Galois representations in the profinite Teichmüller modular groups

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This article was published from
"Geometric Galois Actions I;
Around Grothendieck's Esquisse d'un Programme"
(L.Schneps, P.Lochak eds.)
London Math. Soc. Lect. Note Series 242
(1997) pp.159--173

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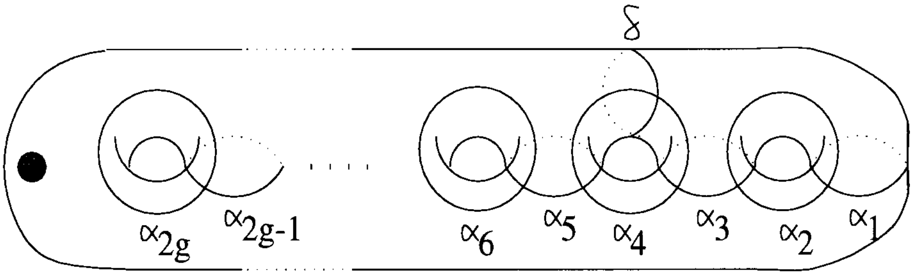
§1. Introduction

Let $M_{g,1}$ be the moduli stack over $\mathbb{Q}$ of the one point marked smooth projective curves of genus $g \geq 1$. Then, the Galois-Teichmüller modular group '$\pi_1(M_{g,1})$' is a group extension of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by the profinite completion of the mapping class group Γ_g^1 of a 1-pointed genus g surface (cf. [Oda]):

$$1 \to \hat{\Gamma}_g^1 \to \pi_1(M_{g,1}) \to G_{\mathbb{Q}} \to 1. \tag{1.1}$$

It is well known that Γ_g^1 has a finite number of generators $a_1, \ldots, a_{2g}$, d which are Dehn twists along simple closed curves $\alpha_1, \ldots, \alpha_{2g}, \delta$ in the following figure respectively (Lickorish [L], Humphries [Hu]).



In this note, we prove the following

Theorem A. *There exists a good splitting homomorphism* $s : G_{\mathbb{Q}} \to \pi_1(M_{g,1})$ *of (1.1) such that the conjugate action* $* \mapsto s(\sigma) * s(\sigma)^{-1}$ ($\sigma \in G_{\mathbb{Q}}$) *transforms the twist generators* $a_1, \ldots, a_{2g}, d$ *of* $\hat{\Gamma}_g^1$ *as follows.*

$$\begin{cases} \sigma(d) &= d^{\chi(\sigma)}, \\ \sigma(a_i) &= \mathfrak{f}_\sigma(y_i, a_i^2)^{-1} a_i^{\chi(\sigma)} \mathfrak{f}_\sigma(y_i, a_i^2) \qquad (1 \leq i \leq 2g). \end{cases}$$

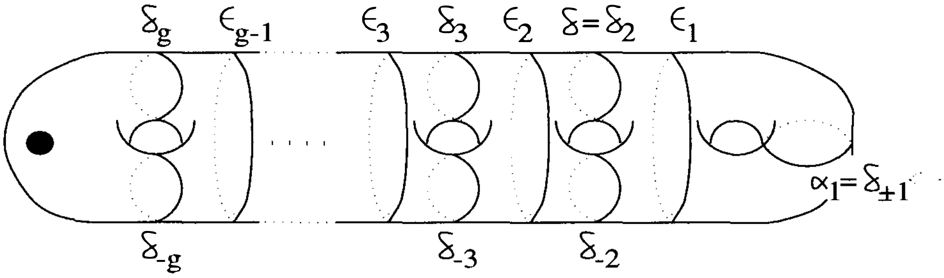
Here, $\chi : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times$ *is the cyclotomic character acting on the roots of unity,*

$$y_1 = 1, \quad y_i = a_{i-1} \cdots a_1 \cdot a_1 \cdots a_{i-1} \quad (2 \leq i \leq 2g),$$

* The author was partially supported by Yoshida Foundation for sciences and technology.

and $\mathfrak{f}_\sigma(X,Y)$ *is a unique "pro-word" in* X, Y *defined as an element of the commutator subgroup of the free profinite group* $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}, \overrightarrow{01})$ *on the standard loops* X, Y *turning around the punctures* $0, 1$ *respectively, on which* $\sigma \in G_{\mathbb{Q}}$ *acts as* $\sigma(X) = X^{\chi(\sigma)}$, $\sigma(Y) = \mathfrak{f}_\sigma(X,Y)^{-1} Y^{\chi(\sigma)} \mathfrak{f}_\sigma(X,Y)$.

Our splitting homomorphism s in Theorem A is provided by a sophisticated use of Deligne's notion of "tangential base points" ([De]). In fact, we construct two tangential base points lying on the hyperelliptic locus $\mathcal{H}_{g,1}$ of $M_{g,1}$. One is the image of a tangential base point on the "braid configuration space", originated from Drinfeld [Dr], Ihara-Matsumoto [IM], which a priori succeeds to a Galois action of desired form on $a_1, \ldots, a_{2g}$. The other is the one induced from a certain maximally degenerate hyperelliptic curve over $\mathbb{Q}[[q]]$ whose special fibre is in the form where the simple closed curves $\delta_{\pm i}$ $(1 \le i \le g)$, ϵ_j $(1 \le j \le g-1)$ indicated below vanish:



We construct such a curve in §3 by using Grothendieck's formal patching technique in a very similar way to Ihara-Nakamura [IN] (cf. also Harbater [Ha]). At the second tangential base point, $G_{\mathbb{Q}}$ acts a priori via the cyclotomic character on the Dehn twists $d_{\pm i}$, e_j corresponding to those simple closed curves $\delta_{\pm i}$, ϵ_j respectively. We then estimate these two tangential base points in the local neighborhood of a maximally degenerate ∞-point of $M_{0,2g+2}$, and show that the differences of the corresponding two Galois actions preserve the $d_{\pm i}, e_j$'s respectively. From this we conclude that the desired Galois action is obtained from the first tangential base point (twisted by a dummy 1-cocycle on the "hyperelliptic involution".) Especially, our proof shows more information on our Galois action:

Theorem A'. *The Galois action given in Theorem A on* $\hat{\Gamma}^1_g$ *transforms the Dehn twists* $d_{\pm *}, e_*$*'s (corresponding to* $\delta_{\pm *}, \epsilon_*$*'s respectively) by the cyclotomic character:*

$$\sigma(d_i) = d_i^{\chi(\sigma)}, \quad \sigma(e_j) = e_j^{\chi(\sigma)} \quad (1 \le |i| \le g,\ 1 \le j \le g-1;\ \sigma \in G_{\mathbb{Q}}).$$

In [Dr], Drinfeld introduced what is called the Grothendieck-Teichmüller group $\widehat{GT}$, into which, thanks to Belyi [Be], $G_{\mathbb{Q}}$ is embedded by the parameters $(\chi(\sigma), \mathfrak{f}_\sigma)$ of Theorem A (cf. [Ih2], [N, Appendix]). The group Γ_g^1 is known to be finitely presented (A.Hatcher-W.Thurston), and the relations for the Humphries' generators are listed in Wajnryb [W]. Only by using defining relations of $\widehat{GT}$, one can check directly that our $G_{\mathbb{Q}}$-action of Theorem A preserves almost all Wajnryb's relations except for the lantern relation (due to M.Dehn, D.Johnson). These calculations are relevant to the problem of approximating $G_{\mathbb{Q}}$ by $\widehat{GT}$ which was taken up also by L.Schneps, P.Lochak at the Luminy conference in several contexts. See the article [S] by L. Schneps in this volume for various background materials on $\widehat{GT}$.

At the same conference, M.Matsumoto posed a remarkable approach to genus 3 case from his E_7-singularity viewpoint, which motivated the author to make the present work. Matsumoto also worked out his resultant article [M2] soon in which another type of tangential base point on M_3 is displayed in connection with the Artin group of type E_7.

§2. Hyperelliptic locus

Let $\mathbb{A}_v^{2g+1}$ be the $(2g+1)$-dimensional affine space over $\mathbb{Q}$ with coordinates $v = (v_1, \ldots, v_{2g+1})$ and $\Delta = \bigcup_{i \neq j} \Delta_{ij}$ be the weak diagonal divisor on it, where $\Delta_{ij} = \{v \mid v_i = v_j\}$. The symmetric group S_{2g+1} acts naturally on $\mathbb{A}_v^{2g+1} - \Delta$, and its quotient variety is in the form of $\mathbb{A}_u^{2g+1}$ minus the discriminant locus D. The points $u = (u_1, \ldots, u_{2g+1}) \in \mathbb{A}_u^{2g+1}$ are identified with the monic polynomials $f_u(x) = x^{2g+1} + u_1 x^{2g} + \cdots + u_{2g+1}$, and $u \notin D$ if and only if the equation $f_u(x) = 0$ has only simple zeros. We have then a family of hyperelliptic curves $\{y^2 = f_u(x)\}_u$ over $\mathbb{A}_u^{2g+1} - D$ each fibre of which has ∞ as a specially attached point. Thus, there exists a representing morphism from $\mathbb{A}_u^{2g+1} - D$ to the hyperelliptic locus $\mathcal{H}_{g,1}$ of the moduli stack $M_{g,1}$ whose point represents, by definition, a hyperelliptic curve Y with one marked point fixed by the hyperelliptic involution. Every such $[Y] \in \mathcal{H}_{g,1}$ can be realized as a double cover of $\mathbb{P}^1$ with $2g+2$ branch points (one of which is distinguished from others as the point ∞) so that there exists a natural morphism $\mathcal{H}_{g,1} \dashrightarrow M_{0,2g+2}/S_{2g+1}$, where $M_{0,2g+2}$ is the moduli of (ordered) $2g+2$-pointed projective lines and S_{2g+1} is the automorphism group of $M_{0,2g+2}$ "fixing the $(2g+2)$-nd marking point ∞". We also have an obvious morphism $\mathbb{A}_v^{2g+1} \setminus \Delta \dashrightarrow M_{0,2g+2}$ mapping v to the class of $(\mathbb{P}^1; v_1, \ldots, v_{2g+1}, \infty)$ so as to fit into the commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{A}_v^{2g+1} \setminus \Delta & & \longrightarrow & & M_{0,2g+2} \\
\downarrow & & & & \downarrow \\
\mathbb{A}_u^{2g+1} \setminus D & \dashrightarrow & \mathcal{H}_{g,1} & \dashrightarrow & M_{0,2g+2}/S_{2g+1}.
\end{array}
\tag{2.1}
$$

Now, the geometric fundamental group of $\mathbb{A}_v^{2g+1} \setminus D$ is the profinite braid group $\hat{B}_{2g+1}$ with standard generators $\sigma_1, \dots, \sigma_{2g}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ ($|i-j| \geq 2$), $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ ($i = 1, \dots, 2g-1$), and its center is a free procyclic subgroup generated by $w_{2g+1} = (\sigma_1 \cdots \sigma_{2g})^{2g+1}$. The lower horizontal arrows of the above diagram induce projections of $\hat{B}_{2g+1}$ leading to

$$\pi_1(M_{0,2g+2}/S_{2g+1} \otimes \overline{\mathbb{Q}}) \cong \hat{B}_{2g+1}/\langle w_{2g+1} \rangle.$$

Moreover, the natural homomorphism $\pi_1(\mathcal{H}_{g,1}) \to \pi_1(M_{g,1})$ maps σ_i to a_i for $i = 1, \dots, 2g$ (cf. [BH]). In $\hat{B}_{2g+1}$, we have a distinguished commutative subgroup generated by $y_i = \sigma_{i-1} \cdots \sigma_1 \cdot \sigma_1 \cdots \sigma_{i-1}$ ($2 \leq i \leq 2g+1$). When mapped into $\pi_1(M_{g,1})$, these y_i ($2 \leq i \leq 2g$) coincide with those of Theorem A, while $w_{2g+1} = y_{2g+1} \cdots y_2$ gives a topological mapping class of a "hyperelliptic involution".

§3. Hyperelliptic stable curve

In this section, we shall construct a certain hyperelliptic curve over $\mathbb{Q}[[q]]$ with a special type of maximal degeneration. Our construction process goes on exactly parallel to that of Ihara-Nakamura [IN] §2, with an additional care to the hyperelliptic involution making the curve be a double-cover of a degenerate projective line (cf. also [Ha]). In [IN], we showed an explicit method for constructing a curve over $\mathbb{Q}[[q_1, \dots, q_{m'}]]$ from a maximally degenerate stable marked curve – "$\mathbb{P}^1_{01\infty}$-diagram" – over $\mathbb{Q}$ and its "distinguished coordinates" of the irreducible components. In this note, we present a variant of this method by introducing a certain $\mathbb{P}^1_{0\pm1\infty}$-diagram Y^0 appearing as a double cover of a standard $\mathbb{P}^1_{01\infty}$-tree X^0. This variant is useful when extending the natural involution on Y^0 to that on the deformed family over $\mathbb{Q}[[q]]$.

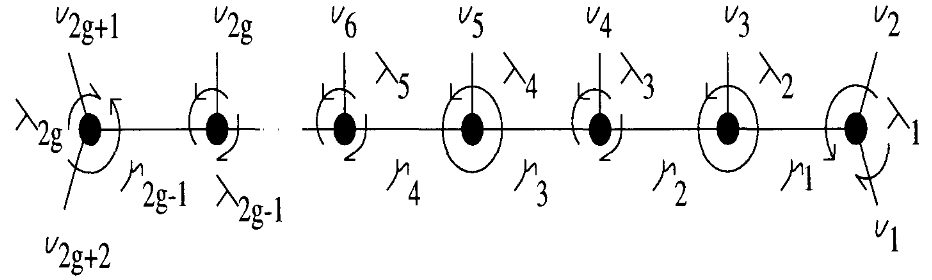
Now, let us start from the definition of X^0. It is a connected stable curve over $\mathbb{Q}$ consisting of rational irreducible components X^0_λ ($\lambda \in \Lambda$), ordinary double points P^0_μ ($\mu \in M$) and marking points Q^0_ν ($\nu \in N$) such that

$$\Lambda = \{\lambda_1, \dots, \lambda_{2g}\}, \ M = \{\mu_1, \dots, \mu_{2g-1}\}, \ N = \{\nu_1, \dots, \nu_{2g+2}\}, \tag{3.1}$$

and the incidence relations are given by

$$\begin{cases} \mu_i/\lambda_i, \ \mu_i/\lambda_{i+1} \quad (1 \leq i \leq 2g-1), \\ \nu_1, \nu_2/\lambda_1, \ \nu_i/\lambda_{i-1} \ (3 \leq i \leq 2g), \ \nu_{2g+1}, \nu_{2g+2}/\lambda_{2g}, \end{cases} \tag{3.2}$$

where μ/λ (resp. ν/λ) means that P^0_μ (resp. Q^0_ν) lies on X^0_λ. The dual graph of X^0 (with "legs" corresponding to Q^0_ν) is as follows.



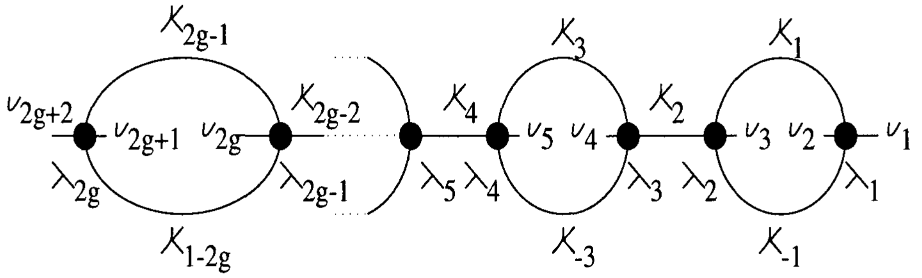
For each incidence pair μ/λ, ν/λ, we introduce distinguished coordinates $t_{\mu/\lambda}$ (resp. $t_{\nu/\lambda}$) of X^0_λ which has value 0 at P^0_μ (resp. Q^0_ν) and $1, \infty$ at the other distinguished points (i.e., double/marking points) on X^0_λ. Regarding the above figure as a plane tree, we introduce such coordinates in the way that the values at the distinguished points on each X^0_λ are anticlockwise arranged in the same cyclic order as $0, 1, \infty$ except for

$$\begin{cases} t_{\mu_{2i}/\lambda_{2i+1}}(Q^0_{\nu_{2i+2}}) = \infty \quad (1 \le i \le g-1), \\ t_{\nu_1/\lambda_1}(Q^0_{\nu_2}) = t_{\nu_{2g+1}/\lambda_{2g}}(Q^0_{\nu_{2g+2}}) = \infty. \end{cases}$$

Next, we construct a double cover Y^0 over X^0 also as a connected stable curve. Its irreducible components Y^0_λ $(\lambda \in \Lambda)$ are again all rational components, and the marking points R^0_ν $(\nu \in N)$ lie on them in the same incidence relations ν/λ as in (3.2) above. But the double points $\{P^0_\kappa\}$ $(\kappa \in K)$ on Y^0 are more complicated. The index set K is taken to be

$$\{\kappa_i\,; |i| \le 2g+1, \text{ odd}\} \cup \{\kappa_i\,; 2 \le i \le 2g-2, \text{ even}\}$$

and the incidence relations are given by $\kappa_j/\lambda_{|j|}, \lambda_{|j|+1}$ for all j.



Since each Y^0_λ has 4 distinguished points, we need to impose some condition on the relative locations of them on each component. This is done by introducing distinguished coordinates $s_{\kappa/\lambda}$, $s_{\nu/\lambda}$ in compatible ways so that their values at the distinguished points are $\{0, \pm 1, \infty\}$. We define them by $s_{\kappa/\lambda}(P^0_\kappa) = 0$, $s_{\nu/\lambda}(R^0_\nu) = 0$ and

$$s_{\kappa_i/\lambda}(P^0_{\kappa_{-i}/\lambda}) = \infty, \; s_{\kappa_i/\lambda_{|i|}}(R^0_{\nu_{|i|+1}}) = s_{\kappa_i/\lambda_{|i|+1}}(R^0_{\nu_{|i|+2}}) = -1 \quad (i = \text{odd}),$$

$$s_{\kappa_i/\lambda}(P^0_{\kappa_{\text{odd}>0}}) = 1, \; s_{\kappa_i/\lambda}(P^0_{\kappa_{\text{odd}<0}}) = -1 \quad (i = \text{even}),$$

$$s_{\nu_i/\lambda}(P^0_{\kappa_{\text{odd}>0}}) = 1, \; s_{\nu_i/\lambda}(P^0_{\kappa_{\text{odd}<0}}) = -1 \quad (1 \le i \le 2g+2).$$

Checking the compatibilities amounts to the fact that the transformations $s \mapsto -s$, $\frac{1}{s}$, $\frac{1-s}{1+s}$ keep $\{0, \pm 1, \infty\}$ invariant. We then define the covering morphism $\varpi : Y^0 \to X^0$ by

$$(3.3) \qquad \begin{cases} P^0_{\kappa_{\pm i}} \mapsto P^0_{\mu_{|i|}} \quad (i = \text{odd}), \\ s_{\kappa_i/\lambda} \mapsto s^2_{\kappa_i/\lambda} = t_{\mu_i/\lambda} \quad (i = \text{ even}), \\ s_{\nu/\lambda} \mapsto s^2_{\nu/\lambda} = t_{\nu/\lambda} \quad (\nu = \nu_1,\, \nu_{2g+2}). \end{cases}$$

Note that $\varpi : Y_0 \to X_0$ is ramified at all the Q^0_ν's and the $P^0_{\mu_{\text{even}}}$'s so that each component of Y^0 is a double cover of the corresponding component of X^0 ramified over exactly two points.

Let us then deform Y^0 to a 1-parameter family $Y/\mathbb{Q}[[q]]$ of hyperelliptic curves, by Grothendieck's formal patching technique ([G] EGA III Sect. 5.4; cf. also [DR], [Ha], [IN]). We prepare the following $\mathbb{Q}[[q]]$-algebras as parts of Y:

$$\text{(a1)} \qquad A_\kappa = \mathbb{Q}[s, s', \frac{1}{1 \pm s}, \frac{1}{1 \pm s'}][[q]]/(ss' - q) \qquad (\kappa \in K),$$
$$s = s_{\kappa/\lambda},\ s' = s_{\kappa/\lambda'}\ (\lambda \neq \lambda'),$$

$$\text{(a2)} \qquad A_\nu = \mathbb{Q}[s, \frac{1}{1 \pm s}][[q]] \qquad (\nu \in N),$$
$$s = s_{\nu/\lambda},$$

$$\text{(a3)} \qquad A_\lambda = \mathbb{Q}[s, \frac{1}{s}, \frac{1}{1 \pm s}][[q]] \qquad (\lambda \in \Lambda),$$
$$s = s_{\nu_{i+1}/\lambda_i}.$$

Then, since $A_\kappa[\frac{1}{s}]/q^N \cong \mathbb{Q}[s, \frac{1}{s}, \frac{1}{1 \pm s}][[q]]/q^N$ etc., the first two kinds of spectrums $\mathrm{Spec}\,(A_\kappa/q^N)$ $(\kappa \in K)$ and $\mathrm{Spec}\,(A_\nu/q^N)$ $(\nu \in N)$ are glued together by identifying their open parts with $\mathrm{Spec}\,(A_\lambda/q^N)$ $(\lambda \in \Lambda)$ along the diagram Y^0 so as to produce a scheme $\mathfrak{Y}^N$ over $\mathbb{Q}[q]/q^N$ $(N \geq 1)$. The resulting sequence $Y^0 = \mathfrak{Y}^1 \subset \mathfrak{Y}^2 \subset \cdots$ over artinian schemes are compatible to form a proper regular formal scheme $\mathfrak{Y}$ over $\mathrm{Spf}\,\mathbb{Q}[[q]]$. We denote the algebraization of $\mathfrak{Y}$ by $Y/\mathbb{Q}[[q]]$, and identify its special fibre with Y^0 in the obvious manner.

Observe that in each step of the above process, we have an involution on $\mathfrak{Y}_N$ interchanging local data compatibly as

$$s \leftrightarrow -s \quad \text{in } A_{\kappa_i} (i : \text{ even}), A_\nu, A_\lambda$$
$$A_{\kappa_i} \leftrightarrow A_{\kappa_{-i}};\ s_{\kappa_i/\lambda} \leftrightarrow s_{\kappa_{-i}/\lambda},\ s_{\kappa_i/\lambda'} \leftrightarrow s_{\kappa_{-i}/\lambda'}\ (i : \text{odd}).$$

These involutions on $\mathfrak{Y}^N$ $(N \geq 1)$ define an involution on $Y/\mathbb{Q}[[q]]$ extending the covering transformation of Y^0/X^0. Moreover, each marking point R^0_ν

has natural extensions $R_\nu^N \in \mathfrak{Y}^N(\mathbb{Q}[q]/q^N)$ and hence $R_\nu \in Y(\mathbb{Q}[[q]])$ fixed under the respective involutions. In particular, the generic fibre Y_η is a complete smooth curve over $\mathbb{Q}((q))$ with $2g+2$ fixed $\mathbb{Q}((q))$-points under an involution, hence is a hyperelliptic curve of the form

$$y^2 = (x - v_1(q)) \cdots (x - v_{2g+1}(q)),$$

where $v_i(q) \in \mathbb{Q}((q))$ corresponds to the branch at $R^0_{\nu_i}$ $(i = 1, \dots, 2g+1)$. These coordinates $v(q) = (v_1(q), \dots, v_{2g+1}(q))$ give a $\mathbb{Q}((q))$-valued point of $\mathbb{A}_v^{2g+1} \setminus \Delta$. We have thus obtained a tangential base point $\vec{v}$ on $\mathbb{A}_v^{2g+1} \setminus \Delta$ induced from the $\mathbb{Q}((q))$-rational point $v(q)$.

§4. Tangential base points

In the previous section, we constructed a deformation of a double cover Y^0 over X^0 using a single deformation parameter q to control all parts of the deformation procedure. In this section, we shall consider another direct construction of an explicit deformation of X^0 — a chain of $\mathbb{P}^1$'s — by allowing each singular point to deform independently by its own deformation parameter q_i. This construction provides a universal deformation of X^0 which will be related with the former deformation later in (4.3).

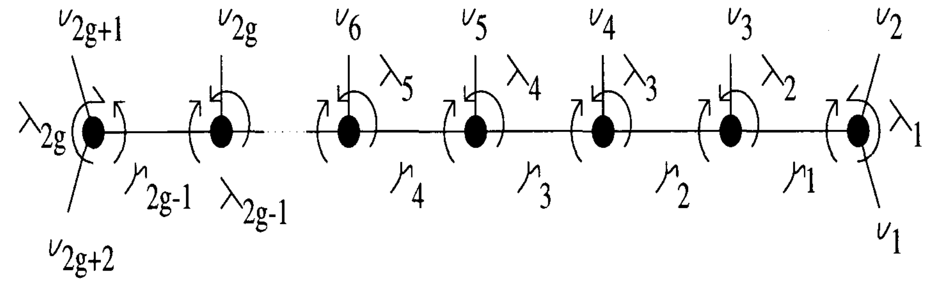
What we wish to be concerned with here is a standard tangential base point $\vec{b}$ on $\mathbb{A}_v^{2g+1} \setminus \Delta$ having the following two properties (4.1) and (4.2).

(4.1) $\vec{b}$ induces a sectional homomorphism $s_{\vec{b}} : G_\mathbb{Q} \to \pi_1(\mathbb{A}_u^{2g+1} \setminus D)$ such that the conjugate action by $s_{\vec{b}}(\sigma)$ $(\sigma \in G_\mathbb{Q})$ on the standard generators $\sigma_1, \dots, \sigma_{2g} \in \hat{B}_{2g+1}$ is given by

$$s_{\vec{b}}(\sigma)\, \sigma_i\, s_{\vec{b}}(\sigma)^{-1} = \mathfrak{f}_\sigma(y_i, \sigma_i^2)^{-1} \sigma_i^{\chi(\sigma)} \mathfrak{f}_\sigma(y_i, \sigma_i^2) \qquad (1 \le i \le 2g),$$

where $y_1 = 1$, $y_i = \sigma_{i-1} \cdots \sigma_1 \cdot \sigma_1 \cdots \sigma_{i-1}$ $(2 \le i \le 2g)$.

(4.2) In $M_{0,2g+2}$, the image of $\vec{b}$ coincides with the tangential base point coming from a 1-parameter family X_b over $\mathbb{Q}[[q]]$ of deformation of X^0 constructed explicitly, as in [IN] §2, from a system of distinguished coordinates $\{r_{\mu/\lambda}\}_{\mu/\lambda}$ such that the values at the points $Q^0_{\nu_2}, \dots, Q^0_{\nu_{2g+1}}$ are always 1 and with $\{r_{\nu/\lambda} := t_{\nu/\lambda}\}_{\nu/\lambda}$. (In [IN], we called $\{r_{\mu/\lambda}\}_{\mu/\lambda}$ a tangential structure on the '$\mathbb{P}^1_{01\infty}$-diagram' X^0.)



This kind of (tangential) base point was suggested by Drinfeld [Dr] after interpreting Grothendieck [G3], whose Galois property (4.1) was established by Ihara-Matsumoto [IM] in detail. For $\vec{b}$ satisfying both (4.1) and (4.2), one may employ the image, via a natural open immersion $M_{0,2g+4} \hookrightarrow \mathbb{A}^{2g+1} \setminus \Delta$, of the tangential base point in $M_{0,2g+4}$ constructed from the similar tangential-structured $(2g+4)$-pointed $\mathbb{P}^1_{01\infty}$-tree as in Ihara-Nakamura [IN]. Here, however, we shall look at a way to attach the property (4.2) to the tangential base point of Ihara-Matsumoto [IM], by introducing a canonical coordinate system of [IN] §2 on the formal neighborhood of the locus of X^0 in the moduli stack $\mathfrak{M}_{0,2g+2}$ of the stable $(2g+2)$-pointed $\mathbb{P}^1$-trees. Namely, gluing the following $\mathbb{Q}[[q_1, \dots, q_{2g-1}]]$-algebras

$$\text{(b1)} \qquad B_{\mu_i} = \mathbb{Q}[r, r', \frac{1}{1-r}, \frac{1}{1-r'}][[q_1, \dots, q_{2g-1}]]/(rr' - q_i),$$
$$r = r_{\mu_i/\lambda_i},\ r' = r_{\mu_i/\lambda_{i+1}}\ (1 \le i \le 2g-1),$$

$$\text{(b2)} \qquad B_{\nu_i} = \mathbb{Q}[r, \frac{1}{1-r}][[q_1, \dots, q_{2g-1}]],$$
$$r = r_{\nu_i/\lambda}\ (1 \le i \le 2g+2),$$

$$\text{(b3)} \qquad B_{\lambda_i} = \mathbb{Q}[r, \frac{1}{r}, \frac{1}{1-r}][[q_1, \dots, q_{2g-1}]],$$
$$r = r_{\mu_j/\lambda_i}\ (1 \le j \le i \le 2g)$$

along X^0, and applying Grothendieck's formal geometry, we obtain a sequence $X^0 = \mathfrak{X}^1 \subset \mathfrak{X}^2 \subset \cdots$ over the sequence of artinian schemes $\{\operatorname{Spec} \mathbb{Q}[[q_1, \dots, q_{2g-1}]]/\mathfrak{q}^N\}_{N \ge 1}$, where $\mathfrak{q} = (q_1, \dots, q_{2g-1})$, and hence a universal deformation $\tilde{X} \to \operatorname{Spec} \mathbb{Q}[[q_1, \dots, q_{2g-1}]]$ of $X^0/\mathbb{Q}$. The representing morphism for this $\tilde{X}$ gives a local coordinate system of the locus of X^0 in $\mathfrak{M}_{0,2g+2}$. Our 1-parameter family $X_{\vec{b}}$ over $\mathbb{Q}[[q]]$ (4.2) is the pull back of $\tilde{X}$ by the diagonal specialization $\mathbb{Q}[[q_1, \dots, q_{2g-1}]] \to \mathbb{Q}[[q]]$ $(\forall q_i \mapsto q)$. Meanwhile, by simple calculations, we see that the local universal family $\tilde{X}$ generically parameterizes $(2g+2)$-pointed projective lines $(\mathbb{P}^1; Q_1, \dots, Q_{2g+2})$ with $Q_1 = 0$, $Q_{2g+1} = 1$, $Q_{2g+2} = \infty$ and $Q_i = q_{i-1} \cdots q_{2g-1}$ $(i = 2, \dots, 2g)$. From this and the relations $Q_i = (v_i - v_1)/(v_{2g+1} - v_1)$, we can conclude that our q_i coincides with Ihara-Matsumoto's "t_i" (cf. [IM] p.179).

Let us compare the images of $\vec{v}$ and $\vec{b}$ on $M_{0,2g+2}$. Recall that $\vec{v}$ corresponds to the 1-parameter family $X_v/\mathbb{Q}[[q]]$ obtained from the sequence $\{\mathfrak{X}_v^N\}_{N\geq 1}$, where $\mathfrak{X}_v^N$ is the quotient of $\mathfrak{Y}^N$ by the hyperelliptic involution. Since this is also a deformation of X^0, there is a specialization homomorphism representing $X_v/\mathbb{Q}[[q]]$ in the form of

$$\begin{aligned} \mathbb{Q}[[q_1,\dots,q_{2g-1}]] &\longrightarrow \mathbb{Q}[[q]] \\ q_i &\longmapsto f_i(q) \qquad (1 \leq i \leq 2g-1), \end{aligned}$$

where $f_i(q)$ is a power series with $f_i(0)=0$.

Lemma (4.3) *(i)* $f_i(q) = q^2 + \{\textit{higher terms}\}$ *(i =even).*
(ii) $f_i(q) = 16q + \{\textit{higher terms}\}$ *(i =odd).*

Proof. Let $\varpi : Y^0 \to X^0$ be the double covering morphism constructed in §3, and let $U_\mu^0 \subset X^0$ denote the affine open Spec $(B_\mu/\mathfrak{q})$. Then, $\mathfrak{X}_v^N|U_\mu^0$ is the spectrum of the ring of invariant functions on $\varpi^{-1}(U_\mu^0)$ mod q^N under the hyperelliptic involution.

(i) When i is even, this ring is $\mathbb{Q}[t,t',\frac{1}{1-t},\frac{1}{1-t'}][[q]]/(tt'-q^2,q^N)$ by (3.3), where $t = t_{\mu_i/\lambda_i}$, $t' = t_{\mu_i/\lambda_{i+1}}$. By assumption, this ring has to be isomorphic to $\mathbb{Q}[r,r',\frac{1}{1-r},\frac{1}{1-r'}][[q]]/(rr'-f_i(q),q^N)$ ($r = r_{\mu_i/\lambda_i}$, $r' = r_{\mu_i/\lambda_{i+1}}$) via some variable transformations of the form $r \equiv \frac{t}{t-1}$, $r' \equiv \frac{t'}{t'-1}$ mod q. Observing this isomorphism localized at (t,t'), we get $f_i(q) = q^2 + O(q^3)$.

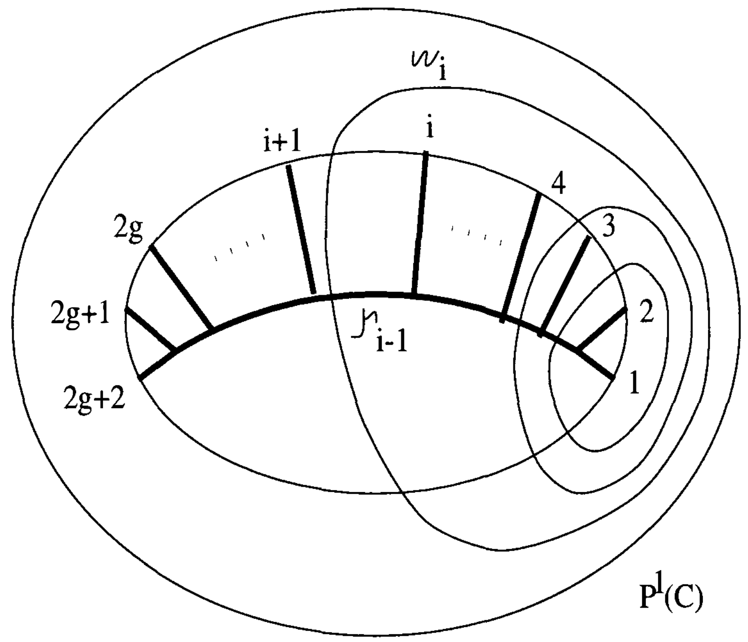
(ii) When i is odd, we may employ a more a posteriori argument. On U_μ^0, the sequence $\{\mathfrak{X}_v^N|U_\mu^0\}_{N\geq 1}$ coincides with that induced from the Tate elliptic curve of level 2 ([DR]) modulo $\{\pm 1\}$. In this case, the Legendre function $\lambda(q) = 16q + \cdots$ ($q = e^{\pi\sqrt{-1}\tau}$) uniformizing $\mathbb{P}^1 - \{0,1,\infty\}$ measures the difference between $\{\mathfrak{X}_v^N|U_\mu^0\}_{N\geq 1}$ and $\{\mathfrak{X}_b^N|U_\mu^0\}_{N\geq 1}$ (cf. [N3] §4). Since different values of $f_i'(0)$ give different deformation rings of $\mathbb{Q}[[t,t']]/(tt')$ over $\mathbb{Q}[q]/q^N$ ($N \geq 2$) near P_μ^0, we conclude that 16 is the exact value. ♢

§5. End of the proof

The fundamental group of the local neighborhood Spec $\mathbb{Q}[[\mathbf{q}]]$ (where $\mathbf{q} = (q_1,\dots,q_{2g-1})$) within $M_{0,2g+2}$ can be identified with $\mathrm{Aut}(\overline{\mathbb{Q}}\{\{\mathbf{q}\}\}/\mathbb{Q}[[\mathbf{q}]])$, where $\overline{\mathbb{Q}}\{\{\mathbf{q}\}\}$ is the union of the rings $k[[q_1^{1/n},\dots,q_{2g-1}^{1/n}]]$ ($n \geq 1$, $[k:\mathbb{Q}] < \infty$). It has an abelian normal subgroup $\hat{\mathbb{Z}}(1)^{2g-1}$ with independent generators $w_2,\dots,w_{2g}$, where $w_{i+1} : q_i^{1/n} \mapsto q_i^{1/n}\zeta_n^{-1}$ ($\zeta_n = e^{2\pi\sqrt{-1}/n}$), $q_j^{1/n} \mapsto q_j^{1/n}$ ($j \neq i$), and fits into the following exact sequence:

$$1 \to \hat{\mathbb{Z}}(1)^{2g-1} \to \mathrm{Aut}(\overline{\mathbb{Q}}\{\{\mathbf{q}\}\}/\mathbb{Q}[[\mathbf{q}]]) \to G_{\mathbb{Q}} \to 1. \tag{5.1}$$

The image of w_i via the natural map $\mathrm{Aut}(\overline{\mathbb{Q}}\{\{\mathbf{q}\}\}/\mathbb{Q}[[\mathbf{q}]]) \to \pi_1(M_{0,2g+2},\vec{b})$ corresponds to the monodromy around the singular divisor '$q_{i-1}=0$' ($2 \leq i \leq 2g$). This is the Dehn twist along a simple closed curve ω_i on the $(2g+2)$-pointed sphere pinching $P^0_{\mu_{i-1}} \in X^0$ (indicated below), and comes from $w'_i = y_2 y_3 \cdots y_i = (\sigma_1 \cdots \sigma_{i-1})^i \in \hat{B}_{2g+1}$.



Our two tangential base points $\vec{b}$ and $\vec{v}$ give different splitting sections s_b, s_v of (5.1) respectively. The $s_b(\sigma)$ ($\sigma \in G_{\mathbb{Q}}$) transforms each Puiseux power series $\sum_{\alpha \in \mathbb{Q}^{2g-1}} a_\alpha \mathbf{q}^\alpha$ to $\sum_{\alpha\in\mathbb{Q}^{2g-1}} \sigma(a_\alpha)\mathbf{q}^\alpha$. On the other hand, we can perceive the action by $s_v(\sigma)$ to be the coefficientwise Galois action on the Puiseux power series after specialized via $q_i^{1/n} \to f_i(q)^{1/n}$ ($n \geq 1$): The specialization process via (4.3) becomes 'power-compatible' after setting $1^{1/n} = \zeta_n$, $2^{1/n} \in \mathbb{R}_{>0}$ (this corresponds to a choice of 'natural' chemin connecting $\vec{v}$ and $\vec{b}$.) Then, for each $\alpha = (\alpha_i) \in \mathbb{Q}^{2g-1}$, $f(\mathbf{q}^\alpha) = \prod_i f_i(q)^{\alpha_i}$ makes sense in $\overline{\mathbb{Q}}\{\{q\}\}$, and $s_v(\sigma)$ transforms it into $e^{2\pi\sqrt{-1}(4\rho_2(\sigma)\sum_{i:\mathrm{odd}}\alpha_i)} f(\mathbf{q}^\alpha)$, where $\rho_2 : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}(1)$ is the Kummer 1-cocycle with $2^{1/n(\sigma-1)} = \zeta_n^{\rho_2(\sigma)}$. Comparing these two operations on Puiseux series, we obtain:

$$s_b(\sigma) = \prod_{\substack{j=2\\ \text{even}}}^{2g} w_j^{4\rho_2(\sigma)} s_v(\sigma).$$

Here, it is noteworthy that, although the i-th component of the "tangent vector" $\vec{v}$ vanishes via f_i for i even, its non-trivial principal term ('q^2', in this case) still works well in carrying Galois properties from the tangential base point $\vec{v}$. The author is indebted to Prof. Deligne for this crucial remark.

Then, let us be back to the diagram (2.1), and let s'_b, s'_v be the splitting homomorphisms of the surjection $\pi_1(\mathcal{H}_{g,1}) \to G_{\mathbb{Q}}$ coming down from the tangential base points $\vec{b}$, $\vec{v}$ on $\mathbb{A}_v^{2g+1} \setminus \Delta$. Considering the above relation in $\pi_1(M_{0,2g+2}/S_{2g+1})$ and lifting it back to $\pi_1(\mathbb{A}_u^{2g+1} \setminus D)$ (2.1), we see

$$s'_b(\sigma) = \prod_{\substack{j=2 \\ \text{even}}}^{2g} (w'_j)^{4\rho_2(\sigma)} \cdot w_{2g+1}^{c_\sigma} \cdot s'_v(\sigma)$$

for some 1-cocycle $c : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}(1)$ $(\sigma \mapsto c_\sigma)$. Let s'_b, s'_v also denote the induced sectional homomorphisms $G_{\mathbb{Q}} \to \pi_1(M_{g,1})$ from (2.1) and $\mathcal{H}_{g,1} \hookrightarrow M_{g,1}$. Then the conjugate actions by $s'_b(\sigma)$ $(\sigma \in G_{\mathbb{Q}})$ on $a_1, \cdots, a_{2g}$ are described just as direct images of (4.1):

$$s'_b(\sigma)\, a_i\, s'_b(\sigma)^{-1} = \mathfrak{f}_\sigma(y_i, a_i^2)^{-1} a_i^{\chi(\sigma)} \mathfrak{f}_\sigma(y_i, a_i^2) \qquad (1 \leq i \leq 2g).$$

On the other hand, the conjugate actions by $s'_v(\sigma)$ $(\sigma \in G_{\mathbb{Q}})$ on $d_{\pm *}, e_*$'s are a priori via the cyclotomic character. The reason is that our $Y^0/\mathbb{Q}$ lies over a representative point of a maximally degenerate locus in the moduli stack $\mathfrak{M}_{g,1}$ of the 1-pointed stable curves of genus g, whose local neighborhood within $M_{g,1}$ has the geometric fundamental group $\hat{\mathbb{Z}}(1)^{3g-2}$ with the commutative $3g-2$ generators $d_{\pm *}, e_*$'s.

We define then the sectional homomorphism $s : G_{\mathbb{Q}} \to \pi_1(M_{g,1})$ of Theorem A by

$$s(\sigma) := w_{2g+1}^{-c_\sigma} s'_b(\sigma) \qquad (\sigma \in G_{\mathbb{Q}}).$$

Since w_{2g+1}(="hyperelliptic involution") commutes with $a_1, \ldots, a_{2g}$, the conjugate action by $s(\sigma)$ on the a_*'s are in the same way as that by $s'_b(\sigma)$, hence in the desired way. As for $d_{\pm *}, e_*$'s, notice that $\{w'_{\text{even}}\}$ and $\{d_{\pm *}, e_*\}$ commute elementwise with each other. Then, we see that $s(\sigma)$ operates on the $d_{\pm *}, e_*$'s in the same way as $s'_v(\sigma)$ by conjugation, i.e., via the cyclotomic character. Thus, Theorems A and A' are both settled.

§6. Complementary notes

This section describes complementary remarks to the results of this note, whose details will be included in a forthcoming paper [N3]. Let $\mathfrak{M}_{g,n}$ denote the stack$_{/\mathbb{Q}}$ of the ordered n-pointed stable curves of genus g, and $M_{g,n} \subset \mathfrak{M}_{g,n}$ its nonsingular locus (Deligne-Mumford [DM], Knudsen [K]). By using Grothendieck-Murre's theory [GM], one can observe behaviors of the fundamental group of the tubular neighborhood in $M_{g,n}$ of the divisor of the form $\mathfrak{M}_{g_1,n_1} \times \mathfrak{M}_{g_2,n_2}$ $(g = g_1 + g_2, n = n_1 + n_2 - 2)$ inside $\pi_1(M_{g,n})$

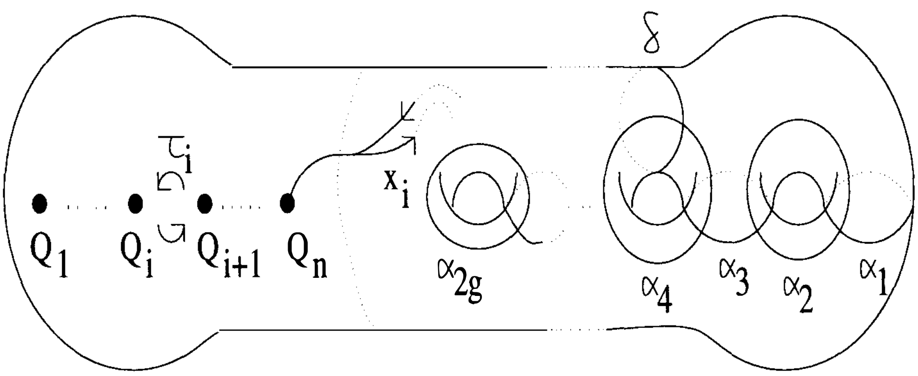
(cf. [N2]). Roughly speaking, the "coupling device" considered in [N2] enables one to relate Galois-Teichmüller modular groups of different genera by "sewing up" two topological types of Riemann surfaces along boundaries.

By looking at the arguments of previous sections along the coupling of $\mathfrak{M}_{g-1,1} \times \mathfrak{M}_{1,2} \subset \mathfrak{M}_{g,1}$, we can see that the indeterminate parameter c_σ in Sect.5 is negligible. Thus, the $G_{\mathbb{Q}}$-action at the base point $\vec{b}$ is essentially the desired one. Meanwhile, the $G_{\mathbb{Q}}$-action at $\vec{v}$ differs from it by the factors $(w'_j)^{4\rho_2(\sigma)}$ ($j \geq 2$, even). Since $\rho_2(\sigma)$ is recovered from the ratio of the upper components of $\mathfrak{f}_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}), (\begin{smallmatrix}1&0\\-2&1\end{smallmatrix})) \in \mathrm{SL}_2(\hat{\mathbb{Z}})$ ([N3] §4), we may say that both $G_{\mathbb{Q}}$-actions on $\hat{\Gamma}_g^1$ can be written in terms of parameters $(\chi(\sigma), \mathfrak{f}_\sigma) \in \widehat{GT}$.

The natural forgetful map $M_{g,n} \to M_{g,0}$ obtained by forgetting the marking points induces the exact sequence

$$1 \to \hat{\Pi}_{g,0}^{(n)} \to \hat{\Gamma}_g^n \to \hat{\Gamma}_g^0 \to 1,$$

where $\hat{\Gamma}_g^n$ (resp. $\hat{\Pi}_{g,0}^{(n)}$) denotes the profinite completion of the mapping class group of an n-pointed genus g surface (resp. of the pure braid group with n-strings on a genus g surface). Note that our $G_{\mathbb{Q}}$-actions on $\hat{\Gamma}_g^1$ induce those on $\hat{\Gamma}_g^0$ by the above forgetful mapping with $n = 1$. Matsumoto [M] studied the Galois action on the profinite braid group for a fixed affine smooth curve, and decomposed it into the Galois actions on $\hat{B}_n$ and the π_1 of the curve. One can also consider his insight in our coupling context as follows. Let us introduce $M_{g,[n]} := M_{g,n}/S_n$, the moduli stack over $\mathbb{Q}$ obtained by letting the marking points unordered. Then the kernel $\hat{\Gamma}_g^{[n]}$ of $\pi_1(M_{g,[n]}) \to G_{\mathbb{Q}}$ includes $\hat{\Gamma}_g^n$ as an open subgroup, and is isomorphic to the profinite completion of the mapping class group of a closed surface of genus g preserving n points $Q_1, \ldots, Q_n$ as a set.



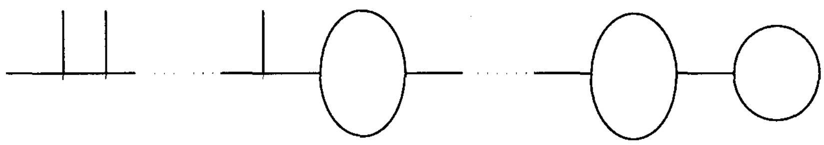
The group $\hat{\Gamma}_g^{[n]}$ has the following three types of generators: (1) $a_1, \ldots, a_{2g}$, d (Dehn twists); (2) $\tau_1, \ldots, \tau_{n-1}$ (braids); (3) $x_1, \ldots, x_{2g}$ (peripheral paths of Q_n around the handles).

By modifying the constructions of this note, one can get a tangential base point attached to the locus of the maximally degenerate marked stable curve whose dual graph (with legs) looks like the following picture, at which $\sigma \in G_{\mathbb{Q}}$ acts on $\hat{\Gamma}_g^{[n]}$ by

(1) $\sigma(a_i) = \mathfrak{f}_\sigma(y_i, a_i^2)^{-1} a_i^{\chi(\sigma)} \mathfrak{f}_\sigma(y_i, a_i^2)$, $\sigma(d) = d^{\chi(\sigma)}$ $(1 \le i \le 2g)$;

(2) $\sigma(\tau_j) = \mathfrak{f}_\sigma(\eta_j, \tau_j^2)^{-1} \tau_j^{\chi(\sigma)} \mathfrak{f}_\sigma(\eta_j, \tau_j^2)$ $(1 \le j \le n-1)$, where $\eta_1 = 1$, $\eta_j = \tau_{j-1} \cdots \tau_1 \cdot \tau_1 \cdots \tau_{j-1}$ $(j \ge 2)$;

(3) $\sigma(x_i)$ $(1 \le i \le 2g)$ are described explicitly in terms of $\widehat{GT}$.



The third part action is, in effect, the main theme of [N3], where, based on [IN], established is a concrete procedure of computing the limit behaviors of exterior Galois representations when (marked) algebraic curves maximally degenerate to various types of marked stable curves consisting of 3-pointed projective lines. This procedure is, as shown in the author's talk at the Luminy conference, described in terms of a graph of profinite groups over the dual graph of the special fibre whose edge/vertex groups are products of free profinite groups of rank 1 or 2 with standard Galois actions.

Acknowledgments. The author would like to thank Prof. P.Lochak, L.Schneps for sharing enjoyable tests of some known relations of twist generators of mapping class groups by direct calculations after Th. A,A'. He is also very grateful to Prof. P.Deligne, V.G.Drinfeld for helpful suggestions on ideas of tangential base points and moduli theory (cf. Sect.4,5), and to Prof. D.Harbater, F.Pop for instructive discussions on patching techniques from different points of view. The result of the present note wouldn't have occurred to the author without his joint work with Prof. Y.Ihara [IN]. The author would like to express his sincere thanks to Prof. Y.Ihara also for several pieces of advice for improving the presentation of this note. He also should like to thank Prof. S.Morita for several helpful comments on literature and topological figures. In addition to them, his hearty thanks should also go to many Japanese colleagues on Galois-π_1-theory and related topologists together with foreign inverse-Galois researchers (cf. [F]), for their mutually related works helped his imagination into the formula of this note.

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