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Moving frames and Eisenstein invariants

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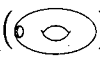
ABSTRACT. We recall combinatorial reconstitution of the periods of Eisenstein series of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, and present some consequence of "moving frames" in a free profinite group.

Plan:
1. Moving frames (review)
2. Eisenstein periods
3. Combinatorics in $\hat{F}_2 = \pi_1^{\text{ét}}($$)$
4. Some applications

1. Review: Moving frames

Suppose we are given a sequence of linear transformations on a vector space V:

$$V \xleftarrow{f_3} V \xleftarrow{f_2} V \xleftarrow{f_1} V.$$

Fix a basis $\epsilon_0 = (e_1, \dots, e_n)$ of V, and let A_i be the representative matrices of f_i $(i = 1, 2, 3)$ respectively in view of the basis ϵ_0. Then, as is well known, the composed transformation $f_3 \circ f_2 \circ f_1$ is represented by the matrix $A_3A_2A_1$.

According to the idea of moving frames, we consider not only the initial basis ϵ_0 but also the moved bases $\epsilon_1 := f_1(\epsilon_0)$ and $\epsilon_2 := f_2f_1(\epsilon_0)$. Then, letting B_i denote the representative matrix of f_i in view of the basis ϵ_{i-1} for $i = 1, 2, 3$, we derive that

$$B_1 = A_1, \quad B_2 = A_1^{-1}A_2A_1, \quad B_3 = A_1^{-1}A_2^{-1}A_3A_2A_1.$$

Consequently we find that the composition $f_3 \circ f_2 \circ f_1$ is represented by the reversely multiplied matrix $B_1B_2B_3$ with respect to ϵ_0.

We have borrowed from Spivak's book [Sp99, Chap. 7] the term "moving frames" as an English translation of E. Cartan's notion "repère mobile". See loc. cit. for more sophisticated applications. A most typical example of that idea may be what is called the Euler angle representation of the space rotations $\mathrm{SO}(3) = \{A \in \mathrm{GL}_3(\mathbb{R}) \mid {}^tA\,A = 1,\ \det(A) = 1\}$, which was most impressively encountered to the author in his youth 1983 upon an occasion of reading [YS, Chap.II, §2]: Define special matrices

$$\mathrm{Rot}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \mathrm{Rot}_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

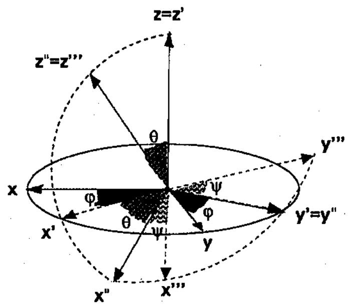
Then, every space rotation in SO(3) can be written as

$$A_{\varphi,\theta,\psi} = \mathrm{Rot}_3(\varphi)\mathrm{Rot}_2(\theta)\mathrm{Rot}_3(\psi) \quad (0 \le \varphi, \psi \le 2\pi,\ 0 \le \theta \le \pi),$$

uniquely with only exceptions $A_{\varphi,\theta,\psi} = A_{\varphi+\alpha,\theta,\psi-\alpha}$ for $\theta \in \{0, \pi\}$ and $\alpha \in \mathbb{R}$. The above composition of three rotation matrices may be interpreted more naturally if it is read from the left to the right moving xyz-coordinates

$$(x, y, z) \xrightarrow{\varphi} (x', y', z') \xrightarrow{\theta} (x'', y'', z'') \xrightarrow{\psi} (x''', y''', z''')$$

as illustrated in the picture.



2. Eisenstein periods

Let $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the complex upper half plane on which $\mathrm{SL}_2(\mathbb{Z})$ acts in the usual way. For each $\boldsymbol{x} = \frac{\boldsymbol{u}}{N} = (\frac{u}{N}, \frac{v}{N}) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we have the holomorphic Eisenstein series of weight 2 and 'label' $\boldsymbol{x}$ on $\mathfrak{H}$ defined by

$$E_2^{(\boldsymbol{x})}(\tau) := \sum_{\boldsymbol{a} \in (\mathbb{Z}/N\mathbb{Z})^2} \frac{e^{2\pi i \det(\frac{\boldsymbol{a}}{\boldsymbol{x}})}}{(2\pi i)^2} \left(\sum_{\substack{(m_1,m_2)\equiv \boldsymbol{a} \\ \bmod N}}{}' \frac{1}{(m_1\tau + m_2)^2} \cdot \frac{1}{|m_1\tau + m_2|^s} \right)_{s\to 0}.$$

The classcial Eisenstein periods of $E_2^{(\boldsymbol{x})}$ for those $\boldsymbol{x} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ are well known to be encoded in what are called the (generalized) Rademacher functions $\Phi_{\boldsymbol{x}} : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}$, which are good extensions of the period mapping $A \mapsto \int_z^{Az} E_2^{(\boldsymbol{x})}(\tau)d\tau$ for $A \in \Gamma(N)$ with $N\boldsymbol{x} \in \mathbb{Z}^2$. The value of $\Phi_{\boldsymbol{x}}(A) \in \mathbb{Q}$ for every $\boldsymbol{x} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $A \in \mathrm{SL}_2(\mathbb{Z})$ is explicitly calculated in terms of Bernoulli polynomials and Dedekind sums (B.Schoeneberg [Sc74]).

Based on our recent work [N13], we can introduce a (profinite) *combinatorial avatar* "$\mathbb{E}_{\boldsymbol{x}}$" of $\Phi_{\boldsymbol{x}} : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}$. Here, we consider the label $\boldsymbol{x}$ to lie in $\mathbb{Q}_f^2 := (\mathbb{Q} \otimes \hat{\mathbb{Z}})^2$ (adelic row vectors) and replace $\mathrm{SL}_2(\mathbb{Z})$ by a certain profinite group $\pi_1^{\text{ét}}(\mathfrak{M})$ which is:

(1) in the form of a semi-direct product $G_{\mathbb{Q}} \ltimes \hat{B}_3$ of two profinite groups, where $\hat{B}_3$ is a central extension of $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ and $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;

(2) equipped with a standard representation $\rho : \pi_1^{\text{ét}}(\mathfrak{M}) \to \mathrm{GL}_2(\hat{\mathbb{Z}})$;

as explained soon in more details. Throughout below, we write $A_\sigma \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ for the transposed matrix of $\rho(\sigma)$:

$$A_\sigma = {}^t\rho(\sigma) \qquad (\sigma \in \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3).$$

The main aim of the present article is to illustrate roughly a use of "moving frames" idea to get the following composition law for our invariant $\mathbb{E}_{\boldsymbol{x}}$:

Theorem 2.1 (Composition law [N16b]). *Let $\boldsymbol{x} \in \mathbb{Q}_f^2 := (\mathbb{Q} \otimes \hat{\mathbb{Z}})^2$. Then,*

$$\mathbb{E}_{\boldsymbol{x}}(\sigma_1\sigma_2) = \mathbb{E}_{\boldsymbol{x}A_{\sigma_2}}(\sigma_1) + \det(A_{\sigma_1})\mathbb{E}_{\boldsymbol{x}}(\sigma_2)$$

holds for $\sigma_1, \sigma_2 \in \pi_1^{\acute{e}t}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3$. □

Before going further, we quickly introduce a relation between the classical period $\Phi_{\boldsymbol{x}}$ and our avatar $\mathbb{E}_{\boldsymbol{x}}$. Just for now, we recall that the discrete Artin braid group B_3 with three strands fits in a central extension

$$\begin{array}{ccccccc} 1 \to \mathbb{Z} \to \widetilde{\mathrm{SL}_2(\mathbb{Z})} \cong & B_3 & \xrightarrow{\rho} & \mathrm{SL}_2(\mathbb{Z}) & \to 1. \\ & \Downarrow & & \Downarrow & \\ & \sigma & \mapsto & \rho(\sigma) & \end{array} \tag{2.2}$$

As seen later in §3, the above ρ extends to a continuous homomorphism

$$\rho : \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3 \longrightarrow \mathrm{GL}_2(\hat{\mathbb{Z}})$$

representing the monodromy actions on the torsion points of an elliptic curve.

If σ lies in the discrete part B_3 of $\hat{B}_3 \subset \pi_1^{\text{ét}}(\mathfrak{M})$, then $\rho(\sigma)$ and A_σ lie in $\mathrm{SL}_2(\mathbb{Z})$.

The following theorem is based on our work [N13].

Theorem 2.3. *One can introduce* $\mathbb{E}_{\boldsymbol{x}}(\sigma) \in \hat{\mathbb{Z}}$ *for* $\sigma \in \pi_1^{ét}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3$ *and* $\boldsymbol{x} \in \mathbb{Q}_f^2$ *in a purely combinatorial way (Fox calculus) so that when* $\boldsymbol{x} \in \mathbb{Q}^2$ *and* $\sigma \in B_3$ *with* $A_\sigma \in \mathrm{SL}_2(\mathbb{Z})$,

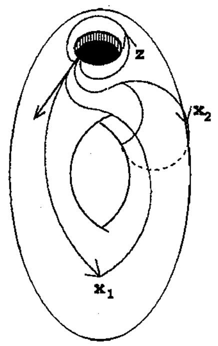
$$\begin{array}{ccccc} \mathbb{E}_{\boldsymbol{x}}(\sigma) & = & -\Phi_{\boldsymbol{x}}(A_\sigma) & + & (\textit{explicit error term}). \\ \Cap & & \Cap & & \Cap \\ \mathbb{Z} & & \mathbb{Q} & & \mathbb{Q} \end{array} \quad \square$$

Remark 2.4. It is noteworthy to observe that the above error term sweeps out the denominator of $\Phi_{\boldsymbol{x}}(A_\sigma) \in \mathbb{Q}$ to obtain an integer value $\mathbb{E}_{\boldsymbol{x}}(\sigma) \in \mathbb{Z}$. The explicit form of the error term '$K_{\boldsymbol{x}}(A_\sigma) - \frac{1}{12}\rho_\Delta(\sigma)$' is calculated in [N13, Th.7.2.3]. As a consequence, it follows, e.g., that the denominator of $\Phi_{(\frac{u}{N},\frac{v}{N})}(A)$ for $A \in \mathrm{SL}_2(\mathbb{Z})$ is bounded by $12N^2$.

3. Combinatorics in $\hat{F}_2 = \hat{\pi}_{1,1}$

In order to introduce our combinatorial avatar of Eisenstein periods, we shall set up the universal elliptic curves $E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\}$ over the parameter space $\mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\}$. We consider both $E \setminus \{O\}$ and $\mathfrak{M}$ as affine varieties over $\mathbb{Q}$. The natural projection $E \setminus \{O\} \to \mathfrak{M}$ is the Weierstrass family of elliptic curves whose structured chart from a viewpoint of anabelian geometry was discussed in [N13, §5]. In summary, we have a tangential section $\tilde{w} : \mathfrak{M} \dashrightarrow E \setminus \{O\}$ (normalized with $t := -2x/y$) and a tangential fiber $\mathrm{Tate}(q) \hookrightarrow E \setminus \{O\}$. Using the van-Kampen construction of the Tate curve, we also introduced standard loops $\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}$ of $\hat{\pi}_{1,1} := \pi_1^{ét}(\mathrm{Tate}(q) \otimes \overline{\mathbb{Q}})$ based at $\mathrm{Im}(\tilde{w}) \cap \mathrm{Tate}(q)$ on $E(\mathbb{C}) \setminus \{O\}$ with $[\mathbf{x}_1, \mathbf{x}_2]\mathbf{z} = 1$ ($[\mathbf{x}_1, \mathbf{x}_2] := \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}$). Note that $\hat{\pi}_{1,1}$ is isomorphic to a free profinite group $\hat{F}_2$ freely generated by $\mathbf{x}_1, \mathbf{x}_2$.

$$\begin{array}{ccc} E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\} & \dashleftarrow & \textcircled{\scriptsize 0}\ \mathrm{Tate}(q) \\ \downarrow\uparrow \tilde{w} & & \downarrow\uparrow \\ \mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\} & \dashleftarrow & \mathrm{Spec}\, \mathbb{Q}((q)) \end{array}$$



It is natural to employ the images of $\mathrm{Spec}\, \mathbb{Q}((q))$ as base points of those étale fundamental groups of individual spaces in the above diagram. Then, we obtain the basic identification :

$$\pi_1^{ét}(E \setminus \{O\}) = \pi_1^{ét}(\mathfrak{M}) \ltimes \hat{\pi}_{1,1}, \quad \pi_1^{ét}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3.$$

In fact, the moduli space $\mathfrak{M}$ is naturally interpreted as the space of (normalized) cubics, and a topological loop in $\pi_1(\mathfrak{M}(\mathbb{C}))$ is a motion of three points on the plane: we may identify $\pi_1(\mathfrak{M}(\mathbb{C}))$ with the Artin braid group B_3 of three strands, consequently, $\pi_1^{\text{ét}}(\mathfrak{M})$ as the semidirect product of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the profinite completion $\hat{B}_3$.

The conjugate action in the above splitting $\pi_1^{\text{ét}}(E \setminus \{O\}) = \pi_1^{\text{ét}}(\mathfrak{M}) \ltimes \hat{\pi}_{1,1}$ induces the monodromy action of $\pi_1^{\text{ét}}(\mathfrak{M})$ on $\hat{\pi}_{1,1} = \hat{F}_2$:

$$\begin{array}{ccccc} \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \rtimes \hat{B}_3 & \xrightarrow{\varphi} & \mathrm{Aut}^*(\hat{F}_2) & \xrightarrow{\mathrm{mod}\, \hat{F}_2'} & \mathrm{GL}(\hat{\mathbb{Z}}^2) \\ \Psi & & \Psi & & \Psi \\ \sigma & \longmapsto & \varphi(\sigma) & \longmapsto & \rho(\sigma) \end{array}$$

where $\mathrm{Aut}^*(\hat{F}_2)$ denotes the group of *special automorphisms* defined by

$$\mathrm{Aut}^*(\hat{F}_2) = \{\sigma \in \mathrm{Aut}(\hat{F}_2) \mid \sigma(\langle \mathbf{z} \rangle) = \langle \mathbf{z} \rangle\}.$$

Given $m \geq 1$, $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M})$ and $(u,v) \in \hat{\mathbb{Z}}^2$, let $\rho(\sigma) = \binom{ab}{cd}$ and set

$$\mathcal{S}_{uv}(\sigma) := \sigma(\mathbf{x}_2^{-v}\mathbf{x}_1^{-u})\mathbf{x}_1^{au+bv}\mathbf{x}_2^{cu+dv} \in \hat{F}_2' := [\hat{F}_2, \hat{F}_2].$$

By Ihara's theory (cf. [I99]), with the class of $\mathcal{S}_{uv}(\sigma)$ in the 2nd derived quotient $\hat{F}_2'/\hat{F}_2''$, we may associate a unique element of the complete group algebra $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] = \varprojlim_m \frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$, where $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ designate the abelianization images of $\mathbf{x}_1, \mathbf{x}_2 \in \hat{F}_2$ respectively. In order to explain this procedure in a more fitting form with the moving frame idea, it is useful to introduce a sequence of maps composed of the Fox derivative ∂_{x_1} with projections

$$\hat{\mathbb{Z}}[[\hat{F}_2]] \xrightarrow{\partial_{x_1}} \hat{\mathbb{Z}}[[\hat{F}_2]] \xrightarrow{\mathrm{ab}} \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] \xrightarrow{\mathrm{mod}\, m} \frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$$

and, writing any element of $\frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$ as $\sum_{i,j=0}^{m-1} c_{ij}\bar{\mathbf{x}}_1^i\bar{\mathbf{x}}_2^j$, define

$$\mathbb{E}_m(\sigma; u, v) := \text{constant term } c_{00} \text{ of } \left[\frac{[\partial_{\mathbf{x}_1}(\mathcal{S}_{uv}(\sigma))]^{\mathrm{ab}}}{\bar{\mathbf{x}}_2 - 1}\right]_{\bar{\mathbf{x}}_1^m = \bar{\mathbf{x}}_2^m = 1} \left(\in \hat{\mathbb{Z}}\right).$$

Cf. [N13, (3.2.3)].

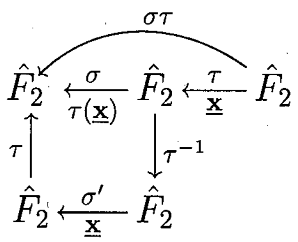
Proposition 3.1 ([N16b], Theorem A). *It holds that*

$$\mathbb{E}_m(\sigma_1\sigma_2; \boldsymbol{u}) = \mathbb{E}_m(\sigma_1; \boldsymbol{u}A_{\sigma_2}) + (\det \rho(\sigma_1)) \cdot \mathbb{E}_m(\sigma_2; \boldsymbol{u})$$

for $\sigma_1, \sigma_2 \in \mathrm{Aut}^*(\hat{F}_2)$ *and* $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$.

Proof motivation of the above composition law: Given any $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M})$, view the data $\mathbb{E}_m(\sigma) := \left[\mathbb{E}_m(\sigma; u, v)\right]_{(u,v)\in\hat{\mathbb{Z}}^2}$ as a profinite tableau on the plane $\hat{\mathbb{Z}}^2$ with entries $\hat{\mathbb{Z}}$. Let us consider traveling in $\hat{F}_2$ (with portable $\mathbb{E}_m$-board in one hand) along the composition of two automorphisms $\sigma \circ \tau \in \mathrm{Aut}^*(\hat{F}_2)$ and observe effects on the $\mathbb{E}_m$. Noting that the definition of $\mathbb{E}_m$ depends entirely on the choice of free generator system $\underline{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2)$ of $\hat{F}_2$, we are urged to look closely at the diagram

(3.2)



and especially at the effect of σ with regard to the moved frame $\tau(\underline{\mathbf{x}}) = (\tau(\mathbf{x}_1), \tau(\mathbf{x}_2))$. In fact, one symbolically finds

$$\mathcal{S}_{\boldsymbol{u}}(\sigma\tau) = \mathcal{S}_{\boldsymbol{u}}(\sigma; \text{"rel.}\tau(\underline{\mathbf{x}})\text{"}) \cdot \mathcal{S}_{\sigma'\boldsymbol{u}}(\tau)$$

which approximately leads to

$$\mathbb{E}_m(\sigma\tau, \boldsymbol{u}) \approx (\det \rho(\tau)) \cdot \mathbb{E}_m(\sigma', \boldsymbol{u}) + \mathbb{E}_m(\tau, \boldsymbol{u}A_{\sigma'}).$$

Proposition 3.1 follows then by rewriting: $\sigma_2 = \sigma' = \tau^{-1}\sigma\tau$, $\sigma_1 = \tau$ so that $\sigma_1\sigma_2 = \sigma\tau$. □

Remark 3.3. In [N13], it is shown that the adelic tableau $\mathbb{E}_m(\sigma) \in \hat{\mathbb{Z}}^{\hat{\mathbb{Z}}^2}$ encodes the image of σ by $\pi_1^{\text{ét}}(\mathfrak{M}) \to \mathrm{Aut}^*(F_2/F_2'')$ (the meta-abelian monodromy).

4. Some applications

Let us briefly pick up a few topics from [N16b].

4.1. **Homogeneity.** The above composition law Proposition 3.1 leads us to the following basic property:

Corollary 4.1 (Homogeneity [N16b] Theorem C). *Let* $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$, $\sigma \in \pi_1^{\acute{e}t}(\mathfrak{M})$. *Then, for each positive integer* $k \in \mathbb{N}$, *it holds that*

$$\mathbb{E}_m(\sigma, \boldsymbol{u}) = \mathbb{E}_{mk}(\sigma, k\boldsymbol{u}).$$

In fact, by virtue of Proposition 3.1, expressing σ as a product of $\sigma_1 \in G_{\mathbb{Q}}$ and $\sigma_2 \in \hat{B}_3$, we may reduce the proof of Corollary to individual cases where $\sigma \in G_{\mathbb{Q}}$ or $\sigma \in \hat{B}_3$. In the latter case, since $B_3 \times \mathbb{Z}^2$ is dense in $\hat{B}_3 \times \hat{\mathbb{Z}}^2$, the result follows from the explicit formula of $\mathbb{E}_{km}(\sigma, k\boldsymbol{u})$ for $\sigma \in B_3$, $\boldsymbol{u} \in \mathbb{Z}^2$ given in Theorem 2.3 (cf. [N13, Th. 7.2.3]). In the former case where $\sigma \in G_{\mathbb{Q}}$, the result follows from an explicit calculation of $\mathbb{E}_m(\sigma, \boldsymbol{u})$ which is based on the Grothendieck-Teichmüller theory on $\pi_1^{\text{ét}}(\text{Tate}(q) \setminus O)$ (see [N16b]).

The above corollary allows us to define the "*adelic Eisenstein function*" $\mathbb{E}_{\boldsymbol{x}}(\sigma)$:

$$\pi_1^{\text{ét}}(\mathfrak{M}) \times \mathbb{Q}_f^2 \ni (\sigma, \boldsymbol{x}) \longmapsto \mathbb{E}_{\boldsymbol{x}}(\sigma) \in \hat{\mathbb{Z}}$$

by assigning $\mathbb{E}_m(\sigma, \boldsymbol{u})$ for any choice of $m \in \mathbb{N}$ and $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$ so that $\boldsymbol{x} = \frac{\boldsymbol{u}}{m} \in \mathbb{Q}_f^2$. Then, Theorem 2.1 is only the reload of Proposition 3.1.

4.2. **Level splitter homomorphism** ([N16b, §7]). Let m, M be positive integers and set $N = \gcd(2, M) \cdot M$. We define the principal congruence subgroup of level N by $\pi_1^{\text{ét}}(\mathfrak{M})[N] := \{\sigma \mid A_\sigma \equiv \mathbf{1} \bmod N\}$. Then, combining results of [N12], [N13] and [N16b], we see that $\mathbb{E}_m(\sigma, \boldsymbol{u}) \bmod M$ has $m \times m$-periodicity in $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$, hence that it induces a homomorphism

$$\mathbb{E}_{m\rceil M} : \pi_1^{\text{ét}}(\mathfrak{M})[mN] \to (\mathbb{Z}/M\mathbb{Z})[(\mathbb{Z}/m\mathbb{Z})^2].$$

Generally, the above level splitter $\mathbb{E}_{m\rceil M}$ affords a non-trivial abelian quotient of $\pi_1^{\text{ét}}(\mathfrak{M})[N]$ and should involve highly arithmetic information about "Eisenstein quotient". We hope to discuss it in more details on some other occasion.

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