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GALOIS RIGIDITY OF PROFINITE FUNDAMENTAL GROUPS

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Every nonsingular complete curve over the algebraic closure $\overline{\mathbf{Q}}$ of the rationals can be realized as a three-point ramified cover over the projective line. This is a famous theorem discovered by G.V. Belyi [Be]. In this striking fact, A. Grothendieck found a clue to a wonderful new world of mathematics, and indicated the possibility of a so-called "anabelian" algebraic geometry in his mysterious note "Esquisse d'un Programme" [G3]. About ten years have passed since copies of [G3] were circulated. Unfortunately, due to his subsequent silence, we do not have precise formulations of the lines suggested in [G3], yet we are increasingly fascinated by its mysterious impression. However, the recent related work by other people seems to enable us to obtain some possible ideas, or at least vague images, on this subject.

Especially, arithmetic studies of "exterior Galois representations" initiated by Y. Ihara [Ih1], from his own motivation independent of Grothendieck, have involved many domestic and foreign mathematicians in this area, and have played a unique role in forming a mathematical crystallization of Grothendieck's dreams. In this article, we shall give an introductory sketch of the present stage of this process from the viewpoint of the "Galois rigidity" phenomenon of a profinite fundamental group which the author has been concerned with for several years. The author would like to apologize that his viewpoint here is restricted so that many other important aspects of Grothendieck's anabelian world are ignored, mainly due to the limitation of his knowledge. Moreover, due to the author's style of exposition, readers may get the impression that the following sections are just a simple enumeration of independent topics, although they are in fact closely related by a single stream of thinking. The author hopes that the reader will accept these limitations of this paper.

§1. Anabelian algebraic geometry ([G3], [G4])

(1.1) Primitive stage $\mathbf{P}^1 - \{0, 1, \infty\}$, dessin d'enfant. Let us denote by X_k the $\mathbf{P}^1_k - \{0, 1, \infty\}$, for a subfield k of the complex number field $\mathbf{C}$. By Belyi's theorem, there appear abundant nonsingular (irreducible) curves over number fields in the family of finite étale covers over $X_{\overline{\mathbf{Q}}}$. They correspond to (connected) unramified finite covers of $X(\mathbf{C})$ $(= \mathbf{C} - \{0, 1\})$ bijectively, and by the Galois theory of covers, their equivalence classes correspond to the conjugacy classes of finite index subgroups of the discrete fundamental group $\pi_1(X(\mathbf{C}))$. Since each cover can be expressed by algebraic equations with coefficients in $\overline{\mathbf{Q}}$, the absolute Galois group

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$G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of the rationals gives rise to permutations on the classes of covers through coefficient transformations.

In terms of algebraic fundamental groups, this is explained as follows. First, the algebraic fundamental group $\pi_1(X_{\overline{\mathbf{Q}}})$ of $X_{\overline{\mathbf{Q}}}$ is the profinite completion of the discrete fundamental group $\pi_1(X(\mathbf{C}))$ (i.e., the compact totally disconnected topological group obtained as the projective limit of the finite quotient groups), and the finite index subgroups of $\pi_1(X(\mathbf{C}))$ correspond bijectively to the open subgroups of $\pi_1(X_{\overline{\mathbf{Q}}})$. On the other hand, the étale fundamental group $\pi_1(X_{\mathbf{Q}})$ of $X_{\mathbf{Q}}$ is a natural group extension of $\pi_1(X_{\overline{\mathbf{Q}}})$ by $G_{\mathbf{Q}}$, fitting into the homotopy exact sequence

$$(1.1.1)\qquad 1 \longrightarrow \pi_1(X_{\overline{\mathbf{Q}}}) \longrightarrow \pi_1(X_{\mathbf{Q}}) \xrightarrow{p_X} G_{\mathbf{Q}} \longrightarrow 1.$$

The operation of $G_{\mathbf{Q}}$ on the set of equivalence classes of finite covers over $X(\mathbf{C})$ is obtained from the conjugacy action of an inverse image $\tilde{\sigma}$ of $\sigma \in G_{\mathbf{Q}}$ via p_X (i.e., a transformation of the form $* \mapsto \tilde{\sigma} * \tilde{\sigma}^{-1}$) on the open subgroups of $\pi_1(X_{\overline{\mathbf{Q}}})$.

A naive observation of Grothendieck at an early stage was that an unramified finite cover $f : Y_{\mathbf{C}} \to X_{\mathbf{C}}$ can be reconstituted from the (bipartite) graph $D_f = f^{-1}([0,1])$ on the closed compactification $[Y_{\mathbf{C}}]$ of $Y_{\mathbf{C}}$, which is named a *dessin d'enfant* (child's drawing) by Grothendieck. (Here, $[0,1] = \{t \in \mathbf{P}^1(\mathbf{C}) \mid t \in \mathbf{R},\ 0 \leq t \leq 1\}$.) This is easy to see, because the monodromy permutation representation by the standard generators $x, y \in \pi_1(X(\mathbf{C}), t_0)$ on the fibre set $f^{-1}(t_0)$ can be described by observing the graph $D_f \subset [Y_{\mathbf{C}}]$, if we take x (resp. y) to be the loop turning around the puncture 0 (resp. 1) which originates at the base point t_0 on the open interval $(0,1)$ and runs along the interval. A dessin d'enfant is an equivalence class of a pair consisting of a connected graph and its embedding in a topological surface enjoying simple axioms, and if it is given, the complex structure of the surface and the morphism to the projective line with a three-point ramification are determined (cf. [JS]). For a given dessin, this morphism is called the associated Belyi function ([SV]), and the genus of the surface or the valency list of the graph (the list of the numbers of edges adjacent to vertices) are called those of the dessin. Thus the dessins correspond bijectively to the equivalence classes of the unramified finite covers of $X_{\mathbf{C}}$. Since the $G_{\mathbf{Q}}$-action on the dessins preserves the genus and the valency list of each dessin, the orbits are finite sets; hence with each dessin an algebraic number field (of finite degree) is associated as the fixed field of the stabilizer of the Galois action on it. It is an interesting problem to identify the associated number field by the combinatorial structure of each dessin. See, e.g., [SV], [Sch], [Cou], [BPZ] for experimental computations. Recalling that for each $j \in \overline{\mathbf{Q}}$ there is a standard elliptic curve with minimum field of definition $\mathbf{Q}(j)$ (see, e.g., [Sh], 4.1), we see that the Galois group $G_{\mathbf{Q}}$ acts faithfully on the set of dessins of genus 1 as a corollary of Belyi's theorem. (Recently, L. Schneps [Sch] also showed that $G_{\mathbf{Q}}$ acts on the genus 0 dessins faithfully.)

We say that an automorphism of a group π is an inner automorphism if it is written as the conjugate action by an element of π. The collection $\mathrm{Int}(\pi)$ of the inner automorphisms forms a normal subgroup of the automorphism group $\mathrm{Aut}(\pi)$, and the quotient group $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Int}(\pi)$ is called the outer automorphism group of π. Taking an inverse image of each element $\sigma \in G_{\mathbf{Q}}$ in the sequence (1.1.1) via p_X, and considering its conjugate action on $\pi_1(X_{\overline{\mathbf{Q}}})$, we obtain an outer automorphism of $\pi_1(X_{\overline{\mathbf{Q}}})$ well-defined for σ, and hence the Galois representation

$$(1.1.2)\qquad \varphi : G_{\mathbf{Q}} \to \mathrm{Out}\pi_1(X_{\overline{\mathbf{Q}}}).$$

This will be called the *exterior Galois representation* arising from (1.1.1).

The faithfulness of the action of $G_{\mathbf{Q}}$ on the dessins described above enables us to get Belyi's striking observation that *the representation φ is injective.* This is remarkable because it suggests the possibility of describing the huge Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by a finite number of parameters of finitely generated profinite groups. We shall return to this topic later in (1.4) in the context of moduli spaces of curves.

(1.2) Galois rigidity of mixed fundamental groups. The above phenomenon means especially that in the "mixed" fundamental group $\pi_1(X_{\mathbf{Q}})$ (1.1.1) which consists of the Galois group $G_{\mathbf{Q}}$ and the fundamental group $\pi_1(X_{\overline{\mathbf{Q}}})$, the arithmetic quotient and the geometric subgroup are fused together in a strong manner far from being just a direct product of groups. Their strong links will be inherited by each of the open subgroups of $\pi_1(X_{\mathbf{Q}})$ in a more and more complicated fashion in our nonabelian fundamental group, where a lot of fundamental groups of arithmetic curves appear as its open subgroups (as explained above).

Now we consider an arbitrary hyperbolic curve C over a number field k. Here "hyperbolic" means that C is uniformized by the complex upper half-plane. Then there exists an exact sequence similar to (1.1.1):

$$(1.2.1) \qquad 1 \longrightarrow \pi_1(C_{\bar{k}}) \longrightarrow \pi_1(C) \xrightarrow{p_{C/k}} G_k \longrightarrow 1.$$

If the Riemann surface associated with C has genus g and n punctures, then the geometric fundamental group $\pi_1(C_{\bar{k}})$ is isomorphic to the profinite completion of the discrete group

$$(1.2.2) \qquad \Pi_{g,n} = \left\langle \begin{matrix} x_1, \ldots, x_g, x_{g+1}, \ldots, x_{2g} \\ z_1, \ldots, z_n \end{matrix} \;\middle|\; [x_1, x_{g+1}] \cdots [x_g, x_{2g}] z_1 \cdots z_n = 1 \right\rangle,$$

and it depends only on the topological type (g, n) of the $\mathbf{C}$-valued points of C. Supposing that the mixed fundamental group $\pi_1(C)$ is structured as a strong fusion of this and the arithmetic quotient G_k, we may expect that the structure of the group extension involves delicate information on the algebraic structure of the curve C/k. In other words, the exterior Galois representation induced from (1.2.1),

$$\varphi_C : G_k \to \mathrm{Out}\pi_1(C_{\bar{k}}),$$

may involve a certain proper structure (a kind of rigidity) from which one may effectively extract algebraic properties of the curve. Pursuing this possibility will lead us to the following series of conjectures.

(1.3) Fundamental conjecture, anabelian Tate conjecture ([G3],[G4]). By Faltings' theorem that solved the Tate conjecture on homomorphisms of abelian varieties ([Fal]), the Galois representations on the 1-dimensional l-adic homology groups determine the isogeny classes of the abelian varieties, and the set of Galois equivariant homomorphisms between them is naturally isomorphic to the l-adic completion of the module of homomorphisms between the corresponding abelian varieties. The 1-dimensional homology of an abelian variety is nothing but its fundamental group. Comparing this with the Galois rigidity of fundamental groups of hyperbolic curves far from being abelian (1.2) leads us to the following Grothendieck conjectures:

(1.3.1) Fundamental conjecture. *The isomorphism class of a hyperbolic curve C over a field k that is finitely generated over the rationals can be reconstituted from the structure of the mixed fundamental group $\pi_1(C)$ as an extension group over the Galois group G_k.*

(1.3.2) Anabelian Tate conjecture. *If C, C' are two hyperbolic curves over a field k finitely generated over the rationals, then the natural mapping*

$$\Phi_{(C,C')} : \mathrm{Hom}_k(C, C') \to \mathrm{Hom}_{G_k}(\pi_1(C), \pi_1(C'))/\mathrm{Int}\pi_1(C'_{\bar{k}})$$

gives a bijection between the dominant k-morphisms and the classes of G_k-compatible open homomorphisms. Here, "$/\mathrm{Int}\pi_1$" means the quotient by the natural actions by the inner automorphisms of π_1.

If J is the Jacobian variety of the smooth compactification of the curve C, then the 1-dimensional homology group $H_1(J_{\bar{k}})$ appears as a natural abelian quotient of $\pi_1(C_{\bar{k}})$. The Tate conjecture proved by Faltings asserts that for Jacobian varieties J, J',

$$\mathrm{Hom}_k(J, J') \otimes \mathbf{Z}_l \xrightarrow{\sim} \mathrm{Hom}_{G_k}(H_1(J_{\bar{k}}), H_1(J'_{\bar{k}})).$$

So, the conjecture (1.3.2) could be regarded as a noncommutative lifting of the Tate conjecture.

There are some affirmative results on the fundamental conjecture (1.3.1) in the cases of genus 0 and genus 1 ([N1], [N6]), whereas there are no examples supporting the anabelian Tate conjecture as it is. For the latter, there is also a formulation by Grothendieck in the case where C is a point. Recently, an approach to the pro-l variant for the case $C = C'$ was given in [NT1]; this will be explained in detail later in §3.

So far, we have mainly discussed curves, although Grothendieck actually presumed that, even in the higher-dimensional varieties, there should be a class of varieties whose geometric properties can be reconstituted from the group structures of their mixed fundamental groups. He called varieties in such a hypothetical class "*anabelian algebraic varieties*", because they should have fundamental groups far from being abelian. Hyperbolic curves, their iterated fibration spaces, or the moduli spaces of curves are suggested as candidates for anabelian varieties.

In number theory, the reconstitution of geometry from group theory has been known for algebraic number fields by the Neukirch-Ikeda-Iwasawa-Uchida theorem. Namely, if k and k' are number fields of finite degree over $\mathbf{Q}$, the isomorphy of the Galois groups $G_k \cong G_{k'}$ implies the isomorphy of fields $k \cong k'$, and the outer automorphism group of G_k recovers the automorphism group of the field k (cf. [Neu], [U]; note that G_k is the fundamental group of the point $\mathrm{Spec}\, k$). It is unknown to the author whether Grothendieck knew this fact when he considered his conjectures above. However, Grothendieck more generally conjectured that the category of the fields finitely generated over $\mathbf{Q}$ should be embedded into the category of profinite groups accompanied by augmentation homomorphisms into $G_{\mathbf{Q}}$ (the fundamental conjecture for birational anabelian algebraic geometry [G4]). For this, there are affirmative results in the relative dimension-one case ([Pol-2]), and some ambitious approaches can be found in [Bogl-3].

(1.4) Playing Lego with Galois-Teichmüller. In the light of the fundamental conjecture (1.3.1), the exterior Galois representation $\varphi : G_{\mathbf{Q}} \to \mathrm{Out}\pi_1(X_{\overline{\mathbf{Q}}})$ (1.1.2) in the primitive stage $X = \mathbf{P}^1 - \{0,1,\infty\}$ acquires fundamental significance, as the Belyi theorem assures that the subquotients of $\pi_1(X_{\mathbf{Q}})$ contain information on all the complete hyperbolic curves definable over number fields. As a first question on φ, we ask the following:

(1.4.1) Problem. *How is the Galois image $\varphi(G_{\mathbf{Q}})$ characterized as a subgroup of* $\mathrm{Out}\,\pi_1(X_{\overline{\mathbf{Q}}})$?

Now $\pi_1(X_{\overline{\mathbf{Q}}})$ is a free profinite group $\hat{F}_2(x,y)$ of rank 2 with free generators x, y taken as in (1.1), and φ can be uniquely lifted to a representation $\tilde{\varphi} : G_{\mathbf{Q}} \to \mathrm{Aut}\pi_1(X_{\overline{\mathbf{Q}}})$ ($\sigma \mapsto \tilde{\varphi}_\sigma$) under the normalization conditions ([Be]): (1) $\tilde{\varphi}_\sigma(x) = x^{\chi(\sigma)}$, (2) $\tilde{\varphi}_\sigma(y) = f_\sigma^{-1} y^{\chi(\sigma)} f_\sigma$ ($f_\sigma \in [\pi_1, \pi_1]$). We shall call this lift $\tilde{\varphi}$ the Belyi lifting of φ. Here $\chi : G_{\mathbf{Q}} \to \hat{\mathbf{Z}}^\times$ is the cyclotomic character obtained from the Galois action on roots of unity, and $[\pi_1, \pi_1]$ denotes the (closure of the) commutator subgroup of $\pi_1(X_{\overline{\mathbf{Q}}}) = \hat{F}_2(x,y)$. Since $f_\sigma \in \hat{F}_2(x,y)$ is uniquely determined by (2) for each element $\sigma \in G_{\mathbf{Q}}$, the problem (1.4.1) is equivalent to the problem of determining the image of the parameter $f : G_{\mathbf{Q}} \to \hat{F}_2(x,y)$. The symmetry of $X = \mathbf{P}^1 - \{0,1,\infty\}$ under the symmetric group of degree 3 forces this parameter to satisfy the two conditions:

$$f_\sigma(x,y) = f_\sigma(y,x)^{-1}; \tag{1.4.2}$$

$$f_\sigma(z,x) z^{b_\sigma} f_\sigma(y,z) y^{b_\sigma} f_\sigma(x,y) x^{b_\sigma} = 1, \tag{1.4.3}$$

where $z = y^{-1}x^{-1}$, $b_\sigma = (\chi(\sigma) - 1)/2$ (see, e.g., [Ih11] or [N3], Appendix).

In [G3], Grothendieck pointed out that $X = \mathbf{P}^1 - \{0,1,\infty\}$ is nothing but the moduli of the 4-pointed genus 0 curves, $M_{0,4}$, and that it should be regarded as the primitive stage ("premier étage") of the tower of the moduli spaces $M_{g,n}$ of n-pointed genus g curves. (We shall simply call $M_{g,n}$ the moduli of type (g,n).) Moreover, he conjectured that the tower of the fundamental group(oids) of the $M_{g,n}$ (the Teichmüller modular tower) can be understood by analyzing only the four types $(g,n) = (0,4), (0,5), (1,1), (1,2)$. V. G. Drinfeld, in his famous paper [Dr], gave a beautiful interpretation of the genus 0 tower in the context of conformal field theory, obtained a third (pentagonal) condition supplementing (1.4.2-3) from a study corresponding to $M_{0,5}$, and defined the *Grothendieck-Teichmüller group "GT"* to be the subgroup of $\mathrm{Aut}\hat{F}_2(x,y)$ of the elements enjoying these three conditions. Since the group GT turned out to have natural actions on the Teichmüller groups of types $(0,n)$ ($n = 4, 5, \dots$), Drinfeld's work supports Grothendieck's conjecture affirmatively in that only the types (0,4) and (0,5) are fundamental in the genus 0 tower of moduli. Although $\tilde{\varphi}(G_{\mathbf{Q}}) \subset GT$ was not proved in [Dr], a geometric proof ([Ih7], [Ih11]) and a group-theoretical proof ([N3], Appendix) of it were published later. Recently, Lochak and Schneps [LS] showed that GT can be regarded as the automorphism group of a certain tower of braid groups, simplifying the categorical treatment of GT given by Drinfeld. Precise definition of the Teichmüller tower including higher genera has not been published yet, though some people say that it would be formed by putting together $M_{g,n}$ in some ways through infinity structures along degeneration of curves, modeling on the duality groupoid of Moore-Seiberg [MS] in physics.

§2. Hyperbolic varieties, morphism-rigidity

(2.1) Center-triviality of profinite fundamental groups. One reason why the fundamental groups of hyperbolic Riemann surfaces are "far from being abelian" is that the subgroups of finite index have trivial centers. This is explained by Gottlieb's theorem [Go] in topology asserting that if a finite simplicial complex is $K(\pi,1)$ and has nonzero Euler characteristic, then its fundamental group has trivial center. Notice that the assumptions of this theorem are obviously inherited by the finite covers of the complex. The finiteness and the $K(\pi,1)$-ness of the complex are equivalent to the (FL)-ness of its fundamental group Γ in the sense of Serre [Se2], i.e., the property that the trivial Γ-module $\mathbf{Z}$ has a resolution of finite length by finitely generated free $\mathbf{Z}[\Gamma]$-modules.

Suppose we are given a nonsingular variety $X_{\bar{k}}$ over an algebraically closed subfield $\bar{k}$ of $\mathbf{C}$ whose associated analytic manifold $X(\mathbf{C})$ satisfies Gottlieb's conditions. Although the profinite fundamental group $\pi_1(X_{\bar{k}})$ is isomorphic to the profinite completion $\hat{\Gamma}$ of $\Gamma=\pi_1(X(\mathbf{C}))$, we still cannot guarantee in general that the open subgroups of $\pi_1(X_{\bar{k}})$ have trivial centers. A sufficient condition given in [N3] (1.3.3) for this is that $\Gamma=\pi_1(X(\mathbf{C}))$ is, in addition, *good* in the sense of Serre [Se], that is, has the property that the natural isomorphism $H^i(\hat{\Gamma},M)\cong H^i(\Gamma,M)$ of cohomology groups holds for every continuous finite $\hat{\Gamma}$-module M. From this we see that the hyperbolic curves and their iterated fibration spaces (Artin neighborhoods of hyperbolic type) have geometric profinite fundamental groups whose open subgroups are always center-free. The fundamental groups of moduli of curves, namely, the Teichmüller modular groups, are virtually of type (FL) with nonzero Euler characterisitic (see [Harv], [Iv3], [HZ]), but it does not seem to be known whether, in general, they are good or not (see [Oda1]). Precise conditions characterizing anabelian algebraic varieties in higher dimensions have not yet been proposed. The author suspects that the center-triviality of open subgroups of profinite fundamental groups would be one of the main features of such varieties. This will be indicated also from our discussions in (2.2) and (3.1) below.

(2.2) An application to Sunada's conjecture ([N5]). In the Nevanlinna theory, a principle enunciated by L. V. Ahlfors says that "the negative curvature of the image manifold restricts a holomorphic mapping" ([Sugaku], 21-N), while in the context of complex geometry, T. Sunada [Sun] conjectured that any surjective holomorphic mapping onto a hyperbolic manifold should have strong rigidity. As an analog of these phenomena in the algebraic context, we shall discuss a homotopical hyperbolicity property of an algebraic variety, which has a slightly different taste from Lang's conjecture on the distribution of rational points in hyperbolic varieties ([Lan],[Voj]).

In fact, through the exterior Galois actions on profinite fundamental groups, we can deduce the following purely geometric statement. A morphism $f: X\to Y$ of nonsingular algebraic varieties over $\mathbf{C}$ is said to have (algebraic) strong rigidity, if every morphism homotopic to f must coincide with f.

(2.2.1) Theorem. *If Y is a finite étale cover of an Artin neighborhood of hyperbolic type, then every morphism $f: X\to Y$ with $\pi_1(f)$ dominant (i.e., having open image)has strong rigidity.*

When Y is "$K(\pi,1)$", a morphism $f : X \dashrightarrow Y$ has strong rigidity if and only if f is mapped injectively by

$$(2.2.2) \qquad \Phi = \Phi^{\mathrm{geom}} : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(\pi_1 X, \pi_1 Y)/\mathrm{Int}\pi_1 Y,$$

i.e., $\Phi^{-1}\Phi(f) = \{f\}$ holds. By a standard descent argument on fields of definition, our problem is reduced to showing that if $f_0 : X_0 \to Y_0$ is a π_1-dominant morphism of algebraic varieties over k_0 finitely generated over $\mathbf{Q}$, then the mapping

$$(2.2.3) \qquad \Phi^{\mathrm{geom}} : \mathrm{Hom}(\overline{X}_0, \overline{Y}_0) \to \mathrm{Hom}(\pi_1\overline{X}_0, \pi_1\overline{Y}_0)/\mathrm{Int}\pi_1\overline{Y}_0$$

is injective at $\bar{f}_0$. (Here, $^-$ means $\otimes\bar{k}_0$.) Moreover, if $\mathrm{Hom}_{\mathrm{Gal}}(\pi_1\tilde{X}_0, \pi_1\tilde{Y}_0)$ denotes the space of Galois-compatible homomorphisms defined as the inductive limit of the spaces of the G_k-compatible homomorphisms $\pi_1(X_0 \otimes k) \dashrightarrow \pi_1(Y_0 \otimes k)$ along the finite extensions k/k_0 in $\bar{k}_0$, then Φ^{geom} can be decomposed as a composite of the two mappings

$$(2.2.3) \qquad \Phi^{\mathrm{arith}} : \mathrm{Hom}(\overline{X}_0, \overline{Y}_0) \dashrightarrow \mathrm{Hom}_{\mathrm{Gal}}(\pi_1\tilde{X}_0, \pi_1\tilde{Y}_0)/\mathrm{Int}\pi_1\overline{Y}_0,$$

$$(2.2.4) \quad \mathrm{Res} : \mathrm{Hom}_{\mathrm{Gal}}(\pi_1\tilde{X}_0, \pi_1\tilde{Y}_0)/\mathrm{Int}\pi_1\overline{Y}_0 \dashrightarrow \mathrm{Hom}(\pi_1\overline{X}_0, \pi_1\overline{Y}_0)/\mathrm{Int}\pi_1\overline{Y}_0.$$

Our theorem is then a consequence of the following two lemmas.

(2.2.5) Lemma (Grothendieck). *If Y is a subvariety of a quasi-abelian variety or a finite étale cover of an Artin neighborhood of hyperbolic type, then Φ^{arith} is injective.*

(2.2.6) Lemma. *If every open subgroup of $\pi_1 Y$ has trivial center, then the map* Res *is injective on the open homomorphisms of π_1.*

The conclusion of Lemma (2.2.5) is needless to say a necessary condition to be assured before one expects (a higher-dimensional version of) the anabelian Tate conjecture. Observe also that one may utilize the profinite Gottlieb theorem (2.1) when checking the assumption of Lemma (2.2.6).

It is also possible to apply our method to the similar problem on morphisms having the same image at a given point. In fact, in an early stage, Grothendieck [G1] considered the pointed version in the case of moduli of abelian varieties, and Borel-Narasimhan [BN] generalized his result in the context of hyperbolic geometry. These kinds of morphism-rigidity together with certain kinds of "boundedness" lead to finiteness theorems of de Franchis type (see, e.g., [ZL], [No], [Suz], [Fa2], [Sa], [ImS], etc.) Parshin [Pa] gave suggestive comments on [G4] from the viewpoint of hyperbolic geometry.

§3. AUTOMORPHISMS, ALGEBRAIC CURVES

(3.1) Galois centralizers. As explained in (1.2), a basic theme of anabelian algebraic geometry is the reconstitution of "geometry" from fundamental groups. But which kind of geometry should be considered in our first stage of investigation? Consulting early stages of the Kobayashi hyperbolic geometry [Ko] or Iitaka's program [Ii], one may learn to begin with "automorphism groups" at the first step toward a new geometry. So, we shall set up our problem as follows.

Let X be a nonsingular algebraic variety over a number field k, and $\mathfrak{C}$ a class of finite groups closed under the formation of subgroups, quotients and group extensions. We define the pro-$\mathfrak{C}$ fundamental groups $\pi_1^{\mathfrak{C}}(X_{\bar{k}})$ and $\pi_1^{\mathfrak{C}}(X)$; the former is the maximal pro-$\mathfrak{C}$ quotient of the geometric fundamental group $\pi_1(X_{\bar{k}})$, and the latter is the quotient of the mixed fundamental group $\pi_1(X)$ fitting into the following natural exact sequence:

$$(3.1.1) \qquad 1 \longrightarrow \pi_1^{\mathfrak{C}}(X_{\bar{k}}) \longrightarrow \pi_1^{\mathfrak{C}}(X) \xrightarrow{p_{X/k}^{\mathfrak{C}}} G_k \longrightarrow 1.$$

We say an automorphism f of $\pi_1^{\mathfrak{C}}(X)$ is G_k-compatible if it satisfies $p_{X/k}^{\mathfrak{C}} = p_{X/k}^{\mathfrak{C}} \circ f$. Write $\mathrm{Aut}_{G_k}\pi_1^{\mathfrak{C}}(X)$ for the collection of G_k-compatible automorphisms of $\pi_1^{\mathfrak{C}}(X)$ and set

$$\mathcal{E}_k^{\mathfrak{C}}(X) := \mathrm{Aut}_{G_k}\pi_1^{\mathfrak{C}}(X)/\mathrm{Int}\pi_1^{\mathfrak{C}}(X_{\bar{k}}).$$

On the other hand, letting

$$(3.1.2) \qquad \varphi_X^{\mathfrak{C}} : G_k \to \mathrm{Out}\pi_1^{\mathfrak{C}}(X_{\bar{k}})$$

be the exterior Galois representation associated with (3.1.1), we denote by

$$\mathrm{Out}_{G_k}\pi_1^{\mathfrak{C}}(X_{\bar{k}})$$

the centralizer of the image $\varphi_X^{\mathfrak{C}}(G_k)$ in $\mathrm{Out}\pi_1^{\mathfrak{C}}(X_{\bar{k}})$, and call it *the Galois centralizer*. Both $\mathcal{E}_k^{\mathfrak{C}}(X)$ and $\mathrm{Out}_{G_k}\pi_1^{\mathfrak{C}}(X_{\bar{k}})$ are naive candidates for "the group of Galois-compatible homotopy equivalence classes" when the variety X is assumed to be more or less negatively curved and hence to be almost $K(\pi, 1)$ by the Hadamard-Cartan principle ([Sugaku], 178 B). We have a natural restriction mapping $\mathcal{E}_k^{\mathfrak{C}}(X) \dashrightarrow \mathrm{Out}_{G_k}\pi_1^{\mathfrak{C}}(X_{\bar{k}})$, which gives a bijection if the center of $\pi_1^{\mathfrak{C}}(X_{\bar{k}})$ is trivial (cf. [N4], 1.5). Thus, when the center triviality of the fundamental group is established, the pro-$\mathfrak{C}$ automorphy version of the anabelian Tate conjecture is reduced to the problem of estimating the following canonical mapping:

$$\Phi_X^{\mathfrak{C}} : \mathrm{Aut}_k(X) \dashrightarrow \mathrm{Out}_{G_k}\pi_1^{\mathfrak{C}}(X_{\bar{k}}).$$

In most cases, X is of log general type and has finite $\mathrm{Aut}_k(X)$ ([Ii]). So our main questions are:

Question (3.1.3). Which kinds of pairs $(X/k, \mathfrak{C})$ can have bijective $\Phi_X^{\mathfrak{C}}$?

Question (3.1.4). More basically, can the Galois centralizer $\mathrm{Out}_{G_k}\pi_1^{\mathfrak{C}}(X_{\bar{k}})$ be finite even when $\pi_1^{\mathfrak{C}}(X_{\bar{k}})$ is (uncountably) infinite?

A verification of Question (3.1.3) for a class of finite groups $\mathfrak{C}$ means a reconstitution of the geometry "$\mathrm{Aut}_k(X)$" from the mixed fundamental group of X. In addition to our discussion in §2 on the injectivity of $\Phi_X^{\mathfrak{C}}$, Borel's criterion [Bor2] is known (see [CR]). The geometric fundamental group $\pi_1^{\mathfrak{C}}(X_{\bar{k}})$ turns out to be a characteristic subgroup of the mixed fundamental group $\pi_1^{\mathfrak{C}}(X)$, and when the former is center-free, the outer automorphism group of the mixed fundamental group is a group extension of the Galois centralizer by a subgroup of $\mathrm{Out}(G_k)$ ([N3], 1.6). By Uchida's theorem (see [Neu], [U]), $\mathrm{Out}(G_k)$ is isomorphic to $\mathrm{Aut}(k)$, which is finite, so Question (3.1.4) is equivalent to asking whether the outer automorphism group of the mixed fundamental group $\pi_1^{\mathfrak{C}}(X)$ is finite.

In the remainder of this section, we shall discuss these questions in the case where X is a hyperbolic curve, and in (4.1) we shall present affirmative examples of higher-dimensional varieties for Question (3.1.3).

(3.2) Pro-l mapping class groups

In order to describe the exterior Galois representation for a curve, we shall explain the "coordinatization" of the representation space. Fix a rational prime l, and let (l) denote the class of finite l-groups. We call the pro-l completion of the surface fundamental group $\Pi_{g,n}$ (1.2.2) the *pro-l fundamental group*, and denote it by $\Pi_{g,n}^{\mathrm{pro}-l}$. The method of "coordinates" in the study of the outer automorphism group of the pro-l fundamental group was initiated by Ihara [Ih1-2] in the genus 0 case, and was developed by Oda, Asada, Kaneko ([Oda2], [AK], [Ka]) and by Nakamura and Tsunogai [NT1-2].

First, we shall call the group of the outer automorphisms of $\Pi_{g,n}^{\mathrm{pro}-l}$ that preserve the union of the conjugacy classes of the cyclic subgroups $\langle z_1\rangle, \dots, \langle z_n\rangle$ the *pro-l mapping class group* (denoted by $\Gamma_{g,n}^{\mathrm{pro}-l}$). In these groups $\Pi_{g,n}^{\mathrm{pro}-l}$, $\Gamma_{g,n}^{\mathrm{pro}-l}$, there are good descending filtrations by normal subgroups $\{\Pi_{g,n}^{\mathrm{pro}-l}(m)\}_{m=1}^{\infty}$, $\{\Gamma_{g,n}^{\mathrm{pro}-l}(m)\}_{m=0}^{\infty}$, called the weight filtrations, with the following properties:

(3.2.1) $\mathrm{gr}^0\Gamma_{g,n}^{\mathrm{pro}-l} \cong \mathrm{GSp}(2g, \mathbf{Z}_l) \times S_n$. (We designate this group simply by $\mathrm{GSp}(2g, n)$, which we understand to be $\mathbf{G}_m \times S_n$ when $g = 0$.)

(3.2.2) For $m \geq 1$, $\mathrm{gr}^m\Pi_{g,n}^{\mathrm{pro}-l}$ and $\mathrm{gr}^m\Gamma_{g,n}^{\mathrm{pro}-l}$ are finitely generated free $\mathbf{Z}_l$-modules on which the group $\mathrm{GSp}(2g, n)$ naturally acts algebraically. The action of the latter comes from the conjugate action inside $\Gamma_{g,n}^{\mathrm{pro}-l}$.

(3.2.3) If a matrix $A \in \mathrm{GSp}(2g)$ has a characteristic polynomial with rational integer coefficients and has eigenvalues with the same complex absolute value N, then the complex absolute values of eigenvalues of the action of A on $\mathrm{gr}^m\Pi_{g,n}^{\mathrm{pro}-l}$ and $\mathrm{gr}^m\Gamma_{g,n}^{\mathrm{pro}-l}$ are all equal to N^m.

(3.2.4) $\bigcap_m \Pi_{g,n}^{\mathrm{pro}-l}(m) = \{1\}$, $\bigcap_m \Gamma_{g,n}^{\mathrm{pro}-l}(m) = \{1\}$.

In fact, we have explicit descriptions of the graded quotient modules $\mathrm{gr}^m\Pi_{g,n}^{\mathrm{pro}-l}$ $(m \geq 1)$ as $\mathrm{GSp}(2g, n)$-modules, through the study by Witt and Labute (see [Bou], [Lab], [KO], [NT1], [AN]). By using these $\mathrm{gr}^m\Pi_{g,n}^{\mathrm{pro}-l}$, we can introduce the *coordinate module* $C^m(2g, n)$ $(m \geq 1)$ by

$$C^m(2g,n) = \begin{cases} \mathrm{Hom}(\mathrm{gr}^1\Pi_{g,n}^{\mathrm{pro}-l}, \mathrm{gr}^{m+1}\Pi_{g,n}^{\mathrm{pro}-l}) \oplus \mathrm{Hom}(\mathbf{Z}_l^n, \mathrm{gr}^m\Pi_{g,n}^{\mathrm{pro}-l}), & m \neq 2, \\ \mathrm{Hom}(\mathrm{gr}^1\Pi_{g,n}^{\mathrm{pro}-l}, \mathrm{gr}^3\Pi_{g,n}^{\mathrm{pro}-l}) \oplus \bigoplus_{j=1}^n \mathrm{Hom}(\mathbf{Z}_l, \mathrm{gr}^2\Pi_{g,n}^{\mathrm{pro}-l}/\langle \bar{z}_j\rangle), & m = 2. \end{cases}$$

There is a natural "diagonal" embedding of $\mathrm{gr}^m\Pi_{g,n}^{\mathrm{pro}-l}$ into $C^m(2g, n)$, and we call the quotient module

$$\overline{C}^m(2g,n) := C^m(2g,n)/\mathrm{gr}^m\Pi_{g,n}^{\mathrm{pro}-l}$$

the *reduced coordinate module.* Then, the graded quotient $\mathrm{gr}^m\Gamma_{g,n}^{\mathrm{pro}-l}$ of the pro-l mapping class group can be embedded into $\overline{C}^m(2g, n)$ as a $\mathrm{GSp}(2g, n)$-module, and the cokernel is isomorphic to $\mathrm{gr}^{m+2}\Pi_{g,n}^{\mathrm{pro}-l}(-1)$. (This embedding is the pro-l analog of the Johnson homomorphism studied by D. L. Johnson, S. Morita in topology. In the discrete case, the problem of determining the cokernel of the Johnson homomorphism is still open; see [J], [Mo1-2], [AN], [N7], [H].) We call such a system of the weight filtration on the pro-l mapping class group equipped with the $\mathrm{GSp}(2g, n)$-equivariant coordinatizations of its graded quotient modules the *graded* *coordinate system* or *weight coordinate system* on the pro-l mapping class group. One can compute explicitly the characters of the representations of $\mathrm{GSp}(2g) \times S_n$ on these $\otimes \mathbf{Q}_l$ which have stable irreducible decompositions for sufficiently large g and n. For example,

$$\begin{aligned}
\mathrm{gr}^1 \Gamma_{g,n}^{\mathrm{pro}-l} \otimes \mathbf{Q}_l &\cong [1^3]_{Sp(2g)} + \{(n)_{S_n} + (n-1,1)_{S_n}\} \otimes [1]_{Sp(2g)}, \\
\mathrm{gr}^2 \Gamma_{g,n}^{\mathrm{pro}-l} \otimes \mathbf{Q}_l &\cong [2^2]_{Sp(2g)} + \{(n)_{S_n} + (n-1,1)_{S_n}\} \otimes [1^2]_{Sp(2g)} \\
&\quad + \{(n)_{S_n} + (n-1,1)_{S_n} + (n-2,2)_{S_n}\} \otimes [0]_{Sp(2g)}, \\
\mathrm{gr}^3 \Gamma_{g,n}^{\mathrm{pro}-l} \otimes \mathbf{Q}_l &\cong [3,1^2]_{Sp(2g)} + [3]_{Sp(2g)} + \{(n)_{S_n} + (n-1,1)_{S_n}\} \otimes [2,1]_{Sp(2g)} \\
&\quad + \{(n-2,1^2)_{S_n} + (n-1,1)_{S_n}\} \otimes [1]_{Sp(2g)}.
\end{aligned}$$

Here $\{\lambda\}_{\mathrm{Sp}(2g)}$ and $(\mu)_{S_n}$ denote the irreducible characters corresponding to the Young diagrams λ and μ respectively. For small g, n, one can apply the specialization rule of Koike and Terada [KT] and the sign rule of Murnaghan [Mu] for computing the irreducible decompositions. For the explicit character formulae, the reader may refer to [AN]. By using a computer, we obtain an atlas of the graded quotient modules of pro-l fundamental groups and pro-l mapping class groups in terms of Young diagrams for $m \leq 17$, 15 respectively ([NT2]).

(3.3) Duplicate "weight" arguments. The problem of estimating Galois centralizers in pro-l fundamental groups of curves was studied in [N2-3], and a general framework was given in [NT1]. Our approach depends on two different kinds of weight filtrations —the weight filtration in the pro-l mapping class group of (3.2) and the anabelian weight filtration explained below.

Let X be a complete nonsingular curve defined over a number field k of genus g, S a finite set of closed points of X with geometric cardinality $|S(\bar{k})| = n$. The complement curve $C = X \setminus S$ is of hyperbolic type if and only if the Euler characteristic $2 - 2g - n$ is negative. We assume this hyperbolicity condition. If the pro-l fundamental group of $C_{\bar{k}}$ is identified with $\Pi_{g,n}^{\mathrm{pro}-l}$ in such a way that $z_1, \ldots, z_n$ represent parabolic elements, then it is well known that the associated exterior Galois representation $\varphi_C^{(l)} : G_k \to \mathrm{Out}\Pi_{g,n}^{\mathrm{pro}-l}$ has its image contained in the pro-l mapping class group $\Gamma_{g,n}^{\mathrm{pro}-l}$ (branch cycle argument [Fr]; cf. also [Matz]). On the other hand, we have

(3.3.1) Theorem. *The Galois centralizer* $\mathrm{Out}_{G_k}\Pi_{g,n}^{\mathrm{pro}-l}$ *is also contained in the pro-l mapping class group* $\Gamma_{g,n}^{\mathrm{pro}-l}$.

This theorem means that the union $\mathcal{Z}$ of the cyclic subgroups conjugate to one of $\langle z_1 \rangle, \ldots, \langle z_n \rangle$ is distinguished from its complement in the free pro-l group $\Pi_{g,n}^{\mathrm{pro}-l}$ under the operation of the Galois group. In fact, the subset $\mathcal{Z} \subset \Pi_{g,n}^{\mathrm{pro}-l}$ can be characterized as the "weight (-2) subset" with "weight (-1) complement" ([N1]; [N3], 2.1). We shall call this characterization property of the inertia subgroups in $\pi_1(C)$ *the anabelian weight filtration*, because the filtered components are not closed under the group operation (a union of conjugacy classes).

By the above theorem, we can encircle the Galois image $\varphi_C^{(l)}(G_k)$ and the Galois centralizer $\mathrm{Out}_{G_k}\Pi_{g,n}^{\mathrm{pro}-l}$ in the net of the coordinate system on the pro-l mapping class group $\Gamma_{g,n}^{\mathrm{pro}-l}$. By definition, the conjugate action of the Galois image is trivial (namely idling) on the Galois centralizer, while it has weight $(-m)$ on each graded quotient $\mathrm{gr}^m\Gamma_{g,n}^{\mathrm{pro}-l}$ $(m \geq 1)$ by the Riemann-Weil hypothesis and (3.2.3). Therefore, the Galois centralizer and the "Torelli subgroup" $\Gamma_{g,n}^{\mathrm{pro}-l}(1) \subset \Gamma_{g,n}^{\mathrm{pro}-l}$ have necessarily trivial intersection. This is the effect of the weight filtration of the pro-l mapping class group of (3.2). Our argument so far can be summarized as follows.

(3.3.2) Theorem. *The Galois centralizer* $\mathrm{Out}_{G_k}\Pi_{g,n}^{\mathrm{pro}-l}$ *is injectively mapped into* $\mathrm{gr}^0\Gamma_{g,n}^{\mathrm{pro}-l} \cong \mathrm{GSp}(2g, n)$.

The classical Hurwitz theorem says that the automorhism group $\mathrm{Aut}_k(C)$ of a curve C is embedded into $\mathrm{Sp}(2g)$. So Theorem (3.3.2) is necessary for everyone who was led to Question (3.1.3-4) after the anabelian Tate conjecture expecting that the Galois centralizer approximates $\mathrm{Aut}(C)$. As explained above, it is a consequence of the two kinds of weight concepts.

(3.4) Torelli-Galois images, and first examples of Galois rigidity. We shall continue to estimate the Galois centralizer

$$\mathrm{Out}_{G_k}\Pi_{g,n}^{\mathrm{pro}-l} \ (\hookrightarrow \mathrm{GSp}(2g, n) \quad \text{by (3.3.2)})$$

associated with the curve C/k. Immediately, we notice that, by Faltings' theorem, the Galois centralizer must be contained in the closure of the endomorphism ring of the Jacobian variety in the l-adic Tate module. But this is not enough for finiteness of the Galois centralizer, since it may still involve scalar multiplications. A natural second step is to observe the commutativity of the Galois centralizer with the Galois images lying inside the Torelli subgroup $\Gamma_{g,n}^{\mathrm{pro}-l}(1)$ of the pro-l mapping class group (we call such Galois images the *Torelli-Galois images*).

If one finds a Torelli-Galois image lying in $\Gamma_{g,n}^{\mathrm{pro}-l}(m)$ but not in $\Gamma_{g,n}^{\mathrm{pro}-l}(m+1)$ for some $m \geq 1$ and identifies its coordinates in $\mathrm{gr}^m\Gamma_{g,n}^{\mathrm{pro}-l}$, then the above commutativity restricts the Galois centralizer within the stabilizer of that image in $\mathrm{GSp}(2g, n)$ through the graded coordinate system. The Torelli-Galois images reflect proper information on the curve rather than its Jacobian variety, and construction and identification of the Torelli-Galois images in the graded coordinate system are basic themes of the theory of exterior Galois representations. It is known that every smooth curve over a number field has nontrivial Torelli-Galois images (see 4.2 below), but it is more desirable that various kinds of Torelli-Galois images should be constructed so that their distribution in the graded coordinate systems on the pro-l mapping class groups can be described explicitly.

The following examples of estimates of Galois centralizers are obtained from various kinds of Torelli-Galois images. (Although Example (3.4.5) includes Examples (3.4.2-3), they were historically obtained from different kinds of Torelli-Galois images.)

Example (3.4.1). $C = \mathbf{P}_k^1 - \{a_1, \ldots, a_n\}$. (Here $a_i \in \mathbf{P}^1(k)$, $i = 1, \ldots, n$; $n \geq 3$.) In this case, the Galois centralizer is isomorphic to a subgroup of S_n.

Example (3.4.2). $C = E - \{a_1, \ldots, a_n\}$. (Here E is an elliptic curve over k with $\mathrm{End}_k(E) \cong \mathbf{Z}$, and $a_i \in E(k)$, $i = 1, \ldots, n$; $n \geq 1$.) In this case, the Galois centralizer is isomorphic to a subgroup of $\{\pm 1\} \times S_n$.

Example (3.4.3). Let $C = X - S$ be as in (3.3), and assume that the Jacobian variety J of X has good reduction at a prime of k but X has stable bad reduction at the same prime of k. If moreover $\mathrm{End}_k(J) \cong \mathbf{Z}$, then the Galois centralizer is isomorphic to a subgroup of $\{\pm 1\} \times S_n$.

Example (3.4.4). Let $C = X - S$ be as in (3.3) and assume that the Jacobian J of X is k-simple and $S(\bar{k})$ has at least two points whose difference in J is non-torsion. Then, the Galois centralizer is isomorphic to a subgroup of S_n.

Example (3.4.5). In general, the Galois centralizer of $C = X - S$ (3.3) is contained in $Sp(2g, \mathbf{Z}_l) \times S_n$. In particular, if the Jacobian J satisfies $\mathrm{End}_k(J) \cong \mathbf{Z}$, the Galois centralizer is isomorphic to a subgroup of $\{\pm 1\} \times S_n$.

Let us explain Torelli-Galois images corresponding to the above examples.

Example (3.4.1) can be essentially reduced to the case of $C = \mathbf{P}^1 - \{0, 1, \infty\}$. In this case, Y. Ihara [Ih1] constructed a natural homomorphism $\gamma : \Gamma_{0,3}^{\mathrm{pro}-l}(1) \to \mathbf{Q}_l[[U, V]]^\times$ from the Torelli subgroup to the power series ring of two variables, and conjectured the explicit formula

$$(3.4.6) \qquad \gamma \circ \varphi_C^{(l)}(\sigma) = \exp \sum_{\substack{m \geq 3 \\ odd}} \frac{\chi_m(\sigma)}{l^{m-1} - 1} \sum_{\substack{i+j=m \\ i,j \geq 1}} \frac{U^i V^j}{i! j!} \quad (\sigma \in G_{\mathbf{Q}(\mu_{l^\infty})}),$$

which was proved independently by Anderson [A], Coleman [C], Ihara-Kaneko-Yukinari [IKY]. Here, $\chi_m : G_{\mathbf{Q}(\mu_{l^\infty})} \to \mathbf{Z}_l$ is a character constructed from a system of circular l-units, which is known to be nontrivial for odd $m \geq 3$ by Soulé [So] (cf. [IS]). From this, we obtain Torelli-Galois images in $\mathrm{gr}^{4k+2}\Gamma_{0,3}^{\mathrm{pro}-l}$ $(k = 1, 2, \dots)$. The above power series has a functional equation and a relation to the arithmetic of Jacobi sums. Moreover, the coefficient characters χ_m are known to be important as l-adic realizations of mixed Tate motives ([Ih7], [De], etc.).

Example (3.4.2) is an application of Torelli-Galois images obtained from a power series in the genus 1 case : $C = E - \{O\}$ (cf. [Bl], [Tsu1], [N6]). This is the analog of the above power series in genus 0. In this case, there is also a natural homomorphism into the power series ring of two variables $\gamma : \Gamma_{1,1}^{\mathrm{pro}-l}(1) \to \mathbf{Q}_l[[U, V]]$, and the following explicit formula holds:

$$(3.4.7) \qquad \gamma \circ \varphi_C^{(l)}(\sigma) = \sum_{\substack{m \geq 2 \\ even}} \frac{1}{1 - l^m} \sum_{\substack{i+j=m \\ i,j \geq 0}} \kappa_{ij}(\sigma) \frac{U^i V^j}{i! j!} \quad (\sigma \in G_{k(E_{l^\infty})}).$$

Here, the coefficient character $\kappa_{ij} : G_{k(E_{l^\infty})} \to \mathbf{Z}_l$ is obtained from a system of certain special values of theta functions. These κ_{ij} are known to be nontrivial (at least partially) so that they give nontrivial Torelli-Galois images in (a subsequence of) $\mathrm{gr}^{2k+2}\Gamma_{1,1}^{\mathrm{pro}-l}$ $(k = 1, 2, \dots)$.

Example (3.4.3) has a different nature from the above two, and depends on the anabelian analog of the good reduction criterion for abelian varieties of Serre-Tate type [ST] realized by T. Oda [Oda2-3] in the exterior pro-l Galois representations (see also Tamagawa [Ta] for open curves). When the claimed assumptions on reductions are satisfied, nontrivial images of inertia subgroups appear in $\mathrm{gr}^2\Gamma_{g,n}^{\mathrm{pro}-l}$.

Example (3.4.4) is an application of the simplest Torelli-Galois images ([NT1]). Given two points $P, Q \in S$, one can construct a Kummer character

$$\kappa_{PQ} : G_{k(J_{l^\infty})} \to T_l(J)$$

from the l-power division points of $P - Q$ in J, which turns out to be related to Torelli-Galois images in $\mathrm{gr}^1\Gamma_{g,n}^{\mathrm{pro}-l}$. If $P - Q \in J$ is non-torsion, the Kummer character gives nontrivial images (see [Ba]).

The last Example (3.4.5) is obtained from the Torelli-Galois images lying in the $\mathrm{Gal}(k(J_{l^\infty})/k(\mu_{l^\infty}))$-fixed part of $\mathrm{gr}^{4k+2}\Gamma_{g,n}^{\mathrm{pro}-l}$ ($k = 1, 2, \dots$), which are assured to exist universally by moduli-theoretic considerations. Later in (4.2), we shall explain them in more detail.

The five kinds of Torelli-Galois images above are primitive ones constructed directly in the graded quotients $\mathrm{gr}^m\Gamma_{g,n}^{\mathrm{pro}-l}$ ($m = 1, 2, \dots$) of the pro-l mapping class groups. Occasionally, it is possible to construct secondary Torelli-Galois images by taking iterated Lie brackets of the above primitive ones in the graded Lie algebra $\bigoplus_m \mathrm{gr}^m\Gamma_{g,n}^{\mathrm{pro}-l}$. Standard methods for showing nontriviality of such Lie brackets are kinds of "coordinate calculus" in graded coordinate systems ([Ih5], [Mats1], [IhT], [Tsu2], [AN], [NT1]).

§4. Galois theory of moduli of algebraic curves

(4.1) Profinite version of the Ivanov-McCarthy rigidity. The history of analogues between Teichmüller modular groups and arithmetic subgroups is long and still developing. In the late 1960s, A. Borel [Bor1] showed that non-exceptional arithmetic subgroups have only finitely many outer automorphisms. (Nowadays, this can be regarded as a consequence of Mostow's rigidity.) The analogue of this on the side of Teichmüller modular groups was shown by N. V. Ivanov [Iv1-2] and J. D. McCarthy [Mc], who utilized the classification theory of Thurston [Th] together with the theory of essential reduction systems of surface mapping classes due to Birman-Lubotzky-McCarthy [BLM]. On the other hand, the moduli space $M_{g,n}$ of the n-pointed smooth curves of genus g is a stack over $\mathbf{Q}$, and its fundamental group $\pi_1(M_{g,n})$ is known to be isomorphic to an extension group of the profinite completion $\hat{\Gamma}_{g,n}$ of the Teichmüller modular group $\Gamma_{g,n}$ by $G_{\mathbf{Q}}$ ([Oda1]). By Uchida's theorem [U], every automorphism of $G_{\mathbf{Q}}$ is inner, so if $\hat{\Gamma}_{g,n}$ has trivial center, then this mixed fundamental group has outer automorphism group isomorphic to the Galois centralizer $\mathrm{Out}_{G_{\mathbf{Q}}}\hat{\Gamma}_{g,n}$ (3.1). It is also known by Royden [R] and Earle-Kra [EK] that the automorphisms of the Teichmüller spaces are only the obvious ones. Therefore, we are led to the following conjecture, in which k is a number field.

Conjecture (4.1.2). *The Galois centralizer* $\mathrm{Out}_{G_k}\hat{\Gamma}_{g,n}$ *is isomorphic to* S_n *if* (g, n) *is a pair of nonnegative integers with* $2 - 2g - n < 0$ *(except for a small number of exceptional pairs).*

In the case where $g = 0$, we may consider the pro-$\mathfrak{C}$ version of this conjecture for various classes of finite groups $\mathfrak{C}$. The space $M_{0,n}$ is in the form of the projective space $\mathbf{P}^{n-3}$ minus certain hyperplanes, and its fundamental group $\Gamma_{0,n}$ is a (quotient by the center of a) pure braid group. So the genus 0 case is easier to treat than the other cases.

(4.1.3) Theorem [N3]. *Assume that a class $\mathfrak{C}$ of finite groups is closed under the formation of quotients, subgroups and group extensions, and satisfies certain admissibility conditions for braid groups (cf. loc. cit.) If the conjecture* $\mathrm{Out}_{G_k}\hat{\Gamma}_{0,4}(\mathfrak{C}) \cong S_3$ *for the primitive stage* $M_{0,4} \cong \mathbf{P}^1 - \{0,1,\infty\}$ *is true, then the conjectures* $\mathrm{Out}_{G_k}\hat{\Gamma}_{0,n}(\mathfrak{C}) \cong S_n$ $(n \geq 5)$ *for* $M_{0,n}$ *are all true. In particular, these conjectures are true for the class* $\mathfrak{C}$ *of finite* l*-groups (*l *a prime): cf.* 3.4.1.

This theorem is proved by showing that every homologically trivial element of the Galois centralizer preserves the conjugacy class of the Dehn twist along a simple closed curve cutting a pantalon off from the n-holed sphere. This can be seen by lifting anabelian weight filtrations (3.3) along the Teichmüller tower of genus 0 in "various" ways.

(4.2) Field towers associated with universal monodromies. Fix a rational prime l. The universal family of curves over the moduli space $M_{g,n}$ has the n-point punctured genus g curves as fibres. If we identify $\Pi_{g,n}$ (1.2.2) with the fundamental group of the fibre curve in a standard way, then taking monodromy we obtain a natural representation

$$\varphi_{g,n} : \pi_1(M_{g,n}) \to \Gamma_{g,n}^{\text{pro}-l} \ (\subset \mathrm{Out}\Pi_{g,n}^{\text{pro}-l}). \tag{4.2.1}$$

In the topological context, this corresponds to an isomorphism by the Dehn-Nielsen theorem, but in our algebraic context of pro-l models over $\mathbf{Q}$, it differs from an isomorphism in a delicate way. Now, pulling back by $\varphi_{g,n}$ the weight filtrations in $\Gamma_{g,n}^{\text{pro}-l}$ to introduce a filtration in $\pi_1(M_{g,n})$ as $\pi_1(M_{g,n})(m) = \varphi_{g,n}^{-1}(\Gamma_{g,n}^{\text{pro}-l}(m))$, and then pushing out the filtration by the natural augmentation homomorphism $p_{g,n} : \pi_1(M_{g,n}) \to G_{\mathbf{Q}}$, we can define a tower of number fields (depending on l):

$$\mathbf{Q} \subset \mathbf{Q}_{g,n}(1) \subset \mathbf{Q}_{g,n}(2) \subset \mathbf{Q}_{g,n}(3) \subset \dots \tag{4.2.2}$$

by $G_{\mathbf{Q}_{g,n}(m)} = p_{g,n}(\pi_1(M_{g,n})(m))$.

(4.2.3) Problem (see [Oda5]). *Does the field tower* $\{\mathbf{Q}_{g,n}(m)\}_{m=1}^{\infty}$ *have stability with respect to* (g,n)*?*

One observes immediately that $\mathbf{Q}_{g,n}(1)$ is the l-cyclotomic field $\mathbf{Q}(\mu_{l^\infty})$, independently of (g,n). This corresponds to the fact that the field of Siegel modular functions of l-power levels has a canonical model over the l-cyclotomic field independently of the ranks of Sp. The above problem is therefore asking about stability properties of natural fields of definition of the "Teichmüller modular function fields" with l-power level structures and "weight structures".

In fact, we can prove the following facts.

(4.2.4) Theorem. *Let* $2-2g-n<0$, $m \geq 1$ *and* $n \geq 1$. *Then,*

(1) $\mathbf{Q}_{g,n}(m)$ *is independent of* n ([NTU]).

(2) $\mathbf{Q}_{gr,n}(m) \subset \mathbf{Q}_{g,n}(m)$ $(r \geq 0,\ 2-2gr-n<0)$ ([N7]).

The number fields $\mathbf{Q}_{g,n}(m)$ here are all pro-l Galois extensions of the l-cyclotomic field and unramified outside l, and are distributed as intermediate fields of the extension $\mathbf{Q}_{0,3}(m) \subset \mathbf{Q}_{1,1}(m)$ in the "genus-multiplicative" way. The graded quotient Galois groups $\mathrm{Gal}(\mathbf{Q}_{g,n}(m+1)/\mathbf{Q}_{g,n}(m))$ $(m \geq 1)$ are finitely generated $\mathbf{Z}_l$-modules whose free $\mathbf{Z}_l$-ranks $r_{g,n}(m)$ turn out to satisfy $r_{0,3}(m) \leq r_{g,n}(m) \leq r_{1,1}(m)$ by a certain weight argument. Since the $r_{0,3}(m)$ $(m = 1, 2, \dots)$ measure the Torelli-Galois images in $\mathrm{gr}^m\Gamma_{0,3}$ which have been estimated as in (3.4.6), the above stability insures the universal existence of the Torelli-Galois images for all hyperbolic affine curves (cf. 3.4.5). Finally, the field $\mathbf{Q}_{0,3}(\infty)$ is known to be generated by the so-called "higher circular l-units" which are produced from $\{0, 1, \infty\}$ by iterated processes of taking l-power roots and cross ratios (Anderson-Ihara [AI]), and is expected to involve interesting number-theoretic properties still veiled in mystery.

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NOTES ADDED IN TRANSLATION

After the original manuscript in Japanese was submitted to Sugaku (March 31, 1994), several essentially related papers appeared. It would be appropriate to summarize some of these developments and their relations to the present article.

(A1). M. Matsumoto [Ma] studied Galois representations in profinite braid groups of curves, and generalized Belyi's injectivity of (1.1.2) to any affine curve over a number field. His method can also be applied in the pro-l context to give an alternative proof of the inclusion $\mathbf{Q}_{0,3}(m) \subset \mathbf{Q}_{g,n}(m)$, which is a special case of Theorem (4.2.4)(2).

(A2). Y. Ihara and the author [IN] studied the local monodromy representations arising from the local neighborhood of a maximally degenerate stable curve. They proved that then $\mathbf{Q}_{g,n}(\infty)$ is independent of (g, n), and the field extensions in (4.2.4)(2) are finite extensions (for $n \geq 1$, but see (A4)). One finds also that the power series (3.4.7) degenerates to the "logarithmic partial derivative" of the power series (3.4.6) in the local neighborhood of the Tate elliptic curve over $\mathbf{Q}$.

(A3). N. Takao and the author [NTa] generalized the pro-l version of Theorem (4.1.3) as follows. Namely, the pro-l Galois centralizers for the higher-dimensional configuration spaces of a curve over a number field can be effectively estimated by that for the base curve. So, in the pro-l case, the questions (3.1.3-4) for the configuration spaces of curves are reduced to those for single curves.

(A4). N. Takao [Tak] analyzed in detail braid-relations between the fundamental group of a complete curve and that of its affine open piece in a braid group, and showed that the condition $n \geq 1$ in Theorem (4.2.4) can be dropped.

(A5). F. Pop [Po3] solved the birational conjecture of anabelian algebraic geometry (cf. 1.3). Namely, the absolute Galois groups of fields finitely generated over $\mathbf{Q}$ determine the fields, and the outer isomorphisms among the Galois groups bijectively correspond to the isomorphisms of the fields.

(A6). A. Tamagawa [Ta2] succeeded in proving the fundamental conjecture of anabelian algebraic geometry (1.3.1) and the anabelian Tate conjecture (1.3.2) in the case $C \cong C'$ for affine curves over number fields and finite fields. S. Mochizuki, then managed to treat the case of proper curves over number fields [Moch1], and also found that Grothendieck's conjecture can hold true for the pro-p fundamental groups of hyperbolic curves over p-adic fields [Moch2]. One can also plug these results into Question (3.1.3) for higher-dimensional configuration spaces of curves by (4.1.3), (A3).

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