RIMS Kôkyûroku Bessatsu
B64 (2017), 33–54

Polylogarithmic analogue of the Coleman-Ihara formula, II

By

Hiroaki Nakamura*, Kenji Sakugawa**
and Zdzisław Wojtkowiak ***

Abstract

In this paper, generalizing our result in Part I, we show a formula that expresses a certain linear sum of Galois polylogarithms as a corresponding sum of the Coates-Wiles homomorphisms multiplied by Coleman's p-adic polylogarithms. These linear sums are characterized by tensor conditions of Zagier's conjecture on Bloch groups.

§ 1. Introduction

Let p be an odd prime and F a finite *unramified* extension of $\mathbb{Q}_p$ with the absolute Galois group $\mathscr{G}_F := \mathrm{Gal}(\overline{F}/F)$. In [NSW], we showed a formula (the polylogarithmic Coleman-Ihara formula) that connects the 'ℓ-adic' Galois polylogarithm $\ell i_m(z)\colon \mathscr{G}_F \to \mathbb{Q}_p$ (for $m \geq 1$ and $\ell = p$) with the Coates-Wiles homomorphism multiplied by Coleman's p-adic polylogarithm $\mathrm{Li}_m^{p\text{-adic}}(z)$ in the case z is a root of unity in F (necessarily of order prime to p). In this paper, we shall extend our result there in a form involving more general $z \in F$ satisfying $|z|_p = |z-1|_p = 1$.

Let us fix a coherent system $\{\zeta_{p^r}\}_{r\geq 0}$ of p-power roots of unity and regard $\zeta_{p^\infty} := (\zeta_{p^r})_{r\geq 0}$ as a $\mathbb{Z}_p$-basis of the Galois module $\mathbb{Z}_p(1) := \varprojlim_r \mu_{p^r}$. We often identify $\mathbb{Q}_p \cong \mathbb{Q}_p(m) := \mathbb{Z}_p(1)^{\otimes m} \otimes_{\mathbb{Z}_p} \mathbb{Q}$ by the fixed basis ζ_{p^∞} of $\mathbb{Z}_p(1)$.

Received March 16, 2015. Revised May 13, 2016 and October 2016.
2010 Mathematics Subject Classification(s): 11G35; 14F30, 14H30
Key Words: *l-adic polylogarithm, Bloch group, Coates-Wiles homomorphism, Non-abeilian p-adic Hodge theory*:
Supported by JSPS KAKENHI Grant number 26287005, 26287006.
*Dept. of Math., Graduate School of Science, Osaka University, Osaka 560-0043, Japan
e-mail: nakamura@math.sci.osaka-u.ac.jp
**Dept. of Math., Graduate School of Science, Osaka University, Osaka 560-0043, Japan
e-mail: k-sakugawa@math.sci.osaka-u.ac.jp
***Université de Nice-Sophia Antipolis, Dépt. of Math., Laboratoire J. A. Dieudonné, U.R.A. au C.N.R.S., No 168, Parc Valrose -B.P.N° 71, 06108 Nice Cedex 2, France
e-mail: wojtkow@math.unice.fr

For general $z \in F \setminus \{0,1\}$, the Galois polylogarithm $\ell i_m(z) = \ell i_{F,m,\gamma}(z)$ is defined as a 1-cochain $\mathscr{G}_F \to \mathbb{Q}_p(m)$ cutting out a certain coefficient of a power series (called the Galois associator) that expresses $\mathscr{G}_F$-transformation of a path $\gamma\colon \overrightarrow{01} \rightsquigarrow z$ on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Meanwhile, in a paper [DW] by J.-C.Douai and the last author, it is shown that a suitable linear sum (divisor) of points on $F \setminus \{0,1\}$ has a good choice of paths from $\overrightarrow{01}$ to the points so that the associated linear sum of Galois polylogarithms gives rise to a 1-cocycle $\mathscr{G}_F \to \mathbb{Q}_p(m)$. Our main result in the present paper extends the Coleman-Ihara formula to those 1-cocycles obtained in this manner.

To be more precise, let $\mathbb{P}^1_{01\infty} := \mathbb{P}^1 - \{0,1,\infty\}$, and let $\mathbb{Z}[\mathbb{P}^1_{01\infty}(F)]$ denote the abelian group formed by the $\mathbb{Z}$-linear sums of symbols $\{z\}$ ($z \in \mathbb{P}^1_{01\infty}(F)$), i.e.,

$$\mathbb{Z}[\mathbb{P}^1_{01\infty}(F)] := \bigoplus_{z \in \mathbb{P}^1_{01\infty}(F)} \mathbb{Z}\{z\}.$$

Following ideas of Beilinson-Deligne [BD] and Zagier [Z], we will define two subgroups $R_m^{\ell\text{-adic}}(F) \subset A_m^{\ell\text{-adic}}(F) \subset \mathbb{Z}[\mathbb{P}^1_{01\infty}(F)]$ and the ℓ-adic Bloch groups

$$B_m^{\ell\text{-adic}}(F) := A_m^{\ell\text{-adic}}(F)/R_m^{\ell\text{-adic}}(F)$$

satisfying the following conditions:

- For any $\xi = \sum_i a_i \{z_i\} \in \mathbb{Z}[\mathbb{P}^1_{01\infty}(F)]$, there exist ($\mathbb{Q}_p$-rational) paths $\gamma_i\colon \overrightarrow{01} \rightsquigarrow z_i$ such that $\sum_i a_i\, \ell i_{F,m,\gamma_i}(z_i)\colon \mathscr{G}_F \to \mathbb{Q}_p(m)$ is a 1-cocycle if and only if $\xi \in A_m^{\ell\text{-adic}}(F)$ (cf. Theorem 3.2). Further, the cohomology class of this 1-cocycle in $H^1(F, \mathbb{Q}_p(m))$ does not depend on the choice of paths.
- For any $\xi = \sum_i a_i \{z_i\} \in A_m^{\ell\text{-adic}}(F)$ and for any collection of paths $\gamma_i\colon \overrightarrow{01} \rightsquigarrow z_i$ such that $\sum_i a_i\, \ell i_{F,m,\gamma_i}(z_i)$ is a 1-cocycle, this 1-cocycle is a 1-coboundary if and only if $\xi \in R_m^{\ell\text{-adic}}(F)$.

Note that the above two properties imply that there is induced a well-defined group homomorphism

$$\mathscr{L}_m^{\ell\text{-adic}}\colon B_m^{\ell\text{-adic}}(F) \to H^1(F, \mathbb{Q}_p(m))$$

which sends the class of $\sum_i a_i\{z_i\}$ to the class of the 1-cocycle $\sum_i a_i\, \ell i_{F,m,\gamma_i}(z_i)$.

On the other hand, in order to relate the ℓ-adic polylogarithms to Coleman's p-adic polylogarithms, we extend the Coleman function ([C1])

$$\mathscr{L}_m^{p\text{-adic}}(z) := \sum_{k=0}^{m-1} \frac{B_k}{k!} \log_p^k(z) \mathrm{Li}_{m-k}^{p\text{-adic}}(z)$$

linearly to

$$\mathscr{L}_m^{p\text{-adic}}\colon \mathbb{Z}[\mathbb{P}^1_{01\infty}(\mathcal{O}_F)] \to F \quad \left(\sum_i a_i\{z_i\} \mapsto \sum_i a_i \mathscr{L}_m^{p\text{-adic}}(z_i) \right).$$

Here, B_k's are Bernoulli numbers with $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$. Note also that, on the region $|z|_p = |z-1|_p = 1$, $\log_p(z)$ and $\mathrm{Li}_k^{p\text{-adic}}(z)$ have their canonical values (cf. e.g., [KN] p.425). Our main result is then:

Theorem 1.1. *Let p be an odd prime, and let F be a finite unramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$ and $\sigma_F \in \mathrm{Gal}(F/\mathbb{Q}_p)$ the Frobenius substitution. Let $m \geq 1$ and let $\phi_{m,F}^{CW} \colon \mathscr{G}_{F(\mu_{p^\infty})} \to F \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(m)$ be the m-th Coates-Wiles homomorphism for the local field F (canonically extended to $\mathscr{G}_{F(\mu_{p^\infty})}$ as in [BK]). Then, for $\xi = \sum_i a_i\{z_i\} \in A_m^{\ell\text{-adic}}(F)$ with $z_i \in \mathbb{P}^1_{01\infty}(\mathcal{O}_F) = \mathcal{O}_F^\times \cap (1+\mathcal{O}_F^\times)$, we have*

$$\mathscr{L}_m^{\ell\text{-adic}}(\xi)(\sigma) = \frac{-1}{(m-1)!} \mathrm{Tr}_{F/\mathbb{Q}_p} \left(\left\{ \left(1 - \frac{\sigma_F}{p^m}\right) \mathscr{L}_m^{p\text{-adic}}(\xi) \right\} \phi_{m,F}^{CW}(\sigma) \right) \tag{1.1}$$

for any $\sigma \in \mathscr{G}_{F(\mu_{p^\infty})}$.

Note that (1.1) implies also an equality of cohomology classes in $H^1(\mathscr{G}_F, \mathbb{Q}_p(m))$ induced from the both sides, as the restriction map to $\mathscr{G}_{F(\mu_{p^\infty})}$ is injective.

If z is a non-trivial root of unity in F then the symbol $\{z\} \in \mathbb{Z}[\mathbb{P}^1_{01\infty}(F)]$ is contained in $A_m^{\ell\text{-adic}}(F)$ and the value $\mathscr{L}_m^{p\text{-adic}}(z)$ coincides with $\mathrm{Li}_m^{p\text{-adic}}(z)$. If z is an element of $\mathbb{P}^1_{01\infty}(\mathcal{O}_F)$, then $\{z\} \in A_1^{\ell\text{-adic}}(F)$ when $m = 1$. The formula in Theorem 1.1 thus generalizes our formulas in Part I [NSW] in these special cases.

The content of this paper is as follows. First, in Section 2, we develop a framework on abstract polylogarithms for a mixed Tate category. Then, in Section 3 (resp. 4), we apply it to Galois theoretic (resp. crystalline) realizations of polylogarithms. Much of Section 4 follows courses of the foundational work by M. Kim [Kim1-3] and is closely related to a recent remarkable paper [DCW] by I. Dan-Cohen and S. Wewers. We then settle the proof of Theorem 1.1 in Section 5. Finally, Appendix will be devoted to proving Proposition 4.5 with technical details.

In a separate paper, we will discuss relations between $B_m^{\ell\text{-adic}}(F)$ and other versions of Bloch groups introduced by D. Zagier [Z] and A. Goncharov [Go1]. It enables us to show that classical examples of elements in Bloch groups also satisfy the assumption of Theorem 1.1. For basic materials leading to our present work, we refer the reader to articles [C2], [Ih], [W1-3] and [NSW] and references therein.

Notation: In this paper, we fix a rational odd prime p. For a field F, write $\overline{F}$ for a fixed separable closure with $\mathscr{G}_F := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. Let $\chi_{\mathrm{cyc}} (= \chi_{\mathrm{cyc}}^{p\text{-adic}}) \colon \mathscr{G}_F \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character. For any topological group G equipped with a continuous action of $\mathscr{G}_F$, $H^1(F, G)$ denotes the continuous first Galois cohomology (set). For a set S, we put $\mathbb{Z}[S] := \oplus_{s \in S} \mathbb{Z}\{s\}$ the free abelian group generated by the symbols $\{s\}$. Let V be a finite dimensional vector space over a field k of characteristic zero equipped with an algebraic action of $\mathbb{G}_{m,k}$. Then, we denote

by $V^{(-2n)}$ the subspace of V on which $\mathbb{G}_{m,k}$ acts via the n-th power of the standard character $\mathrm{std} := \mathrm{id}_{\mathbb{G}_{m,k}} \colon \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$. For a commutative ring K, let $\mu(K)$ denote the maximal torsion subgroup of $K^\times$. If R is a K-algebra and X is a K-scheme, then we denote by X_R or $X \otimes_K R$ the base change of X to $\mathrm{Spec}(R)$. For an affine scheme X, we denote by $\mathcal{O}(X)$ the ring of regular functions on X. We write $\mathbb{P}^1_{01\infty}$ for the scheme $\mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathrm{Spec}\, \mathbb{Z}[T, \frac{1}{T}, \frac{1}{T-1}]$. For K as above, the set of K-rational points $\mathbb{P}^1_{01\infty}(K)$ is identified with $K^\times \cap (1 + K^\times)$.

Acknowlegement: The authors would like to thank the referee for several crucial remarks which were useful to improve the presentation of this paper. This work was supported by JSPS KAKENHI Grant Numbers: 26287005, 26287006.

§ 2. Abstract polylogarithms on a mixed Tate category

§ 2.1. Set up: mixed Tate category

Let k be a field of characteristic zero and $\mathcal{M}$ a *mixed Tate category over* k with the invertible object $k(1)$ (cf. [Go2] Appendix). Let $\pi_1(\mathcal{M}, \omega)$ be the Tannakian fundamental group of $\mathcal{M}$ with respect to the (canonical) fiber functor $\omega \colon X \mapsto \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(k(-n), \mathrm{gr}^W_{2n}(X))$. As is well-known, there exists a natural splitting $\pi_1(\mathcal{M}, \omega) = \mathbb{G}_{m,k} \ltimes U(\mathcal{M})$ where $U(\mathcal{M})$ is the pro-unipotent radical of $\pi_1(\mathcal{M}, \omega)$. Write $U(\mathcal{M}) = \varprojlim_\alpha U_\alpha$ with an inverse system $\{U_\alpha\}_\alpha$ of unipotent algebraic groups over k. The *fundamental Lie algebra* $\mathrm{Lie}(\mathcal{M})$ of $\mathcal{M}$ is defined to be the inverse limit of Lie algebras $\mathrm{Lie}(U_\alpha)$. The action of $\mathbb{G}_{m,k}$ on $U(\mathcal{M})$ defines a positive grading on the fundamental coLie algebra

$$\mathrm{coLie}(\mathcal{M}) := \varinjlim_\alpha \mathrm{Hom}_k(\mathrm{Lie}(U_\alpha), k) = \bigoplus_{n=1}^{\infty} \mathrm{coLie}(\mathcal{M})^{(2n)}$$

of $\mathcal{M}$, where $\mathrm{coLie}(\mathcal{M})^{(2n)}$ is the subspace of $\mathrm{coLie}(\mathcal{M})$ on which $\mathbb{G}_{m,k}$ acts by the $(-n)$-th power of the standard character $\mathrm{std} := \mathrm{id}_{\mathbb{G}_{m,k}}$.

We denote the dual of the Lie bracket $[\ ,\ ] \colon \wedge^2 \mathrm{Lie}(\mathcal{M}) \to \mathrm{Lie}(\mathcal{M})$ by

$$d_{\mathcal{M}} \colon \mathrm{coLie}(\mathcal{M}) \to \bigwedge^2 \mathrm{coLie}(\mathcal{M})$$

and call it the co-bracket on $\mathrm{coLie}(\mathcal{M})$.

Lemma 2.1 ([BD, Section 2.1])**.** *We have a canonical identification*

$$\mathrm{coLie}(\mathcal{M})^{(2n), d_{\mathcal{M}}=0} = \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(n)) \otimes \mathrm{std}^{-n}$$

as $\mathbb{G}_{m,k}$-modules. Here, we regard $\mathrm{Ext}^1_{\mathcal{M}}(k(0), k(n))$ as a k-vector space equipped with the trivial action of $\mathbb{G}_{m,k}$. □

§ 2.2. Abstract Albanese maps and polylogarithms

Now, let us formulate a concept of polylogarithmic quotient in $\mathcal{M}$. First, we introduce

$$\mathfrak{p}_m^{\mathcal{M}} := k(1) \ltimes \big(\bigoplus_{i=1}^{m} k(i)\big)$$

to designate the Lie algebra object in $\mathcal{M}$ such that $k(1)$ acts on $\oplus_{i=1}^{m} k(i)$ by the canonical homomorphism $k(1) \otimes k(i) \xrightarrow{\sim} k(i+1)$ for $i < m$ and annihilates $k(m)$.

For any Lie algebra object L of $\mathcal{M}$ such that $\omega(L)$ is nilpotent, we shall denote by $\exp(L)$ (or sometimes L^{CH}) the associated algebraic group object in $\mathcal{M}$ with the Campbell-Hausdorff product.

Definition 2.2. *For $m \geq 0$, we define the m-th* polylogarithmic quotient $\mathscr{P}_m^{\mathcal{M}}$ *in $\mathcal{M}$, which is an algebraic group in $\mathcal{M}$, by*

$$\mathscr{P}_m^{\mathcal{M}} := \exp(\mathfrak{p}_m^{\mathcal{M}}) = \exp\left(k(1) \ltimes \big(\bigoplus_{i=1}^{m} k(i)\big)\right) = \left(k(1) \ltimes \big(\bigoplus_{i=1}^{m} k(i)\big)\right)^{\mathsf{CH}}.$$

We understand $\mathscr{P}_0^{\mathcal{M}}$ as $k(1)$.

Recall that $H^1(\mathcal{M}, \mathscr{P}_n^{\mathcal{M}})$ is the set of isomorphism classes of right $\mathscr{P}_n^{\mathcal{M}}$-torsors in $\mathcal{M}$. The set $\{H^1(\mathcal{M}, \mathscr{P}_n^{\mathcal{M}})\}_{n\geq 0}$ forms an inverse system with respect to n, as $\{\mathscr{P}_n^{\mathcal{M}}\}_{n\geq 0}$ forms an inverse system of algebraic groups in $\mathcal{M}$.

Now, we give a concept of series of abstract Albanese maps:

Definition 2.3 (Abstract Albanese maps)**.** *Let K be a commutative ring and S a subset of $\mathbb{P}^1(K)$. Suppose that $\underline{\mathrm{Alb}} = \{\mathrm{Alb}_n \colon S \to H^1(\mathcal{M}, \mathscr{P}_n^{\mathcal{M}})\}_{n\geq 0}$ is a set of maps compatible with n, namely, the following diagram commutes for each $n > 0$:*

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{Alb}_n} & H^1(\mathcal{M}, \mathscr{P}_n^{\mathcal{M}}) \\ & \searrow^{\mathrm{Alb}_{n-1}} & \downarrow \\ & & H^1(\mathcal{M}, \mathscr{P}_{n-1}^{\mathcal{M}}). \end{array}$$

We say that $\underline{\mathrm{Alb}}$ is a series of abstract Albanese maps on S for $\mathcal{M}$, if the following two conditions hold:

- (**Hom**) *The restriction of* Alb_0 *to $S_1 := S \cap \mathbb{G}_m(K)$ factors through an injective homomorphism from $\langle S_1 \rangle \otimes_{\mathbb{Z}} \mathbb{Q}$ to $H^1(\mathcal{M}, k(1))$, where $\langle S_1 \rangle$ is the subgroup of $\mathbb{G}_m(K)$ generated by S_1.*
- (**Ref**) *For each $z \in S$ with $1 - z \in S$, $\mathrm{Alb}_1(z) = (\mathrm{Alb}_0(z), \mathrm{Alb}_0(1-z))$ in $H^1(\mathcal{M}, \mathscr{P}_1^{\mathcal{M}}) \cong H^1(\mathcal{M}, \mathscr{P}_0^{\mathcal{M}}) \oplus H^1(\mathcal{M}, \mathscr{P}_0^{\mathcal{M}})$. Here we naturally identify $\mathscr{P}_1^{\mathcal{M}} = \exp(k(1) \ltimes k(1))$ with $\mathscr{P}_0^{\mathcal{M}} \oplus \mathscr{P}_0^{\mathcal{M}} = k(1) \oplus k(1)$.*

To cut out polylogarithm parts from a non-abelian cohomology class, we make use of the following lemma. For nilpotent Lie algebras L_1, L_2 over k, define the set of *exterior Lie homomorphisms* $\mathrm{Hom}^{\mathrm{ext}}_{k\text{-Lie}}(L_1, L_2)$ by

$$\mathrm{Hom}^{\mathrm{ext}}_{k\text{-Lie}}(L_1, L_2) := L_2^{\mathsf{CH}} \backslash \mathrm{Hom}_{k\text{-Lie}}(L_1, L_2).$$

Here, L_2^{CH} is the set of k-valued points of $\exp(L_2)$ with the Campbell-Hausdorff product, and its action on the set of Lie homomorphisms $\mathrm{Hom}_{k\text{-Lie}}(L_1, L_2)$ is induced by the adjoint action of L_2^{CH} on L_2.

Lemma 2.4. *There exist canonical maps*

$$r_m \colon H^1(\mathcal{M}, \mathscr{P}_m^{\mathcal{M}}) \to H^0(\mathbb{G}_{m,k}, \mathrm{Hom}^{\mathrm{ext}}_{k\text{-}Lie}(\mathrm{Lie}(\mathcal{M}), \omega(\mathfrak{p}_m^{\mathcal{M}}))) \to \mathrm{coLie}(\mathcal{M})^{(2m+2\epsilon)}$$

for $m \geq 0$, where ϵ denotes 0 and 1 respectively when $m > 0$ and $m = 0$.

To show the above lemma, let us take a standard basis $\{e_0, e_1, \ldots, e_m\}$ of $\omega(\mathfrak{p}_m^{\mathcal{M}})$:

$$\omega(\mathfrak{p}_m^{\mathcal{M}}) = \mathrm{Hom}_{\mathcal{M}}(k(1), k(1)) \ltimes \oplus_{i=1}^m \mathrm{Hom}_{\mathcal{M}}(k(i), k(i)) = ke_0 \bigoplus (ke_1 \oplus \cdots \oplus ke_m) \tag{2.1}$$

by employing elements corresponding to $1 \in k \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}}(k(i), k(i))$. It follows that $e_i = (\mathrm{ad}e_0)^{i-1}(e_1)$ for $i \geq 1$. It is also crucial to consider the dual object $\mathfrak{p}_m^{\mathcal{M},\vee} := \mathrm{coLie}(\mathscr{P}_m^{\mathcal{M}}) \in \mathrm{Obj}(\mathcal{M})$ that has the basis $\{e_0^*, e_1^*, \ldots, e_m^*\} \subset \omega(\mathfrak{p}_m^{\mathcal{M},\vee})$ dual to $\{e_i\}_i$.

Proof of Lemma 2.4. Let $H^1(\pi_1(\mathcal{M}, \omega), \omega(\mathscr{P}_m^{\mathcal{M}}))$ be the first rational cohomology of $\pi_1(\mathcal{M}, \omega)$ with coefficients in $\omega(\mathscr{P}_m^{\mathcal{M}})$. (For a quick account on the first rational cohomology, we refer the reader to Appendix A6.2.) The first map factors through restriction map

$$H^1(\mathcal{M}, \mathscr{P}_m^{\mathcal{M}}) \cong H^1(\pi_1(\mathcal{M}, \omega), \omega(\mathscr{P}_m^{\mathcal{M}})) \to H^0(\mathbb{G}_{m,k}, H^1(U(\mathcal{M}), \omega(\mathscr{P}_m^{\mathcal{M}})))$$

via the isomorphism of $H^1(U(\mathcal{M}), \omega(\mathscr{P}_m^{\mathcal{M}}))) \cong \mathrm{Hom}^{\mathrm{ext}}_{k\text{-Lie}}(\mathrm{Lie}(\mathcal{M}), \omega(\mathfrak{p}_m^{\mathcal{M}}))$. The second map is defined as follows. Take an exterior Lie homomorphism l contained in $H^0(\mathbb{G}_{m,k}, \mathrm{Hom}^{\mathrm{ext}}_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{M}), \omega(\mathfrak{p}_m^{\mathcal{M}})))$ and let $h \colon \mathrm{Lie}(\mathcal{M}) \to \omega(\mathfrak{p}_m^{\mathcal{M}})$ be a Lie homomorphism which represents l. The second map sends l to $e_{m+\epsilon}^* \circ h|_{\mathrm{Lie}(\mathcal{M})^{(-2m-2\epsilon)}}$. To settle the well-definedness of r_m, it suffices to see that $e_{m+\epsilon}^* \circ h|_{\mathrm{Lie}(\mathcal{M})^{(-2m-2\epsilon)}}$ does not depend on the choice of the representative h of l. If $m = 0$, $\omega(\mathfrak{p}_0^{\mathcal{M}})$ is an abelian Lie algebra, hence $\mathrm{Hom}^{\mathrm{ext}}_{k\text{-Lie}} = \mathrm{Hom}_{k\text{-Lie}}$; r_0 is well-defined. For $m > 0$, the assertion follows inductively from the fact that $\omega(k(m))$ is contained in the center of $\omega(\mathfrak{p}_m^{\mathcal{M}})$.

□

Definition 2.5 (Abstract polylogarithm)**.** *Given a series of abstract Albanese maps* $\underline{\mathrm{Alb}} = \{\mathrm{Alb}_n \colon S \to H^1(\mathcal{M}, \mathscr{P}_n^{\mathcal{M}})\}_{n \geq 0}$ *on* $S \subset \mathbb{P}^1(K)$ *for a mixed Tate category* $\mathcal{M}$*, we call the composite*

$$r_m \circ \mathrm{Alb}_m \colon S \to \mathrm{coLie}(\mathcal{M})^{(2m+2\epsilon)} \quad (m \geq 0,\ \epsilon = \max\{0, 1-m\})$$

the m-th abstract polylogarithm. We extend this map to $\mathbb{Z}[S]$ linearly and denote it by

$$\mathscr{L}_m\colon \mathbb{Z}[S] \to \mathrm{coLie}(\mathcal{M})^{(2m+2\epsilon)}.$$

As shown in [BD] Prop. 2.3, we have the following differential formula

$$d_{\mathcal{M}} \circ \mathscr{L}_m = \mathscr{L}_{m-1} \wedge \mathscr{L}_0 \qquad (m \geq 1). \tag{2.2}$$

§ 2.3. Bloch groups

Let K be a commutative ring and let $S := \mathbb{P}^1_{01\infty}(K) = K^\times \cap (1 + K^\times)$. Suppose we are given a series of abstract Albanese maps $\underline{\mathrm{Alb}} = \{\mathrm{Alb}_n\colon S \to H^1(\mathcal{M}, \mathscr{P}_n^{\mathcal{M}})\}_{n\geq 0}$ on S for a fixed mixed Tate category $\mathcal{M}$ over a field k.

Definition 2.6. *For $m \geq 1$, we define two subgroups R_m and A_m of $\mathbb{Z}[S]$ as follows*: *First R_m (called the space of functional equations of Alb_m) is defined by*

$$\mathsf{R}_m := \mathrm{Ker}\left(\mathscr{L}_m\colon \mathbb{Z}[S] \to \mathrm{coLie}(\mathcal{M})^{(2m)}\right).$$

We shall denote by $\{x\}_m$ the image of $x \in S$ in $\mathbb{Z}[S]/\mathsf{R}_m$. Second A_1 is defined to be $\mathbb{Z}[S]$ and A_m is defined to be the kernel of δ_m for each $m \geq 2$, where $\delta_2\colon \mathbb{Z}[S] \to \wedge^2 K^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\delta_m\colon \mathbb{Z}[S] \to ((\mathbb{Z}[S]/\mathsf{R}_{m-1}) \otimes_{\mathbb{Z}} K^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ $(m > 2)$ are given by

$$\delta_m(\{x\}) := \begin{cases} (1-x) \wedge x & \text{if } \ m = 2, \\ \{x\}_{m-1} \otimes x & \text{if } \ m > 2, \end{cases}$$

for all $x \in S = \mathbb{P}^1_{01\infty}(K)$.

Lemma 2.7. *The abelian group R_m is a subgroup of A_m. Moreover, each $\xi \in \mathbb{Z}[\mathbb{P}^1_{01\infty}(K)]$ is contained in A_m if and only if $\mathscr{L}_m(\xi)$ is contained in $\mathrm{coLie}(\mathcal{M})^{(2m), d_{\mathcal{M}}=0}$ $= \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(m))$.*

Proof. By the assumptions (Hom) and (Ref) in Definition 2.3, the assertion of the lemma holds when $m = 1$. Recall that $\langle S \rangle$ denotes the subgroup of $\mathbb{G}_m(K)$ generated by S. For any $m > 1$, put

$$T_m := \begin{cases} (\bigwedge^2 \langle S \rangle) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } m = 2, \\ ((\mathbb{Z}[S]/\mathsf{R}_{m-1}) \otimes_{\mathbb{Z}} K^\times) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } m > 2. \end{cases}$$

Then, we have the following commutative diagram from (2.2), where $\varepsilon = 0$ or 1 when $m > 2$ or $m = 2$ respectively:

$$\begin{array}{ccc} \mathbb{Z}[S] & \xrightarrow{\quad\delta_m\quad} & T_m \\ \downarrow{\scriptstyle \mathscr{L}_m} & & \downarrow{\scriptstyle \mathscr{L}_{m-1-\varepsilon} \wedge \mathscr{L}_0} \\ \mathrm{coLie}(\mathcal{M})^{(2m)} & \xrightarrow{\quad d_{\mathcal{M}}\quad} & \bigwedge^2 \mathrm{coLie}(\mathcal{M})^{(2m-2)}. \end{array} \tag{2.3}$$

Note that we also use (Ref) for proving the commutativity of the diagram when $m = 2$. The injectivity of the right vertical map follows from the definition of R_j. We conclude the lemma from the diagram chasing. □

Definition 2.8 (Bloch groups)**.** *For $m \geq 1$, we define*

$$\mathsf{B}_m := \mathsf{A}_m/\mathsf{R}_m,$$

where $\mathsf{R}_m \subset \mathsf{A}_m \subset \mathbb{Z}[S]$ are as above.

Proposition 2.9. *The abstract polylogarithm $\mathscr{L}_m$ defines a well-defined* injective *homomorphism*

$$\mathscr{L}_m \colon \mathsf{B}_m \hookrightarrow \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(m)).$$

Proof. We show the assertion inductively. If $m = 1$, there is nothing to prove because $\mathrm{coLie}(\mathcal{M})^{(2)} = \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(1)) \otimes \mathrm{std}^{-2}$. Thus, we assume $m > 1$. By the assumption that $\mathsf{R}_m = \mathrm{Ker}(\mathscr{L}_m)$, $\mathscr{L}_m$ induces a well-defined injective homomorphism $\mathbb{Z}[S]/\mathsf{R}_m \to \mathrm{coLie}(\mathcal{M})^{(2m)}$. That the image of A_m lies in $\mathrm{coLie}(\mathcal{M})^{(2m),d_{\mathcal{M}}=0} = \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(m))$ follows from Lemma 2.7. □

Note 2.10. It is crucial to note that $\mathrm{Ext}^1_{\mathcal{M}}(k(0), k(m))$ is a k-vector space whereas B_m is only a $\mathbb{Z}$-module: In particular, if k is strictly bigger than $\mathbb{Q}$, then it is a subtle question to ask whether the above $\mathscr{L}_m$ extends to a k-linear *injection* $\mathsf{B}_m \otimes k \to \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(m))$. Suppose now that K is a number field of finite degree and let r_1 and r_2 be the numbers of real and complex places of K, respectively. The case where $\mathcal{M}$ is the category of real mixed Hodge-Tate structures is classical and the above question may be reduced to a version of Zagier's conjecture that $\mathscr{L}_m$ induces an isomorphism of $\mathsf{B}_m \otimes \mathbb{R}$ to the Deligne cohomology $H^1_{\mathcal{D}}(K \otimes \mathbb{R}, \mathbb{R}(m))^+ \cong \mathbb{R}^{d_m}$ for $m > 1$ (where $d_m = r_1 + r_2$ or r_2 according as m is odd or even respectively) discussed in [BD], [Go1], [Z]. In the case where $\mathcal{M}$ is the category of mixed Tate ℓ-adic Galois representations of $\mathscr{G}_K$, a relevant question is posed in [DW] and investigated in [S2].

§ 3. ℓ-adic Galois polylogarithms

In this section, we fix a field K of characteristic zero such that the p-adic cyclotomic character $\chi^{p\text{-adic}}_{\mathrm{cyc}} \colon \mathscr{G}_K \to \mathbb{Z}_p^\times$ has an open image. We denote by $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_K)$ the category of continuous representations of $\mathscr{G}_K$ on finite dimensional $\mathbb{Q}_p$-vector spaces. Let $\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K) \subset \mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_K)$ be the full subcategory consisting of those $\mathscr{G}_K$-modules with weight filtrations having finite sums of $\mathbb{Q}_p(n) := \mathbb{Q}_p(1)^{\otimes n}$ $(n \in \mathbb{Z})$ as the graded pieces. As is easily seen, $\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K)$ forms a mixed Tate category over $\mathbb{Q}_p$.

§ 3.1. Unipotent Albanese maps of M. Kim

As constructed in Definition 2.2, the polylogarithmic quotient $\mathscr{P}_m^{\mathrm{et}}$ in $\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K)$ is an algebraic group over $\mathbb{Q}_p$ of the form $\mathbb{Q}_p(1) \ltimes \oplus_{n=1}^m \mathbb{Q}_p(n)$. As is well-known, $\mathscr{P}_m^{\mathrm{et}}$ is a quotient of the unipotent fundamental group $\pi_1^{\mathrm{un}}(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}; \overrightarrow{01})$ of $\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}$ which is the Tannakian fundamental group of the category of unipotent $\mathbb{Q}_p$-smooth sheaves over $\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}$ (cf. e.g. [Kim2, §2]). Let $u_m \colon \pi_1^{\mathrm{un}}(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}; \overrightarrow{01}) \twoheadrightarrow \mathscr{P}_m^{\mathrm{et}}$ be the canonical $\mathscr{G}_K$-equivariant surjection. Then, for any K-rational base point z of $\mathbb{P}^1_K \setminus \{0,1,\infty\}$, we define the right $\mathscr{P}_m^{\mathrm{et}}$-torsor $\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z)$ in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_K)$ by

$$\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z) := u_{m,*}(\pi_1^{\mathrm{un}}(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}; \overrightarrow{01}, z)).$$

Here, we allow z to be tangential. According to [W1, §4-5], $\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z)$ has a natural structure of $\mathscr{P}_m^{\mathrm{et}}$-torsor in $\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K)$. Similarly, we can define $\mathscr{P}_m^{\mathrm{et}}(z', z)$ for the other K-rational base point z' as the push-out of the path torsor $\pi_1^{\mathrm{un}}(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}; z', z)$ by u_m (cf. [Kim2, p.100, l.27]).

Definition 3.1 ([Kim2, Section 2])**.** *The* etale unipotent Albanese map

$$\mathrm{Alb}_{K,m}^{\mathrm{et}} \colon \mathbb{P}^1_{01\infty}(K) \to H^1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K), \mathscr{P}_m^{\mathrm{et}})$$

is defined by

$$\mathrm{Alb}_{K,m}^{\mathrm{et}}(z) := [\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z)] \quad (z \in \mathbb{P}^1_{01\infty}(K)).$$

By construction, the sequence of etale unipotent Albanese maps $\{\mathrm{Alb}_{K,m}^{\mathrm{et}}\}_{m\geq 0}$ forms an inverse system with respect to m. As is well-known (cf. e.g., [NW, Prop. 1]), the first two etale Albanese maps are given by

$$\text{(3.1)} \qquad \begin{cases} \mathrm{Alb}_{K,0}^{\mathrm{et}}(z) \text{ is the Kummer 1-cocycle attached to } z, \\ \mathrm{Alb}_{K,1}^{\mathrm{et}}(z) = (\mathrm{Alb}_{K,0}^{\mathrm{et}}(z), \mathrm{Alb}_{K,0}^{\mathrm{et}}(1-z)) \end{cases}$$

for $z \in \mathbb{P}^1_{01\infty}(K)$. Thus, we may apply the formalism of abstract Albanese maps in the previous section to $\{\mathrm{Alb}_{K,m}^{\mathrm{et}}\}_m$, and can linearize $\mathrm{Alb}_{K,m}^{\mathrm{et}}$ to obtain

$$\mathscr{L}_m^{\ell\text{-adic}} \colon \mathbb{Z}[\mathbb{P}^1_{01\infty}(K)] \to \mathrm{coLie}(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K))^{(2m)}.$$

§ 3.2. ℓ-adic Bloch groups

Define subgroups $R_m^{\ell\text{-adic}}(K) \subset A_m^{\ell\text{-adic}}(K) \subset \mathbb{Z}[\mathbb{P}^1_{01\infty}(K)]$ and the ℓ-adic Bloch groups $B_m^{\ell\text{-adic}}(K) := A_m^{\ell\text{-adic}}(K)/R_m^{\ell\text{-adic}}(K)$ from $\{\mathrm{Alb}_{K,m}^{\mathrm{et}}\}_m$ according to Definition 2.6 and Definition 2.8. Then, by Proposition 2.9, the above $\mathscr{L}_m^{\ell\text{-adic}}$ induces

$$\mathscr{L}_m^{\ell\text{-adic}} \colon B_m^{\ell\text{-adic}}(K) \hookrightarrow \mathrm{Ext}^1_{\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K)}(\mathbb{Q}_p, \mathbb{Q}_p(m)) = H^1(K, \mathbb{Q}_p(m)).$$

Now, we shall relate $\mathscr{L}_m^{\ell\text{-adic}}$ to ℓ-adic polylogarithms. Recall that we have two fiber functors $\omega_0, \omega\colon \mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K) \to \mathrm{Vec}_{\mathbb{Q}_p}$, the forgetful functor ω_0 associating the underlying vector space of any $\mathscr{G}_K$-module in $\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K)$ and the canonical functor ω associating the graded sum of its 'weight pieces'. Denote by $U(K)$ the prounipotent radical of the Tannakian fundamental group $\pi_1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K), \omega) = \mathbb{G}_{m,\mathbb{Q}_p} \ltimes U(K)$. Pick any isomorphism Γ between ω and ω_0 such that the induced isomorphism $\pi_1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K), \omega) \cong \pi_1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K), \omega_0)$ is compatible with the projections to $\mathbb{G}_{m,\mathbb{Q}_p}$. Then, since $\pi_1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K), \omega_0)$ is canonically isomorphic to the weighted completion of $\mathscr{G}_K$ with respect to $\chi_{\mathrm{cyc}}^{p\text{-adic}}\colon \mathscr{G}_K \to \mathbb{G}_m(\mathbb{Q}_p)$, the representation $\rho\colon \mathscr{G}_{K(\mu_{p^\infty})} \to U(K)(\mathbb{Q}_p)$ induced from Γ has a Zariski dense image (cf. [HM, Proposition 7.1]). Let $\mathfrak{u}_K$ be the Lie algebra of the image of ρ so that $\log(\rho)\colon \mathfrak{u}_K \xrightarrow{\sim} \mathrm{Lie}(U(K))$. Then, the natural grading of $\mathrm{Lie}(U(K))$ and $\log(\rho)$ give rise to the weight filtration $W_{-2n}\mathscr{G}_K$ of $\mathscr{G}_K$ satisfying $\mathrm{gr}^W_{-2n}\mathscr{G}_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathrm{gr}^W_{-2n}(\mathfrak{u}_K)$ for each $n \in \mathbb{Z}$ (cf. [HM, Section 7.3]). Note that the weight filtration on $\mathscr{G}_K$ and the homomorphisms $\mathrm{gr}^W_{-2n}\mathscr{G}_K \to \mathrm{gr}^W_{-2n}\mathrm{Lie}(U(K)) = \mathrm{Lie}(U(K))^{(-2n)}$ induced from $\log(\rho)$ do not depend on the choice of $\Gamma\colon \omega \cong \omega_0$ because the representation ρ is determined up to inner automorphisms of elements of $U(K)(\mathbb{Q}_p)$.

Let z be a K-rational point of $\mathbb{P}^1_{01\infty}(K)$. We mean by a $\mathbb{Q}_p$-*rational path* from $\overrightarrow{01}$ to z an element of $\pi_1^{\mathrm{un}}(\mathbb{P}^1_K \setminus \{0,1,\infty\}; \overrightarrow{01}, z)(\mathbb{Q}_p)$. (This is nothing but what is called a $\mathbb{Q}_\ell$-path (for $\ell = p$) in [DW, p.66]). For any integer $m \geq 1$ and a $\mathbb{Q}_p$-rational path γ from $\overrightarrow{01}$ to z, the ℓ-adic (Galois) polylogarithm $\ell i_{K,m,\gamma}(z)$ is by definition a function $\mathscr{G}_K \to \mathbb{Q}_p$ obtained by assigning to $\sigma \in \mathscr{G}_K$ the coefficient of $(\mathrm{ad}\, X)^{m-1}(Y)$ in expansion of the Lie formal series $\log(\gamma \cdot \sigma(\gamma)^{-1})$ in the complete universal enveloping algebra of $\pi_1^{\mathrm{un}}(\mathbb{P}^1_K \setminus \{0,1,\infty\}, \overrightarrow{01})(\mathbb{Q}_p)$ with two (standard) free generators e^X, e^Y chosen as in [NSW, §2.1].

Let $\xi = \sum_i a_i\{z_i\}$ be an element of $\mathbb{Z}[\mathbb{P}^1_{01\infty}(K)]$. Then, by definition, $\mathscr{L}_m^{\ell\text{-adic}}(\xi)$ is a $\mathbb{Q}_p$-linear homomorphism $\mathscr{L}_m^{\ell\text{-adic}}(\xi)\colon \mathrm{Lie}(U(K))^{(-2m)} \to \mathbb{Q}_p$. Meanwhile, any ℓ-adic polylogarithm $\ell i_{K,m,\gamma}(z)\colon \mathscr{G}_K \to \mathbb{Q}_p$ vanishes on $W_{-2m-2}\mathscr{G}_K$, and induces a group homomorphism from $W_{-2}\mathscr{G}_K/W_{-2m-2}\mathscr{G}_K$ to $\mathbb{Q}_p$. By the definition of ℓ-adic polylogarithms, the restriction of $\sum_i a_i \ell i_{K,m,\gamma_i}(z_i)$ to $\mathrm{gr}^W_{-2m}\mathscr{G}_K$ coincides with the pull-back of $\mathscr{L}_m^{\ell\text{-adic}}(\xi)$ by the above homomorphism $\mathrm{gr}^W_{-2m}\mathscr{G}_K \to \mathrm{Lie}(U(K))^{(-2m)}$. The following is a consequence of [DW, Theorem 2.3].

Theorem 3.2. *Let $\xi = \sum_i a_i\{z_i\}$ be an element of $\mathbb{Z}[\mathbb{P}^1_{01\infty}(K)]$.*

(i) *For ξ to lie in $A_m^{\ell\text{-adic}}(K)$, it is necessary and sufficient that there exist $\mathbb{Q}_p$-rational paths $\gamma_i\colon \overrightarrow{01} \rightsquigarrow z_i$ such that the linear sum $\sum_i a_i \ell i_{K,m,\gamma_i}(z_i)\colon \mathscr{G}_K \to \mathbb{Q}_p(m)$ is a 1-cocycle.*

(ii) *Suppose $\xi \in A_m^{\ell\text{-adic}}(K)$. Then, the cohomology class of a 1-cocycle given in* (i) *is uniquely determined independently of the choice of paths $\{\gamma_i\}$ and coincides with $\mathscr{L}_m^{\ell\text{-adic}}(\xi)$.* □

§ 4. Coleman's p-adic polylogarithms

In this section, we encode Coleman's p-adic polylogarithms in the framework of Tannakian category. Fix a finite extension F of $\mathbb{Q}_p$ and let F_0 be the maximal subfield of F unramified over $\mathbb{Q}_p$. We denote by $\mathscr{MF}_F^{\mathrm{ad}}(\varphi)$ the category of weakly admissible filtered φ-modules over F in the sense of Fontaine (cf. [Fo, Section 4, Definition 4.4.3]). Remark that $\mathscr{MF}_F^{\mathrm{ad}}(\varphi)$ is a Tannakian category over $\mathbb{Q}_p$ (cf. [CF, Thèoréme A]).

§ 4.1. Mixed Tate filtered φ-modules

First, we introduce a p-adic Hodge theoretic analogue of $\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_K)$. Let $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$ be the category of crystalline representation of $\mathscr{G}_F$, which is known to be a Tannakian category ([Fo] Prop. 1.5.1; cf. e.g. also [Y] 4.1). Recall that the functor

$$D_{\mathrm{crys},F}\colon \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F) \xrightarrow{\sim} \mathscr{MF}_F^{\mathrm{ad}}(\varphi)$$

has a natural quasi-inverse, which we will denote $V_{\mathrm{crys}}\colon \mathscr{MF}_F^{\mathrm{ad}}(\varphi) \to \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$. For each integer n, we put $\zeta_{p^\infty}^{\otimes n}/t^n$ which is a basis of the underlying F_0-vector space of $F_0\langle n\rangle := D_{\mathrm{crys}}(\mathbb{Q}_p(n)) \in \mathrm{Obj}(\mathscr{MF}_F^{\mathrm{ad}}(\varphi))$. Here, $t \in B_{\mathrm{crys}}$ is a p-adic period '$2\pi\sqrt{-1}$' defined by the fixed coherent system $\{\zeta_{p^r}\}_{r\geq 0}$ of p-power roots of unity. By using this fixed basis $\zeta_{p^\infty}^{\otimes n}/t^n$, we sometimes identify $F_0\langle n\rangle$ with F_0 (and accordingly $F\langle n\rangle = F$).

Definition 4.1. *We define the full subcategory $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$ of $\mathscr{MF}_F^{\mathrm{ad}}(\varphi)$ to be the minimal full subcategory of $\mathscr{MF}_F^{\mathrm{ad}}(\varphi)$ containing $\{F_0\langle n\rangle\}_{n\in\mathbb{Z}}$ and closed under extensions.*

Since $V_{\mathrm{crys}}(F_0\langle n\rangle) = \mathbb{Q}_p(n)$, we have:

$$\begin{aligned}\mathrm{Ext}^1_{\mathscr{MT}_F^{\mathrm{ad}}(\varphi)}(F_0\langle 0\rangle, F_0\langle n\rangle) &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)}(\mathbb{Q}_p(0), \mathbb{Q}_p(n))\\ &= H^1_f(F, \mathbb{Q}_p(n)) \xrightarrow{\sim} D_{\mathrm{dR}}(\mathbb{Q}_p(n))/F^0\, D_{\mathrm{dR}}(\mathbb{Q}_p(n)),\end{aligned}$$

where the last isomorphism is the Bloch-Kato logarithmic map. Consequently, we find that the pair $(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), F_0\langle 1\rangle)$ is a mixed Tate category over $\mathbb{Q}_p$. We shall write $\omega\colon \mathscr{MT}_F^{\mathrm{ad}}(\varphi) \to \mathrm{Vec}_{\mathbb{Q}_p}$ for the canonical fiber functor.

§ 4.2. Crystalline polylogarithmic quotient

We introduce the crystalline polylogarithmic quotient.

Definition 4.2. *We define the polylogarithmic quotient $\mathscr{P}_m^{\mathrm{crys}}$, which is an algebraic group in $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$, to be* $\exp\left(F_0\langle 1\rangle \ltimes \oplus_{n=1}^m F_0\langle n\rangle\right)$.

Recall that the polylogarithmic quotient $\mathscr{P}_m^{\mathrm{crys}}$ is a quotient of the unipotent crystalline fundamental group $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}, \overrightarrow{01})$ where k is the residue field of F. Let $u_m \colon \pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}, \overrightarrow{01}) \twoheadrightarrow \mathscr{P}_m^{\mathrm{crys}}$ be the canonical surjection. For each $z \in \mathbb{P}^1_{01\infty}(\mathcal{O}_F)$, define the right torsor $\mathscr{P}_m^{\mathrm{crys}}(\overrightarrow{01}, z)$ under the algebraic group $\mathscr{P}_m^{\mathrm{crys}}$ over F_0 in the usual sense by

$$\mathscr{P}_m^{\mathrm{crys}}(\overrightarrow{01}, z) := u_{m,*}(\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; \overrightarrow{01}, z)).$$

We can define $\mathscr{P}_m^{\mathrm{crys}}(z, z')$ as a push-out of $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; z, z')$ for every pair of $\mathcal{O}_F$-rational base points z, z' similarly.

As the ring of regular functions of $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; \overrightarrow{01})$ has a structure of ind-filtered φ-module over F (cf. [De]), the structure ring of each torsor $\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}(\overrightarrow{01}, z))$ is also an ind-filtered φ-module over F. It is ind-admissible for each $z \in \mathbb{P}^1_{01\infty}(\mathcal{O}_F)$ according to [S1, Proposition 3.3, Remark 6]. Moreover, $\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}(\overrightarrow{01}, z))$ is an object of the ind-category of $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$ (cf. [S1, Corollary 1]). Thus, we obtain the *crystalline-de Rham unipotent Albanese map* of M. Kim type:

$$\mathrm{Alb}_{F,m}^{\text{cr-dR}} \colon \mathbb{P}^1_{01\infty}(\mathcal{O}_F) \to H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_m^{\mathrm{crys}})$$

defined by $\mathrm{Alb}_{F,m}^{\text{cr-dR}}(z) := [\mathscr{P}_m^{\mathrm{crys}}(\overrightarrow{01}, z)]$. At this stage then, it is noteworthy to realize the target set of $\mathrm{Alb}_{F,m}^{\text{cr-dR}}$ identified as

$$H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_m^{\mathrm{crys}}) \cong \mathscr{P}_m^{\mathrm{crys}}(F) = \left(F\langle 1\rangle \ltimes \prod_{n=1}^{m} F\langle n\rangle \right)^{\mathsf{CH}} \cong F \times \prod_{n=1}^{m} F. \tag{4.1}$$

See [S1, Remark 6, Corollary 1] for a detailed account with more general results.

The following fact is crucial to characterize the crystalline-de Rham Albanese map $\mathrm{Alb}_{F,m}^{\text{cr-dR}} \colon \mathbb{P}^1_{01\infty}(\mathcal{O}_F) \to H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_m^{\mathrm{crys}})$.

Lemma 4.3 (cf. [Fu, Theorem 2.3], [Kim2]). *Let z be an element of $\mathbb{P}^1_{01\infty}(\mathcal{O}_F)$. Under the canonical isomorphism* (4.1)*, the crystalline-de Rham Albanese map is calculated as follows:*

$$\mathrm{Alb}_{F,m}^{\text{cr-dR}}(z) = \left(\log_p(z); -\mathrm{Li}_1^{p\text{-adic}}(z), \ldots, -\mathrm{Li}_m^{p\text{-adic}}(z) \right). \quad \square \tag{4.2}$$

Here, $\log_p$ is the p-adic logarithm and $\mathrm{Li}_n^{p\text{-adic}}$ is Coleman's n-th p-adic polylogarithm. Remark that these functions are uniquely determined on the region $z \in \mathbb{P}^1_{01\infty}(\mathcal{O}_F)$. As $\mathrm{Li}_1^{p\text{-adic}}(z) = -\log_p(1-z)$ (cf. [Fu, p.263]), we see that $\{\mathrm{Alb}_{F,m}^{\text{cr-dR}}\}_m$ satisfies our axioms of abstract Albanese maps (Definition 2.3) on $S = \mathbb{P}^1_{01\infty}(\mathcal{O}_F)$.

§ 4.3. p-adic polylogarithms on Bloch groups

Applying our abstract formalism in Section 2 to the series of crystalline-de Rham unipotent Albanese maps on $\mathbb{P}^1_{01\infty}(\mathcal{O}_F)$ for the mixed Tate category $\mathscr{MT}^{\mathrm{ad}}_F(\varphi)$, we obtain, for each $m \geq 0$, the linearization homomorphism

$$\mathscr{L}^{\mathrm{ad}}_m \colon \mathbb{Z}[\mathbb{P}^1_{01\infty}(\mathcal{O}_F)] \to \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m)}.$$

Theorem 4.4. *Suppose that F is a finite* unramified *extension of $\mathbb{Q}_p$, and put*

$$A^{\ell\text{-adic}}_m(\mathcal{O}_F) := A^{\ell\text{-adic}}_m(F) \cap \mathbb{Z}[\mathbb{P}^1_{01\infty}(\mathcal{O}_F)].$$

Then:

(1) *The homomorphism $\mathscr{L}^{\mathrm{ad}}_m$ induces a homomorphism*

$$A^{\ell\text{-adic}}_m(\mathcal{O}_F) \to \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m),d=0} = \mathrm{Ext}^1_{\mathscr{MT}^{\mathrm{ad}}_F(\varphi)}(F\langle 0\rangle, F\langle m\rangle).$$

(2) *For every $\xi = \sum_i a_i\{z_i\} \in A^{\ell\text{-adic}}_m(\mathcal{O}_F)$, we have*

$$\mathscr{L}^{\mathrm{ad}}_m(\xi) = -\mathscr{L}^{p\text{-adic}}_m(\xi) \left(:= -\sum_i a_i \sum_{k=0}^{m-1} \frac{B_k}{k!} \log^k_p(z_i) \mathrm{Li}^{p\text{-adic}}_{m-k}(z_i) \right)$$

under the canonical identification $\mathrm{Ext}^1_{\mathscr{MT}^{\mathrm{ad}}_F(\varphi)}(F\langle 0\rangle, F\langle m\rangle) = F$.

We postpone the proof of Theorem 4.4 (1) until Section 5.

Proof of Theorem 4.4 (2) assuming (1).

Let $\xi = \sum_i a_i\{z_i\} \in A^{\ell\text{-adic}}_m(\mathcal{O}_F)$. We shall compute its image by

$$\mathscr{L}^{\mathrm{ad}}_m \colon \mathbb{Z}[\mathbb{P}^1_{01\infty}(\mathcal{O}_F)] \to \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m)}$$

that is the formal linearization of the composition $r_m \circ \mathrm{Alb}^{\text{cr-dR}}_{F,m}$ of the two maps: $r_m \colon \mathscr{P}^{\mathrm{crys}}_m(F) \to \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m)}$ and

$$\mathrm{Alb}^{\text{cr-dR}}_{F,m} \colon \mathbb{P}^1_{01\infty}(\mathcal{O}_F) \to H^1(\mathscr{MT}^{\mathrm{ad}}_F(\varphi), \mathscr{P}^{\mathrm{crys}}_m) \cong \mathscr{P}^{\mathrm{crys}}_m(F).$$

We need to take care of the fact that the latter map $\mathrm{Alb}^{\text{cr-dR}}_{F,m}$ is not a homomorphism (i.e., only a map of sets) for $m \geq 2$. By virtue of our assuming Theorem 4.4 (1), $\mathscr{L}^{\mathrm{ad}}_m(\xi)$ is contained in $\mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m),d=0} = \mathrm{Ext}^1_{\mathscr{MT}^{\mathrm{ad}}_F(\varphi)}(F_0\langle 0\rangle, F_0\langle m\rangle) \cong F\langle m\rangle$. Now, pointwisely for each z_i, Lemma 4.3 implies $\mathrm{Alb}^{\text{cr-dR}}_{F,m}(z_i)$ is given to be $(\log_p(z_i); -\mathrm{Li}^{p\text{-adic}}_1(z_i), \ldots, -\mathrm{Li}^{p\text{-adic}}_m(z_i))$ in $\mathscr{P}^{\mathrm{crys}}_m(F) = \left(F\langle 1\rangle \ltimes \prod_{i=1}^m F\langle i\rangle\right)^{\mathsf{CH}} \cong F \times \prod_{i=1}^m F$. Consider the underlying Lie algebra

$$\mathfrak{p}^{\mathrm{crys}}_m := F\langle 1\rangle \ltimes \oplus_{i=1}^m F\langle i\rangle = F\varepsilon_0 \bigoplus (F\varepsilon_1 \oplus \cdots \oplus F\varepsilon_m) \tag{4.3}$$

with a basis $\varepsilon_0 := \zeta_{p^\infty}/t$, $\varepsilon_i := \zeta_{p^\infty}^{\otimes i}/t^i$ $(1 \le i \le m)$. Noticing $\varepsilon_i = (\mathrm{ad}\varepsilon_0)^{i-1}(\varepsilon_1)$ for $i \ge 1$ and the Baker-Campbell-Hausdorff formula: $\exp(a\varepsilon_0)\cdot\exp\left(\sum_{i\ge1}\mu_i\varepsilon_i\right) = \exp(a\varepsilon_0 + \sum_{j\ge1}\lambda_j\varepsilon_j)$ with $\sum_{i=1}^\infty \mu_i T^i = \frac{e^{aT}-1}{aT}\sum_{j=1}^\infty \lambda_j T^j$ (cf. [BD] (1.5.4)), we find that the image of $\mathrm{Alb}^{\text{cr-dR}}_{F,m}(z_i)$ under $\log\colon \mathscr{P}^{\mathrm{crys}}_m(F) \to \mathfrak{p}^{\mathrm{crys}}_m$ is given by

$$\log\circ\mathrm{Alb}^{\text{cr-dR}}_{F,m}(z_i) = \log_p(z_i)\varepsilon_0 - \sum_{j=1}^m \mathscr{L}^{p\text{-adic}}_j(z_i)\varepsilon_j \quad (\in \mathfrak{p}^{\mathrm{crys}}_m). \tag{4.4}$$

To proceed the computation, we make use of the following key proposition. Let $\mathrm{proj}_m\colon \mathfrak{p}^{\mathrm{crys}}_m = F\langle1\rangle \ltimes \oplus_{i=1}^m F\langle i\rangle \to F\langle m\rangle$ denote the projection homomorphism to the last component. Then,

Proposition 4.5. *There exists a retraction homomorphism*

$$\mathsf{ret}_m\colon \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m)} \to \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m),d=0}$$

(*i.e., it restricts to the identity on* $\mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m),d=0}$) *that makes the following diagram commute*:

$$\begin{array}{ccc} H^1(\mathscr{MT}^{\mathrm{ad}}_F(\varphi), \mathscr{P}^{\mathrm{crys}}_m(F)) \cong \mathscr{P}^{\mathrm{crys}}_m(F) & \xrightarrow{r_m} & \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m)} \\ \Big\downarrow \log & & \Big\downarrow \mathsf{ret}_m \ \cup \\ \mathfrak{p}^{\mathrm{crys}}_m = F\langle1\rangle \ltimes \oplus_{i=1}^m F\langle i\rangle & \xrightarrow{\mathrm{proj}_m} & F\langle m\rangle = \mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m),d=0}. \end{array}$$

We shall postpone a proof of this proposition until Appendix. Using (4.4) and Proposition 4.5, we find

$$\mathsf{ret}_m\circ\mathscr{L}^{\mathrm{ad}}_m(z_i) = \mathsf{ret}_m\circ r_m\circ\mathrm{Alb}^{\text{cr-dR}}_{F,m}(z_i) = \mathrm{proj}_m\circ\log\circ\mathrm{Alb}^{\text{cr-dR}}_{F,m}(z_i) = -\mathscr{L}^{p\text{-adic}}_m(z_i).$$

By taking the linear combination $\xi = \sum_i a_i\{z_i\}$, we therefore obtain

$$\mathsf{ret}_m\circ\mathscr{L}^{\mathrm{ad}}_m(\xi) = -\mathscr{L}^{p\text{-adic}}(\xi).$$

At this stage, however, the assumption $\xi \in A^{\ell\text{-adic}}_m(\mathcal{O}_F)$ tells that $\mathscr{L}^{\mathrm{ad}}_m(\xi)$ already lies in $\mathrm{coLie}(\mathscr{MT}^{\mathrm{ad}}_F(\varphi))^{(2m),d=0}$ on which ret_m is just the identity map. Thus, $\mathscr{L}^{\mathrm{ad}}_m(\xi) = -\mathscr{L}^{p\text{-adic}}(\xi)$ as desired in Theorem 4.4 (2) . □

§ 5. Coleman-Ihara formula

§ 5.1. Comparison of $\mathrm{Alb}^{\mathrm{et}}_{F,m}$ and $\mathrm{Alb}^{\text{cr-dR}}_{F,m}$.

Throughout this section, we assume that F is a finite unramified extension of $\mathbb{Q}_p$ and look closely at the two Albanese maps of Minhyong Kim ([Kim1], [Kim2]):

$$\begin{aligned} &\mathrm{Alb}^{\mathrm{et}}_{F,m}\colon \mathbb{P}^1_{01\infty}(F) \to H^1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F), \mathscr{P}^{\mathrm{et}}_m),\\ &\mathrm{Alb}^{\text{cr-dR}}_{F,m}\colon \mathbb{P}^1_{01\infty}(\mathcal{O}_F) \to \mathscr{P}^{\mathrm{crys}}_m(F) \cong H^1(\mathscr{MT}^{\mathrm{ad}}_F(\varphi), \mathscr{P}^{\mathrm{crys}}_m). \end{aligned}$$

These two maps are fitting in the following fundamental diagram:

(5.1)
$$\begin{array}{ccccc} & & \mathbb{P}^1_{01\infty}(\mathcal{O}_F) & \xrightarrow{\mathrm{Alb}^{\text{cr-dR}}_{F,m}} & \mathscr{P}^{\mathrm{crys}}_m(F) \\ & \swarrow^{\mathrm{Alb}^{\mathrm{et}}_{F,m}} & \vdots & \nearrow_{D_{\mathrm{crys},F}} & \\ H^1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F), \mathscr{P}^{\mathrm{et}}_m) & \supset & H^1(\mathcal{MT}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F), \mathscr{P}^{\mathrm{et}}_m) & & \end{array}$$

where $\mathcal{MT}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F) := \mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F) \cap \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. M. Kim showed in [Kim2] the commutativity (and the existence of dotted arrow) of the above type diagram for quotients of unipotent fundamental groups of smooth curves with good reduction based at usual points. As mentioned above, the same type of diagrams with tangential base points are generally unknown to exist in literatures. We shall prove the commutativity of the above diagram (5.1) together with factorization by the dotted arrow, first for $z \in \mu(F)$ by using our result of previous paper [NSW], and then extend it to general z by applying a comparison theorem of path torsors proved by M. Olsson [O1], [O2]. Note that Olsson's comparison theorem is proved only for path torsors between usual base points.

Review on Bloch-Kato exponential maps. In [S1, Theorem 1.1], [Kim3, Prop. 1.4], we studied a general theory extending the Bloch-Kato exponential map to any unipotent algebraic group object G in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. It relates (right cosets of) the F-valued points of $D_{\mathrm{crys},F}(G)$ to the non-abelian Galois cohomology set $H^1_f(F, G(\mathbb{Q}_p))$. If we apply this theory to $G = \mathscr{P}^{\mathrm{et}}_m$, then the exponential map $\exp_{\mathscr{P}^{\mathrm{et}}_m}$ provides a bijection from the set of F-rational points of $D_{\mathrm{crys},F}(G) = \mathscr{P}^{\mathrm{crys}}_m$ to the corresponding cohomology:

(5.2)
$$\exp_{\mathscr{P}^{\mathrm{et}}_m} : \mathscr{P}^{\mathrm{crys}}_m(F) \to H^1(\mathcal{MT}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F), \mathscr{P}^{\mathrm{et}}_m(\mathbb{Q}_p)).$$

In the most classical case of $G = \mathbb{Q}_p(n)$, it is reduced to

$$\exp_{\mathbb{Q}_p(n)} : F\langle n\rangle \xrightarrow{\sim} H^1_f(F, \mathbb{Q}_p(n)) = \mathrm{Ext}^1_{\mathcal{MT}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)}(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$$

which has the following explicit description (what is known as *the Bloch-Kato explicit reciprocity law* [BK] Theorem 2.1 and (4.8.2)):

(5.3)
$$\exp_{\mathbb{Q}_p(n)}\left(a \cdot \frac{\zeta_{p^\infty}^{\otimes n}}{t^n}\right) = \frac{1}{(n-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1 - \frac{\sigma_F}{p^n}\right) \cdot a\right\} \phi^{CW}_{n,F}\right)$$

for $n \geq 1$ and $a \in F$. Here, σ_F is the arithmetic Frobenius automorphism on F (unramified over $\mathbb{Q}_p$) and $\phi^{CW}_{n,F}$ is the Coates-Wiles homomorphism.

The commutativity of the diagram (5.1) is rephrased as follows:

Proposition 5.1. (i) *The path torsor $\mathscr{P}^{\mathrm{et}}_m(\overrightarrow{01}, z)$ is an affine scheme lying in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. Hence, $\mathrm{Alb}^{\mathrm{et}}_{F,m}$ factors through $\mathbb{P}^1_{01\infty}(\mathcal{O}_F) \to H^1_f(F, \mathscr{P}^{\mathrm{et}}_m(\mathbb{Q}_p))$.*

(ii) $D_{\mathrm{crys},F} \circ \mathrm{Alb}^{\mathrm{et}}_{F,m} = \mathrm{Alb}^{\text{cr-dR}}_{F,m}$.

Before settling a proof of Proposition 5.1, we now recall from [Gi, Chap.III, Definition 1.3.1], the concept of a composition of torsors: If X (resp. X') is an affine scheme with a right (resp. left) action of an affine group scheme G in a Tannakian category, then the contraction $X \wedge^G X'$ is defined as the quotient of $X \times X'$ by the natural G-action defined by $(x, x') \mapsto (xg, g^{-1}x')$. This construction gives a (G_1, G_2)-bitorsor $X \wedge^G X'$ from a (G_1, G)-bitorsor X and a (G, G_2)-bitorsor X'.

Proof of Proposition 5.1. (i) First, consider the case $z = z_0 \in \mu(F) \setminus \{1\}$. In this case, by the equality (5.3), the assertion of Proposition 5.1 is equivalent to the assertion Theorem 1.1 for $\xi = \{z_0\} \in A_m^{\ell\text{-adic}}(F)$. Hence, it is reduced to [NSW, Theorem 1.1]. Indeed, we have shown that the isomorphism class of the torsor $\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z_0)$ coincides with the image of $\mathrm{Alb}_{F,m}^{\text{cr-dR}}(z_0)$ under the non-abelian exponential map (5.2), hence that $\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z_0)$ lies in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$. Next, we show the commutativity of the diagram for general z. Let z be an arbitrary element of $\mathbb{P}^1_{01\infty}(\mathcal{O}_F)$, and consider a composition of paths $\overrightarrow{01} \rightsquigarrow z_0$ and $z_0 \rightsquigarrow z$ passing an intermediate point $z_0 \in \mu(F) \setminus \{1\}$. As seen above, we know $\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z_0) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$. On the other hand, we also know the path torsor $\mathscr{P}_m^{\mathrm{et}}(z_0, z)$ between usual points lies in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$, as proved in greater generality by M. Olsson [O1, Theorem 1.11]. Therefore, one can execute the path composition constructions: $\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z_0) \wedge \mathscr{P}_m^{\mathrm{et}}(z_0, z) \xrightarrow{\sim} \mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z)$, and find $\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z)$ lies in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$. This settles the existence of the dotted arrow in the diagram (5.1).

(ii) It remains to show the commutativity of the right hand triangle. Since the functor $D_{\mathrm{crys},F}$ commutes with any composition of torsors, we have

$$\begin{aligned} &D_{\mathrm{crys},F}(\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z_0)) \wedge D_{\mathrm{crys},F}(\mathscr{P}_m^{\mathrm{et}}(z_0, z)) \\ &\cong D_{\mathrm{crys},F}(\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z_0) \wedge \mathscr{P}_m^{\mathrm{et}}(z_0, z)) \cong D_{\mathrm{crys},F}(\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z)). \end{aligned} \tag{5.4}$$

According to Olsson's comparison theorem, we have $D_{\mathrm{crys},F}(\mathscr{P}_m^{\mathrm{et}}(z_0, z)) \cong \mathscr{P}_m^{\mathrm{crys}}(z_0, z)$ in $\mathscr{MF}_F^{\mathrm{ad}}(\varphi)$. On the other hand, we have already shown that $D_{\mathrm{crys},F}(\mathscr{P}_m^{\mathrm{et}}(\overrightarrow{01}, z_0)) \cong \mathscr{P}_m^{\mathrm{crys}}(\overrightarrow{01}, z_0)$ in $\mathscr{MF}_F^{\mathrm{ad}}(\varphi)$ in the first step. Hence, the left hand side of (5.4) is canonically isomorphic to $\mathscr{P}_m^{\mathrm{crys}}(\overrightarrow{01}, z)$, hence the assertion. □

§ 5.2. Proof of Main Theorem

In this subsection, we give a proof of Theorem 1.1. First, we give a linearization of Kim's fundamental commutative diagram (5.1) (cf. Proposition 5.1). Recall that the Fontaine functor $V_{\mathrm{crys}} \colon \mathscr{MF}_F^{\mathrm{ad}}(\varphi) \to \mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ induces a functor between two mixed Tate category over $\mathbb{Q}_p$: $V_{\mathrm{crys}} \colon \mathscr{MT}_F^{\mathrm{ad}}(\varphi) \to \mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F)$. It is easily checked that V_{crys} is compatible with the canonical fiber functors of two mixed Tate categories. Thus, by Tannakian duality, we have a homomorphism $\pi_1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F), \omega) \to \pi_1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \omega)$ of pro-algebraic groups over $\mathbb{Q}_p$. Since V_{crys} is fully-faithful, this homomorphism is

surjective. Hence, we obtain an injective homomorphism of coLie algebras

$$V_{\mathrm{crys},*}\colon \mathrm{coLie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi)) \hookrightarrow \mathrm{coLie}(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F)).$$

Proposition 5.2. *Under the same assumption of Proposition* 5.1*, we have the commutative diagram*:

$$\begin{array}{ccccc}
\mathbb{Z}[\mathbb{P}^1_{01\infty}(\mathcal{O}_F)] & \xrightarrow{\mathscr{L}_m^{\mathrm{ad}}} & \mathrm{coLie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(2m)} & \hookleftarrow & F\langle m\rangle \\
\downarrow & & \downarrow{\scriptstyle V_{\mathrm{crys},*}} & & \downarrow{\scriptstyle \exp_{\mathbb{Q}_p(m)}} \\
\mathbb{Z}[\mathbb{P}^1_{01\infty}(F)] & \xrightarrow{\mathscr{L}_m^{\ell\text{-adic}}} & \mathrm{coLie}(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F))^{(2m)} & \hookleftarrow & H^1(F,\mathbb{Q}_p(m))
\end{array}$$

Here, $\exp_{\mathbb{Q}_p(m)}$ *is the Bloch-Kato exponential map.*

Proof. According to Proposition 5.1, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{P}^1_{01\infty}(\mathcal{O}_F) & \xrightarrow{\mathrm{Alb}^{\text{cr-dR}}_{m,F}} & H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi),\mathscr{P}_m^{\mathrm{crys}}) & \xrightarrow{r_m} & \mathrm{coLie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(2m)} \\
& \searrow^{\mathrm{Alb}^{\mathrm{et}}_{F,m}} & \downarrow{\scriptstyle V_{\mathrm{crys},*}} & & \downarrow{\scriptstyle V_{\mathrm{crys},*}} \\
& & H^1(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F),\mathscr{P}_m^{\mathrm{et}}) & \xrightarrow{r_m} & \mathrm{coLie}(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F))^{(2m)}.
\end{array}$$

Here, r_m, $V_{\mathrm{crys},*}$ are induced maps from Lemma 2.4 and V_{crys} respectively, and the restriction of $V_{\mathrm{crys},*}$ to $F\langle m\rangle$ coincides with $\exp_{\mathbb{Q}_p(m)}$ by [S1, Theorem 1.1 (ii)] (cf. [Kim3, Proposition 1.4]). Extending the above commutative diagram linearly, we conclude the proposition. □

Remark 5.3. *By virtue of the above proposition, we see that both series of Albanese maps* $\{\mathrm{Alb}^{\text{cr-dR}}_{F,m}\}_m$ *and* $\{\mathrm{Alb}^{\mathrm{et}}_{F,m}\}_m$ *define the same subgroups* $\mathsf{R}_m \subset \mathsf{A}_m \subset \mathbb{Z}[\mathbb{P}^1_{01\infty}(\mathcal{O}_F)]$ *giving the same Bloch groups arising from* $A_m^{\ell\text{-adic}}(\mathcal{O}_F)$ *of Theorem* 4.4.

Proof of Theorem 4.4 (1): As the degree 2-part of the fundamental coLie algebra coincides with $\mathrm{Ext}^1_{\mathscr{MT}_F^{\mathrm{ad}}(\varphi)}(F\langle 0\rangle, F\langle 1\rangle)$, there is nothing to prove when $m = 0, 1$. Hence, we may assume that $m > 1$. Pick any element ξ of $A_m^{\ell\text{-adic}}(\mathcal{O}_F)$. Then, by Lemma 2.7, the image of ξ in $\mathrm{coLie}(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F))^{(2m)}$ lies in $H^1(F,\mathbb{Q}_p(m)) = \mathrm{coLie}(\mathcal{MT}_{\mathbb{Q}_p}(\mathscr{G}_F))^{(2m),d=0}$. Noting that $\exp_{\mathbb{Q}_p(m)}$ is an isomorphism from $D_{\mathrm{dR}}(\mathbb{Q}_p(m))$ to $H^1(F,\mathbb{Q}_p(m))$, we see that $\mathscr{L}_m^{\mathrm{ad}}(\xi)$ is contained in $F\langle m\rangle \cong \mathrm{coLie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(2m),d=0}$ from the injectivity of the middle vertical homomorphism of the diagram in Proposition 5.2. □

Proof of Theorem 1.1: Let $A_m^{\ell\text{-adic}}(\mathcal{O}_F)$ be the same as in Theorem 4.4 and take an element $\xi \in A_m^{\ell\text{-adic}}(\mathcal{O}_F)$. Then, by Proposition 5.2 and Theorem 4.4 (2), we have the following equality in $H^1(F,\mathbb{Q}_p(m))$:

$$\mathscr{L}_m^{\ell\text{-adic}}(\xi) = \exp_{\mathbb{Q}_p(m)}(\mathscr{L}_m^{\mathrm{ad}}(\xi)) = \exp_{\mathbb{Q}_p(m)}(-\mathscr{L}_m^{p\text{-adic}}(\xi)). \tag{5.5}$$

By (5.3), the RHS of (5.5) is equal to $\frac{-1}{(m-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{(1-\frac{\sigma_F}{p^m})\mathscr{L}_m^{p\text{-adic}}(\xi)\right\}\phi_{m,F}^{CW}\right)$. This completes the proof of Theorem 1.1. □

§ 6. Appendix: Proof of Proposition 4.5

A6.1. In this Appendix, we assume that F is unramified over $\mathbb{Q}_p$. Let M be an object in $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$ and $M=\oplus_{n\in\mathbb{Z}}M^{[n]}$ the slope decomposition of M as a φ-module. Remark that the graded F-vector space $M=\oplus_{n\in\mathbb{Z}}M^{[n]}$ has a natural $\mathbb{Q}_p$-structure as $M^{[n]}=M^{\varphi=p^n}\otimes_{\mathbb{Q}_p}F$ $(n\in\mathbb{Z})$. Now we fix a $\mathbb{Q}_p$-basis $\{\alpha_1,\dots,\alpha_d\}$ of F where $d=[F:\mathbb{Q}_p]$.

Note that giving a Hodge filtration on the graded F-vector space $\oplus_n M^{[n]}=\left(\oplus_n M^{\varphi=p^n}\right)\otimes_{\mathbb{Q}_p}F$ is equivalent to giving a collection of F-linear endomorphisms $\{N_k: M^{[*]}\to M^{[*-k]}\}_{k\geq 1}$ on M. Write p_i for the projection $F=\oplus_i\alpha_i\mathbb{Q}_p\to\alpha_i\mathbb{Q}_p$ and define $N_{k,\alpha_i}:=\alpha_i^{-1}(\mathrm{id}_{V_\bullet}\otimes p_i)\circ N_k|_{V_\bullet}$ for the graded $\mathbb{Q}_p$-vector space $V_\bullet=\oplus_n V_n:=\oplus_n M^{\varphi=p^n}$. Therefore, giving an object M of $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$ is equivalent to giving a tuple $\{V_\bullet,N_{k,\alpha_i}\}_{1\leq i\leq d,k\in\mathbb{Z}_{>0}}$ such as

- A finite dimensional graded $\mathbb{Q}_p$-vector space $V_\bullet=\oplus_n V_n$,
- $\mathbb{Q}_p$-linear endomorphisms N_{k,α_i} of $V_\bullet$ of degree $-k$.

Given the above collection $\{V_\bullet,N_{k,\alpha_i}\}$, the Hodge filtration on $V_\bullet\otimes_{\mathbb{Q}_p}F$ is explicitly delineated by

$$F^n(V_\bullet\otimes_{\mathbb{Q}_p}F)=\left[\exp\Big(\sum_{k=1}^{\infty}\sum_{i=1}^{d}\alpha_i N_{k,\alpha_i}\Big)\right]\left(\bigoplus_{j\geq n}V_j\otimes_{\mathbb{Q}_p}F\right). \tag{6.1}$$

Example 6.1.

(1) *Let $V=V_n=\mathbb{Q}_p e_n$ and $N_{k,\alpha_i}=0$ for all i,k. Then, the corresponding admissible filtered φ-module is $F\langle -n\rangle$.*

(2) *Let $x=\sum_{i=1}^d x_i\alpha_i\in F$ with $x_i\in\mathbb{Q}_p$ and let V be the two-dimensional graded vector space $V_0\oplus V_{-1}=\mathbb{Q}_p e_0\oplus\mathbb{Q}_p e_{-1}$. We define $N_{1,\alpha_i}: V_0\to V_{-1}$ by $N_{1,\alpha_i}(e_0)=x_i e_{-1}$ and define $N_{k,\alpha_i}=0$ if $k>1$. Then, the corresponding admissible filtered φ-module M to $(V,\{N_{k,\alpha_i}\})$ is two-dimensional F-vector space $M:=Fe_0\oplus Fe_{-1}$ equipped with the semi-linear action φ: $e_0\mapsto e_0$, $e_{-1}\mapsto p^{-1}e_{-1}$ and with the filtration $F^\bullet M$ defined by*

$$F^iM=\begin{cases}M & \text{if } i\leq -1,\\ F(e_0+xe_{-1}) & \text{if } i=0,\\ 0 & \text{if } i\geq 1.\end{cases} \tag{6.2}$$

The admissible filtered φ-module M is an extension of $F\langle 0\rangle$ by $F\langle 1\rangle$ and represents the isomorphism class $x\in F=\mathrm{Ext}^1_{\mathscr{MT}_F^{\mathrm{ad}}(\varphi)}(F\langle 0\rangle,F\langle 1\rangle)$.

Lemma 6.2. *The fundamental Lie algebra* $\mathrm{Lie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))$ *is canonically isomorphic to the complete free Lie algebra* $\mathrm{Lie}\langle\langle N_{k,\alpha_i}\rangle\rangle_{k>0,\,1\leq i\leq d}$ *over* $\mathbb{Q}_p$ *generated by those symbols* N_{k,α_i}*, where completion is taken with respect to the quotients modulo the ideals generated by Lie monomials in* N_{k,α_i}*'s whose total weights in* k *are bounded below.*

Proof. This lemma is more or less well-known (e.g., [DCW] 4.4 when $F=\mathbb{Q}_p$). Indeed, for $\mathcal{M}=\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$, let $\pi_1(\mathcal{M},\omega)=\mathbb{G}_m\ltimes U(\mathcal{M})$ be the splitting of §2.1 providing the pro-unipotent radical $U(\mathcal{M})$ and the fundamental Lie algebra $\mathrm{Lie}(\mathcal{M})$. Let $\mathcal{L}_n$ be the quotient of $\mathrm{Lie}\langle\langle N_{k,\alpha_i}\rangle\rangle_{k>0,\,1\leq i\leq d}$ modulo the ideal generated by the Lie monomials of weights $>n$. Then, $\mathcal{L}_n$ is a finite dimensional $\mathbb{Q}_p$-vector space which has a natural grading by the *negative of* total weights (i.e., $\mathcal{L}_n=\bigoplus_{k=1}^n V_{-k}$ with $V_{-k}:=\sum_{i=1}^d\mathbb{Q}_pN_{k,\alpha_i}+\sum_{i,j=1}^d\mathbb{Q}_p[N_{k-1,\alpha_i},N_{1,\alpha_j}]+\cdots$etc.), and has a collection of endomorphisms by $\mathrm{ad}(N_{k,\alpha_i})\colon\mathcal{L}_n\to\mathcal{L}_n$ of degree $-k$ $(k=1,2,\dots)$. Noting this and the above description of $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$, we see that $\mathcal{L}_n$ $(n\geq 1)$ are objects in $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$ and that the representation of $\mathrm{Lie}(\mathcal{M})$ (resp. $\mathbb{G}_m$) on them are given through those adjoint endomorphisms $\{\mathrm{ad}(N_{k,\alpha_i})\}_{k,i}$ (resp. through the weight gradation). Letting $n\to\infty$ along with those Lie homomorphisms $\mathrm{Lie}(\mathcal{M})\to\mathrm{End}_{\mathbb{Q}_p}(\mathcal{L}_n)$ yields a Lie algebra isomorphism $\mathrm{Lie}(\mathcal{M})\xrightarrow{\sim}\mathrm{Lie}\langle\langle N_{k,\alpha_i}\rangle\rangle_{k>0,\,1\leq i\leq d}$. □

Remark 6.3. *By construction, the image* $\overline{N_{k,\alpha_i}}$ *of* N_{k,α_i} *in the abelianization*

$$\mathrm{Lie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{\mathrm{ab}}=\prod_{l=1}^{\infty}\mathrm{Ext}^1_{\mathscr{MT}_F^{\mathrm{ad}}(\varphi)}(F\langle 0\rangle,F\langle l\rangle)^{\vee}=\prod_{l=1}^{\infty}\mathrm{Hom}_{\mathbb{Q}_p}(F\langle l\rangle,\mathbb{Q}_p)$$

is characterized by: $\overline{N_{k,\alpha_i}}(\alpha_j\zeta_{p^\infty}^{\otimes l}/t^l)=\begin{cases}0, & \text{if } j\neq i \text{ or } l\neq k;\\ 1, & \text{if } j=i \text{ and } l=k.\end{cases}$

A6.2. We recall the first rational cohomology of an algebraic group quickly. Let π be a pro-algebraic group over $\mathbb{Q}_p$ and G an algebraic group over $\mathbb{Q}_p$ with an algebraic action of π. Then, a rational 1-cocycle c of π with coefficients in G is a morphism of schemes $c\colon\pi\to G$ satisfying the usual 1-cocycle relation. We say that two rational 1-cocycles c,c' are equivalent if there exists $g\in G(\mathbb{Q}_p)$ such that $c'(\sigma)={}^{\sigma}gc(\sigma)g^{-1}$ for any $\sigma\in\pi$. We denote by $H^1(\pi,G)$ the set of equivalence classes of rational 1-cocycles of π with coefficients in G and call it the first rational cohomology of π with coefficients in G. It is easily checked that $H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi),\mathscr{P}_m^{\mathrm{crys}})$ is canonically isomorphic to the first rational cohomology $H^1(\pi_1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi),\omega),\omega(\mathscr{P}_m^{\mathrm{crys}}))$ where $\omega\colon\mathscr{MT}_F^{\mathrm{ad}}(\varphi)\to\mathrm{Vec}_{\mathbb{Q}_p}$ is the canonical fiber functor.

A6.3. The following lemma enables us to interpret the pointed set $H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi),\mathscr{P}_m^{\mathrm{crys}})$ as a set of the homomorphisms between two Lie algebras.

Lemma 6.4. *We have a canonical isomorphism of pointed sets*

$$\psi_m\colon\mathrm{Hom}_{\mathbb{G}_{m,\mathbb{Q}_p}\text{-Lie}}(\mathrm{Lie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi)),\omega(\mathfrak{p}_m^{\mathrm{crys}}))\xrightarrow{\sim}H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi),\mathscr{P}_m^{\mathrm{crys}})$$

satisfying $r_m \circ \psi_m(f) = \mathrm{Res}_{\mathrm{Lie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(-2m)}}(f)$.

Proof. Let f be an element of $\mathrm{Hom}_{\mathbb{G}_{m,\mathbb{Q}_p}\text{-Lie}}(\mathrm{Lie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi)), \omega(\mathfrak{p}_m^{\mathrm{crys}}))$. Then, we define a rational 1-cocycle $c_f \colon \pi_1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \omega) = \mathbb{G}_{m,\mathbb{Q}_p} \ltimes U(\mathscr{MT}_F^{\mathrm{ad}}(\varphi)) \to \omega(\mathscr{P}_m^{\mathrm{crys}})$ by $(x, y) \mapsto {}^x\exp(f)(y)$. Here, $\exp(f) \colon U(\mathscr{MT}_F^{\mathrm{ad}}(\varphi)) \to \omega(\mathscr{P}_m^{\mathrm{crys}})$ is the algebraic group homomorphism defined by the Lie homomorphism f. We define $\psi_m(f)$ to be the cohomology class $[c_f]$ of c_f. By construction, ψ_m are compatible with the short exact sequence of pointed sets

$$1 \to H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), F\langle m\rangle) \to H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_m^{\mathrm{crys}}) \to H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_{m-1}^{\mathrm{crys}}) \to 1$$

(cf. (4.1)). Then, by induction on m, we can show that ψ_m are isomorphisms of pointed sets for all m. □

A6.4. We now calculate the composite of the isomorphism in Lemma 6.4 and the isomorphism induced from (4.1): $\log \circ \psi_m \colon H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_m^{\mathrm{crys}}) \cong \mathscr{P}_m^{\mathrm{crys}}(F) \xrightarrow{\sim} \mathfrak{p}_m^{\mathrm{crys}}$.

Proposition 6.5. *For* $f \in \mathrm{Hom}_{\mathbb{G}_{m,\mathbb{Q}_p}\text{-}Lie}(\mathrm{Lie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi)), \omega(\mathfrak{p}_m^{\mathrm{crys}}))$, *we have*

$$\log \circ \psi_m(f) = \sum_{k=1}^{m} \sum_{1 \le i \le d} f(N_{k,\alpha_i}) \otimes \alpha_i.$$

Here, $\mathfrak{p}_m^{\mathrm{crys}} = F\langle 1\rangle \oplus (\oplus_{j=1}^m F\langle j\rangle)$ *is identified with* $\omega(\mathfrak{p}_m^{\mathrm{crys}}) \otimes_{\mathbb{Q}_p} F$ *via*

$$\iota \colon \omega(\mathfrak{p}_m^{\mathrm{crys}}) \otimes_{\mathbb{Q}_p} F \xrightarrow{\sim} \mathfrak{p}_m^{\mathrm{crys}}; \quad \big(e_l \otimes 1 \mapsto \varepsilon_l,\ l = 0, \dots, m\big),$$

where $\{e_l\}$, $\{\varepsilon_l\}$ *are fixed bases in* (2.1), (4.3) *respectively.*

Proof. We denote by $\lambda \in \mathfrak{p}_m^{\mathrm{crys}}$ the image of f under the isomorphism in the proposition and by $\mathscr{P}_m^{\mathrm{crys}}(\lambda)$ the corresponding $\mathscr{P}_m^{\mathrm{crys}}$-torsor. Recall from [S1, Proof of Proposition 3.3] that the underlying φ-scheme structure of $\mathscr{P}_m^{\mathrm{crys}}(\lambda)$ is $\mathscr{P}_m^{\mathrm{crys}}$ and the Hodge filtration on $\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}(\lambda)) = \mathcal{O}(\mathscr{P}_m^{\mathrm{crys}})$ is given by the 'left translation' by $\exp(\lambda)$, i.e., $F^n(\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}(\lambda))) = [\exp(\lambda)]^\sharp F^n(\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}))$. On the other hand, it follows from (6.1) that the Hodge filtration on the ind-object $\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}(\lambda))$ of $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$ is given by:

$$F^n(\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}(\lambda))) = \left[\exp\left(\sum_{k,i} f(N_{k,\alpha_i}) \otimes \alpha_i\right)\right]^\sharp \left(F^n(\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}))\right)$$

with the identification $\mathcal{O}(\mathscr{P}_m^{\mathrm{crys}}(\lambda)) \cong \mathcal{O}(\omega(\mathscr{P}_m^{\mathrm{crys}})) \otimes_{\mathbb{Q}_p} F$ given via $\zeta_{p^\infty}^{\otimes l}/t^l$ $(l \in \mathbb{Z})$. Thus the two points $\exp(\lambda)$, $\exp\left(\sum_{k,i} \alpha_i f(N_{k,\alpha_i})\right) \in \mathscr{P}_m^{\mathrm{crys}}(F)$ define the same torsor in $H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_m^{\mathrm{crys}})$, and coincide with each other according to (4.1). □

Proof of Proposition 4.5: We define the map $\mathsf{ret}_m\colon \mathrm{coLie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(2m)} \to F\langle m\rangle$ by

$$\mathsf{ret}_m(\mathfrak{f}) := \sum_{i=1}^{d} \alpha_i\, \mathfrak{f}(N_{m,\alpha_i}) \frac{\zeta_{p^\infty}^{\otimes m}}{t^m}$$

for each $\mathfrak{f} \in \mathrm{coLie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(2m)}$, i.e., for $\mathfrak{f}\colon \mathrm{Lie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(-2m)} \to \mathbb{Q}_p$. Then, on $F\langle m\rangle = \mathrm{coLie}(\mathscr{MT}_F^{\mathrm{ad}}(\varphi))^{(2m),d=0}$, we see from Remark 6.3 that the map ret_m restricts to identity. The commutativity of the diagram in Proposition 4.5 follows from Lemma 6.4 and Proposition 6.5. This completes the proof of Proposition 4.5.

References

[BD] A. Beilinson, P. Deligne, Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs, *Proc. Sympos. Pure Math.*, **55** (1994), Part 2, 97–121.

[BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift, Vol. I, 333-400, *Progr. Math.*, **86**, Birkhauser Boston, Boston, MA, 1990.

[C1] R. Coleman, Dilogarithms, regulators, and p-adic L-functions, *Invent. Math.*, **69** (1982), 171–208.

[C2] R. Coleman, Anderson-Ihara theory: Gauss sums and circular units, *Adv. Stud. Pure Math.*, **17** (1989), 55–72.

[CF] P. Colmez, J-M. Fontaine, Constructions des représentations p-adiques semi-stables, *Invent. Math.* **140** (2000) 1–43.

[DCW] I. Dan-Cohen, S. Wewers, Mixed Tate motives and the unit equation, *Int. Math. Res. Notices* [doi: 10.1093/imrn/rnv239], in press.

[De] P. Deligne, Le groupe fondamental de la droite projective moins trois points, Galois groups over $\mathbb{Q}$, 79-297, *Math. Sci. Res. Inst. Publ.*, **16**, Springer, 1989.

[DW] J.-C. Douai, Z. Wojtkowiak, Descent for l-adic polylogarithms, *Nagoya Math. J.* **192** (2008), 59–88.

[Fo] J. M. Fontaine, Représentations p-adiques semistables, Periodes p-adiques (Bures-sur-Yvette, 1988). *Asterisque* No. 223 (1994), 113–184.

[Fu] H. Furusho, p-adic multiple zeta values I: p-adic multiple polylogarithms and the p-adic KZ equation, *Invent. Math.*, **155** (2004), 223–286; II: Tannakian interpretations, *Amer. Journal of Math.*, **129** (2007), 1105–1144.

[Gi] J. Giraud, Cohomologie non abelienne, Die Grundlehren der mathematischen Wissenschaften, Band 179, Springer (1971).

[Go1] A. B. Goncharov, Geometry of configurations, polylogarithms, and motivic cohomology, *Adv. Math.* **114** (1995), no. 2, 197–318.

[Go2] A. B. Goncharov, Galois symmetries of fundamental groupoids and noncommutative geometry, *Duke Math. J.* **128** (2005), no. 2, 209–284.

[HM] R. Hain, M. Matsumoto, Weighted completion of Galois groups and Galois actions on the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, *Compositio Math.* **139** (2003), no. 2, 119–167.

[Ih] Y. Ihara, Braids, Galois groups, and some arithmetic functions, Proc. Intern. Congress of Math. Kyoto 1990, 99–120.

[Kim1] M. Kim, The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel, *Invent. Math.* **161** (2005), no. 3, 629–656.

[Kim2] M. Kim, The unipotent Albanese map and Selmer varieties for curves, *Publ. Res. Inst. Math. Sci.***45** (2009), no. 1, 89–133.

[Kim3] M. Kim, Tangential localization for Selmer varieties, *Duke Math. J.* **161** (2012), no. 2, 173–199.

[KN] M. Kolster, T. Nguyen Quang Do, Syntomic regulators and special values of p-adic L-functions. *Invent. math.*, **133** (1998), 417–447.

[NSW] H. Nakamura, K. Sakugawa, Z.Wojtkowiak, Polylogarithmic analogue of the Coleman-Ihara formula, I, *Osaka J. Math.* **54** (2017), in press.

[NW] H. Nakamura and Z. Wojtkowiak, On explicit formulae for l-adic polylogarithms, *Proc. Sympos. Pure Math.*, **70** (2002), 285–294.

[O1] M. Olsson, Towards Non-abelian p-adic Hodge Theory in the Good Reduction Case, *Mem. Amer. Math. Soc.* **210**, (2011), no. 990.

[O2] M. Olsson, The bar construction and affine stacks, *Comm. Algebra* **44** (2016), no. 7, 3088–3121.

[P] B. Perrin-Riou, Représentations p-adiques ordinaires, avec un appendice par Luc Illusie, Periodes p-adiques (Bures-sur-Yvette, 1988). *Asterisque* No. 223 (1994), 185–220.

[S1] K. Sakugawa, On a non-abelian generalization of the Bloch–Kato exponential map, *Math. J. Okayama Univ.*, **59** (2017), 41–70.

[S2] K. Sakugawa, The upper bound of the space of ℓ-adic polylogarithms in the first Galois cohomology, Preprint, submitted.

[W1] Z. Wojtkowiak, On l-adic iterated integrals I: Analog of Zagier conjecture, *Nagoya Math. J.* **176** (2004), 113–158.

[W2] Z. Wojtkowiak, On l-adic iterated integrals II: Functional equations and l-adic polylogarithms. *Nagoya Math. J.* **177** (2005), 117–153.

[W3] Z. Wojtkowiak, On l-adic iterated integrals III, Galois actions on fundamental groups, *Nagoya math. J.*, **178** (2005), 1–36.

[Y] G. Yamashita, Bounds for the dimensions of p-adic multiple L-value spaces, *Doc. Math.*, Extra volume: Andrei A. Suslin sixtieth birthday, (2010), 687–723.

[Z] D. Zagier, Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields, Arithmetic algebraic geometry (Texel, 1989), 391–430, *Progr. Math.*, **89**, Birkhauser, 1991.