respectively the restriction, transgression and inflation maps. Suppose first $N$ is a positive integer divisible by $n(n-1)$. Then, (4.11) yields the exact sequence

$$0 \to \mathbb{Z}/n(n-1)\mathbb{Z} \xrightarrow{\overline{\mathrm{tg}}_N} H^2(\mathcal{B}_n, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\mathrm{inf}_N} H^2(B_n, \mathbb{Z}/N\mathbb{Z})\ (\cong \mathbb{Z}/2\mathbb{Z}), \tag{4.12}$$

where $\mathbb{Z}/n(n-1)\mathbb{Z}$ is regarded as the cokernel of the restriction $\mathrm{res}_N : \mathrm{Hom}(B_n, \mathbb{Z}/N\mathbb{Z}) \to \mathrm{Hom}(C_n, \mathbb{Z}/N\mathbb{Z})$ followed by the factorization $\overline{\mathrm{tg}}_N$ of transgression $\mathrm{tg}_N$. Let us vary $N$ multiplicatively. The goodness of $\mathcal{B}_n$ (in the sense of Serre) together with

[NSW, Corollary 2.7.6] allows us to interpret $H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n) = \varprojlim_N H^2(\mathcal{B}_n, \mathbb{Z}/N\mathbb{Z})$ after identification $C_n = \zeta_n^{\mathbb{Z}} \cong \mathbb{Z}$ with trivial (conjugate) action of $\mathcal{B}_n$. The term $\mathrm{coker}(\mathrm{res}_N) \cong \mathbb{Z}/n(n-1)\mathbb{Z}$ in (4.12) is constant in the projective system along $N \in \mathbb{N}$ divisible by $n(n-1)$. On the other hand, we have (#): $\varprojlim_N H^2(B_n, \mathbb{Z}/N\mathbb{Z}) = \{0\}$. In fact, since $B_n^{\mathrm{ab}} \cong \mathbb{Z}$, in the long exact sequence associated with $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2N\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z} \to 0$, we find that $H^1(B_n, \mathbb{Z}/2N\mathbb{Z}) \to H^1(B_n, \mathbb{Z}/N\mathbb{Z})$ is surjective, hence that the former arrow in $H^2(B_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(B_n, \mathbb{Z}/2N\mathbb{Z}) \to H^2(B_n, \mathbb{Z}/N\mathbb{Z})$ gives an isomorphism between groups of order two so that the latter arrow is 0-map. This settles (#) which concludes the first assertion $H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n) \cong \mathbb{Z}/n(n-1)\mathbb{Z}$.

It remains to show that the class $[\mu]$ has order $n(n-1)$ in $H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n)$. For an integer $d > 0$, let $[\mu_d] \in H^2(\mathcal{B}_n, \mathbb{Z}/d\mathbb{Z})$ be the class of factor sets corresponding to the central extension

$$1 \to C_n/C_n^d (\cong \mathbb{Z}/d\mathbb{Z}) \to B_n/C_n^d \to \mathcal{B}_n \to 1. \tag{4.13}$$

It is known that $[\mu_d]$ is the transgression image of the projection $\mathrm{pr}_d : C_n \to C_n/C_n^d$ regarded as an element of $H^1(C_n, C_n/C_n^d)$, i.e.,

$$[\mu_d] = \mathrm{tg}_d(\mathrm{pr}_d) \in \mathrm{Im}(\mathrm{tg}_d) \subset H^2(\mathcal{B}_n, \mathbb{Z}/d\mathbb{Z}), \tag{4.14}$$

where $C_n/C_n^d \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}$ is given by $\zeta_n \mapsto 1$ (cf. e.g., [Sz, Chap. 2 Section 9 (9.4)]). Let us observe that the extension (4.13) splits if and only if $n(n-1) \in (\mathbb{Z}/d\mathbb{Z})^\times$. In fact, a system of lifts of the generators $\bar{\sigma}_i \in \mathcal{B}_n$ $(i = 1, \dots, n-1)$ can be written in the form of images of $\sigma_i \zeta_i^{a_i} \in B_n$ in $B_n/C_n^d$ $(a_i \in \mathbb{Z})$. It is easy to see that they satisfy the braid relations modulo $C_n^d$ if and only if $a_1 \equiv \cdots \equiv a_{n-1}$ and $1 + n\sum_i a_i \equiv 0$ in $\mathbb{Z}/d\mathbb{Z}$ (cf. (4.4)). This condition to be held by a collection $\{a_i\}_i$ is equivalent to $n(n-1) \in (\mathbb{Z}/d\mathbb{Z})^\times$ as desired. Let $p$ be a prime dividing $n(n-1)$ and consider $[\mu_p] \in H^2(\mathcal{B}_n, \mathbb{Z}/p\mathbb{Z})$. It follows from the above observation that $[\mu_p] \neq 0$. Since the restriction map $\mathrm{res}_p : H^1(B_n, \mathbb{Z}/p\mathbb{Z}) \to H^1(C_n, \mathbb{Z}/p\mathbb{Z})$ is trivial under the assumption $p \mid n(n-1)$, the transgression $\mathrm{tg_p}$ injects $H^1(C_n, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ into $H^2(\mathcal{B}_n, \mathbb{Z}/p\mathbb{Z})$ whose image is generated by $[\mu_p] \neq 0$. But for any multiple $N$ of $n(n-1)$, the class $[\mu_N] \in \mathrm{Im}(\mathrm{tg}_N)(\cong \mathbb{Z}/n(n-1)\mathbb{Z}) \subset H^2(\mathcal{B}_n, C_n/C_n^N)$ is mapped to $[\mu_p] \in H^2(\mathcal{B}_n, C_n/C_n^p)$ via the reduction of central extensions induced from the surjective homomorphism $B_n/C_n^N \twoheadrightarrow B_n/C_n^p$ in virtue of (4.14). In particular, the reduction map $\mathrm{Im}(\mathrm{tg}_N) \to \mathrm{Im}(\mathrm{tg}_p)$ is given simply by the mod $p$ surjection between the cyclic groups:

$$\begin{array}{cccc} [\mu_N] & \in \mathrm{Im}(\mathrm{tg}_N) & (\cong \mathbb{Z}/n(n-1)\mathbb{Z}) & \subset H^2(\mathcal{B}_n, C_n/C_n^N) \\ \downarrow & & \downarrow \bmod p & \downarrow \\ 0 \neq [\mu_p] & \in \mathrm{Im}(\mathrm{tg}_p) & (\cong \mathbb{Z}/p\mathbb{Z}) & \subset H^2(\mathcal{B}_n, C_n/C_n^p). \end{array} \tag{4.15}$$

Since the class $[\mu] \in H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n)$ is the common limit of those $[\mu_N]$, it follows that $[\mu]$ generates the $p$-primary component of the cyclic group $H^2_{\mathrm{cont}}(\widehat{\mathcal{B}}_n, \widehat{C}_n) \cong \mathbb{Z}/n(n-1)\mathbb{Z}$ for every prime $p \mid n(n-1)$, hence gives a generator of it. $\square$

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