Landen's trilogarithm functional equation and ℓ-adic Galois multiple polylogarithms

In memory of Toshie Takata

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Abstract. The Galois action on the pro-ℓ étale fundamental groupoid of the projective line minus three points with rational base points gives rise to a non-commutative formal power series in two variables with ℓ-adic coefficients, called the ℓ-adic Galois associator. In the present paper, we focus on how Landen's functional equation of trilogarithms and its ℓ-adic Galois analog can be derived algebraically from the S_3-symmetry of the projective line minus three points. Twofold proofs of the functional equation will be presented, one is based on the chain rule for the associator power series and the other is based on Zagier's tensor criterion devised in the framework of graded Lie algebras. In the course of the second proof, we are led to investigate ℓ-adic Galois multiple polylogarithms appearing as regular coefficients of the ℓ-adic Galois associator. As an application, we show an ℓ-adic Galois analog of Oi-Ueno's functional equation between $Li_{1,\ldots,1,2}(1-z)$ and $Li_k(z)$'s $(k = 1, 2, \ldots)$.

Contents

1. Introduction

The study of polylogarithms, especially their functional equations, originated in the late 18th century by Euler, Landen, and others. The classical polylogarithm they studied is a complex function defined by the following power series

$$Li_k(z) := \frac{z}{1^k} + \frac{z^2}{2^k} + \frac{z^3}{3^k} + \cdots \ (|z| < 1).$$

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For $k=2$, it is called the dilogarithm, and for $k=3$, it is called the trilogarithm. The multiple polylogarithm $Li_{\mathbf{k}}(z)$ for a multi-index $\mathbf{k}=(k_1\dots,k_d)\in\mathbb{N}^d$ generalizes $Li_k(z)$, which is defined by the power series

$$Li_{\mathbf{k}}(z):=\sum_{0<n_1<\cdots<n_d}\frac{z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}}\quad(|z|<1).$$

Note that $Li_k(z)=Li_{(k)}(z)$. The collection $\{Li_{\mathbf{k}}(z)\}_{\mathbf{k}}$ satisfies certain recursive differential equations, from which follows that each $Li_{\mathbf{k}}(z)$ has an iterated integral expression that can be analytically continued to the universal cover of the three punctured Riemann sphere $\mathbf{P}^1(\mathbb{C})-\{0,1,\infty\}$ (cf. e.g. [F04, Lemma 1.5]). There are known a number of functional equations between these functions evaluated at points with suitably chosen tracking paths from the unit segment $(0,1)$ on $\mathbf{P}^1(\mathbb{C})-\{0,1,\infty\}$. For example, the following formulas are typical:

$$Li_2(z)+Li_2(1-z)=\zeta(2)-\log(z)\log(1-z),\tag{1}$$

$$Li_2(z)+Li_2\left(\frac{z}{z-1}\right)=-\frac{1}{2}\log^2(1-z),\tag{2}$$

$$\begin{aligned}&Li_3(z)+Li_3(1-z)+Li_3\left(\frac{z}{z-1}\right)\\&\quad=\zeta(3)+\zeta(2)\log(1-z)-\frac{1}{2}\log(x)\log^2(1-z)+\frac{1}{6}\log^3(1-z).\end{aligned}\tag{3}$$

The former (1) is due to Leonhard Euler [E1768] and the latter two (2)-(3) are due to John Landen [L1780]. See Lewin's book [L81] for many other functional equations for polylogarithms. As for multiple polylogarithms, in [Oi09]-[OU13], Shu Oi and Kimio Ueno showed the following functional equation:

$$\sum_{j=0}^{k-1}Li_{k-j}(z)\frac{(-\log z)^j}{j!}+Li_{\underbrace{1,\dots,1}_{k-2\text{ times}},2}(1-z)=\zeta(k)\qquad(k\geq 2).\tag{4}$$

Let ℓ be a fixed prime. The ℓ-adic Galois multiple polylogarithm

$$Li^{\ell}_{\mathbf{k}}(z)\left(=Li^{\ell}_{\mathbf{k}}(\gamma_z:\overrightarrow{01}\rightsquigarrow z)\right):G_K\to\mathbb{Q}_\ell$$

is a function on the absolute Galois group $G_K:=\mathrm{Gal}(\overline{K}/K)$ of a subfield K of $\mathbb{C}$ defined, for $\mathbf{k}=(k_1\dots,k_d)\in\mathbb{N}^d$ and an ℓ-adic étale path γ_z from $\overrightarrow{01}$ to a K-rational (tangential) point z on $\mathbb{P}^1-\{0,1,\infty\}$, as a certain (signed) coefficient of the non-commutative formal power series

$$\mathfrak{f}^{\gamma_z}_\sigma(X,Y)\in\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle\quad(\sigma\in G_K)\tag{5}$$

called the ℓ-adic Galois associator. The functions $Li^{\ell}_{\mathbf{k}}(z)$ were originally introduced and called *the ℓ-adic iterated integrals* in a series of papers by Zdzisław Wojtkowiak (cf. e.g., [W0]-[W3]). In particular,

$$\boldsymbol{\zeta}^{\ell}_{\mathbf{k}}(\sigma):=Li^{\ell}_{\mathbf{k}}(\delta:\overrightarrow{01}\rightsquigarrow\overrightarrow{10})(\sigma)\tag{6}$$

for the standard path δ along the unit interval $(0,1)\subset\mathbb{R}$. For $z\in K$ with a path $\gamma_z:\overrightarrow{01}\rightsquigarrow z$, we also write

$$\rho_z(=\rho_{\gamma_z}):G_K\to\mathbb{Z}_\ell$$

for the Kummer 1-cocycle of the ℓ-th power roots $\{z^{1/\ell^n}\}_n$ determined by γ_z.

In [NW12], Wojtkowiak and the first named author of the present paper devised Zagier's tensor criterion for functional equations as a means to calculate exact forms of identities with lower degree terms for both complex and ℓ-adic Galois polylogarithms. Applying the method, we established a few examples of functional equations in both polylogarithms. In particular, the above (1) and (2) were shown to have the following ℓ-adic Galois counterparts:

$$Li_2^\ell(z)(\sigma) + Li_2^\ell(1-z)(\sigma) = \boldsymbol{\zeta}_2^\ell(\sigma) - \rho_z(\sigma)\rho_{1-z}(\sigma), \tag{7}$$

$$Li_2^\ell(z)(\sigma) + Li_2^\ell\left(\frac{z}{z-1}\right)(\sigma) = -\frac{\rho_{1-z}(\sigma)^2 + \rho_{1-z}(\sigma)}{2} \tag{8}$$

for $\sigma \in G_K$ (cf. [NW12]. See (30)-(31) and Proposition 4.2 below for adjustments of notations.)

The purpose of this paper is to provide algebraic proofs of (3) and (4) which can be used to obtain their ℓ-adic Galois analogs reading as follows:

Theorem 1.1 (ℓ-adic Galois analog of the Landen trilogarithm functional equation)**.** There are suitable paths $\overrightarrow{01} \rightsquigarrow 1-z, \overrightarrow{01} \rightsquigarrow \frac{z}{z-1}$ associated to a given path $\gamma_z : \overrightarrow{01} \rightsquigarrow z$ such that the following functional equation

$$\begin{aligned}
&Li_3^\ell(z)(\sigma) + Li_3^\ell(1-z)(\sigma) + Li_3^\ell\left(\frac{z}{z-1}\right)(\sigma) \\
&= \boldsymbol{\zeta}_3^\ell(\sigma) - \boldsymbol{\zeta}_2^\ell(\sigma)\rho_{1-z}(\sigma) + \frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)^2 - \frac{1}{6}\rho_{1-z}(\sigma)^3 \\
&\quad - \frac{1}{2}Li_2^\ell(z)(\sigma) - \frac{1}{12}\rho_{1-z}(\sigma) - \frac{1}{4}\rho_{1-z}(\sigma)^2
\end{aligned}$$

holds for $\sigma \in G_K$.

Theorem 1.2 (ℓ-adic Galois analog of the Oi-Ueno functional equation)**.**

$$\sum_{j=0}^{k-1} Li_{k-j}^\ell(z)(\sigma)\frac{\rho_z(\sigma)^j}{j!} + Li^\ell_{\underbrace{1,\dots,1}_{k-2 \text{ times}},2}(1-z)(\sigma) = \boldsymbol{\zeta}_k^\ell(\sigma) \quad (\sigma \in G_K).$$

Remark 1.3. In [S21], the second named author showed that the functional equation (7) has an application to a reciprocity law of the triple mod-$\{2,3\}$ symbols of rational primes via Ihara-Morishita theory (cf. [HM19]). Theorem 1.2 was shortly announced in a talk by the first named author at online Oberwolfach meeting ([N21]). After the present paper was worked out, the second named author obtained a generalization of Theorem 1.2 to higher multi-indices ([S23a]), and showed an ℓ-adic version of Spence-Kummer's trilogarithm functional equation from which various formulas including Theorem 1.1 can be derived by specializations (cf. [S23b, Remark 4.3.]).

The contents of this paper will be arranged as follows: After a quick setup in §2 on the notations of standard paths on $\mathbf{P}^1 - \{0, 1, \infty\}$, in §3 we discuss complex and ℓ-adic Galois associators as formal power series in two non-commuting variables, and define the multiple polylogarithms as their coefficients of certain monomials. We then review in the complex analytic context that (3) and (4) can be derived from algebraic relations (chain rules) of associators along simple compositions of paths. With this line in mind, we prove Theorems 1.1 and 1.2 in the ℓ-adic Galois case by tracing arguments in parallel ways to the complex case. In §4, after shortly recalling polylogarithmic characters introduced in a series of collaborations by Wojtkowiak and the first named author, we present $\mathbb{Z}_\ell$-integrality test for ℓ-adic Galois Landen's equation obtained in Theorem 1.1. Section 5 turns to an alternative approach to functional equations of polylogarithms based on a set of tools devised in [NW12] to enhance Zagier's tensor criterion for functional equations into a concrete form. Then we give alternative proofs of (3) and Theorem 1.1 with this method. In Appendix A, we exhibit lower degree terms of the complex and ℓ-adic Galois associators as a convenient reference from the text. Appendix B summarizes a set of computational tools from [NW12] that converts a tensor criterion of functions into a polylogarithmic identity.

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2. **Set up**

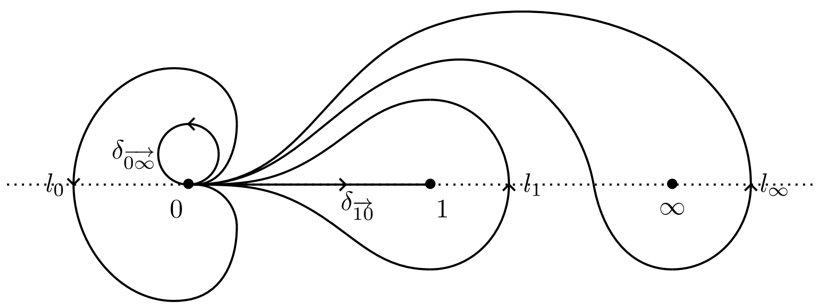
Fix a prime number ℓ. Let K be a subfield of the complex number field $\mathbb{C}$, $\overline{K}$ the algebraic closure of K in $\mathbb{C}$, and $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. Let $U := \mathbf{P}^1_K - \{0, 1, \infty\}$ be the projective line minus three points over K, $U_{\overline{K}}$ the base-change of U via the inclusion $K \hookrightarrow \overline{K}$, and $U^{\mathrm{an}} = \mathbf{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ the complex analytic space associated to the base-change of $U_{\overline{K}}$ via the inclusion $\overline{K} \hookrightarrow \mathbb{C}$.

In the following, we shall write $\overrightarrow{01}$ for the standard K-rational tangential base point on U. Let z be a K-rational point of U or a K-rational tangential base point on U. We consider $\overrightarrow{01}$, z also as points on $U_{\overline{K}}$ or U^{an} by inclusions $K \hookrightarrow \overline{K}$ and $\overline{K} \hookrightarrow \mathbb{C}$. (*Note:* We admit the particular case $K = \mathbb{C}$ where $G_K = \{1\}$. It is also possible to start with a specific complex point $z \in \mathbb{C} - \{0, 1\}$ so that any field K with $\mathbb{Q}(z) \subset K \subset \mathbb{C}$ fits into our setup.)

Let $\pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, z)$ be the set of homotopy classes of topological paths on U^{an} from $\overrightarrow{01}$ to z, and let $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}; \overrightarrow{01}, z)$ be the pro-ℓ-finite set of pro-ℓ étale paths on $U_{\overline{K}}$ from $\overrightarrow{01}$ to z. Note that there is a canonical comparison map

$$\pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, z) \to \pi_1^{\ell\text{-ét}}(U_{\overline{K}}; \overrightarrow{01}, z)$$

that allows us to consider topological paths on U^{an} as pro-ℓ étale paths on $U_{\overline{K}}$.



The dashed line represents $\mathbf{P}^1(\mathbb{R}) - \{0, 1, \infty\}$.
The upper half-plane is above the dashed line.

Let l_0, l_1, l_∞ be the topological paths on U^{an} with base point $\overrightarrow{01}$ circling counterclockwise around $0, 1, \infty$, respectively. Then, $\{l_0, l_1\}$ is a free generating system of the topological fundamental group $\pi_1^{\mathrm{top}}(U^{\mathrm{an}}, \overrightarrow{01}) := \pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, \overrightarrow{01})$ or the pro-ℓ étale fundamental group $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01}) := \pi_1^{\ell\text{-ét}}(U_{\overline{K}}; \overrightarrow{01}, \overrightarrow{01})$. Then, $\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, \overrightarrow{01})$ is a free group of rank 2 generated by $\{l_0, l_1\}$ and $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$ is a free pro-ℓ group of rank 2 topologically generated by $\{l_0, l_1\}$.

Fix a topological path $\gamma_z \in \pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, z)$ on U^{an} from $\overrightarrow{01}$ to z. Moreover, let $\delta_{\overrightarrow{10}} \in \pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, \overrightarrow{10})$ be the topological path on U^{an} from $\overrightarrow{01}$ to $\overrightarrow{10}$ along the real interval, and let $\delta_{\overrightarrow{0\infty}} \in \pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, \overrightarrow{0\infty})$ be the topological path on the upper half-plane in U^{an} from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$.

Let $\phi, \psi \in \mathrm{Aut}(U^{\mathrm{an}})$ be automorphisms of U^{an} defined by

$$\phi(t) = 1 - t, \quad \psi(t) = \frac{t}{t-1}, \tag{9}$$

and introduce specific paths from $\overrightarrow{01}$ to $1-z$ and to $\frac{z}{z-1}$ by

$$\begin{cases} \gamma_{1-z} & := \delta_{\overrightarrow{10}} \cdot \phi(\gamma_z) \in \pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, 1-z), \\ \gamma_{\frac{z}{z-1}} & := \delta_{\overrightarrow{0\infty}} \cdot \psi(\gamma_z) \in \pi_1^{\mathrm{top}}\left(U^{\mathrm{an}}; \overrightarrow{01}, \dfrac{z}{z-1}\right). \end{cases} \tag{10}$$

Here, paths are composed from left to right.

For any field F, we shall write $F\langle\langle X, Y\rangle\rangle$ for the ring of non-commutative power series in the non-commuting indeterminates X, Y with coefficients in F. Every element $f(X,Y) \in F\langle\langle X, Y\rangle\rangle$ can be expanded as a formal sum over the free monoid M generated by X, Y. We call an element $w \in \mathrm{M}$ a word in X, Y, and denote by $\mathtt{Coeff}_w(f(X,Y)) \in F$ the coefficient of a word $w \in \mathrm{M}$ in $f(X,Y) \in F\langle\langle X, Y\rangle\rangle$:

$$f(X,Y) = \sum_{w \in \mathrm{M}} \mathtt{Coeff}_w(f(X,Y)) \cdot w. \tag{11}$$

In particular, for the unit element $w = 1$ of M, $\mathtt{Coeff}_1(f(X,Y))$ denotes the constant term of $f(X,Y)$.

3. **Associators and multiple polylogarithms**

Recall that the multiple polylogarithms appear as coefficients of the non-commutative formal power series in two variables, determined as the basic solution of the KZ equation (Knizhnik-Zamolodchikov equation) on $\mathbf{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ (cf. [Dr90]). More precisely, let $G_0(X,Y)(z)\,(= G_0(X,Y)(\gamma_z))$ be the fundamental solution of the formal KZ equation

$$\frac{d}{dz} G(X,Y)(z) = \left(\frac{X}{z} + \frac{Y}{z-1}\right) G(X,Y)(z)$$

on $\mathbf{P}^1(\mathbb{C}) - \{0, 1, \infty\}$, which is an analytic function with values in $\mathbb{C}\langle\langle X, Y\rangle\rangle$ characterized by the asymptotic behavior $G_0(X,Y)(\gamma_z) \approx \exp(X \log(z))$ $(z \to 0)$ and analytically continued to the universal cover of $\mathbf{P}^1(\mathbb{C}) - \{0, 1, \infty\}$. Here, $\log(z) := \int_{\delta_{\overrightarrow{01}}^{-1} \cdot \gamma_z} \frac{1}{t} dt$.

One can expand $G_0(X,Y)(\gamma_z)$ with the notation (11) as:

$$G_0(X,Y)(\gamma_z) = 1 + \sum_{w \in \mathrm{M}\setminus\{1\}} \mathtt{Coeff}_w(G_0(X,Y)(\gamma_z)) \cdot w. \tag{12}$$

The multiple polylogarithm $Li_{\mathbf{k}}(z)\,(= Li_{\mathbf{k}}(\gamma_z))$ associated to a tuple $\mathbf{k} = (k_1 \ldots, k_d) \in \mathbb{N}^d$ and a topological path γ_z from $\overrightarrow{01}$ to z is equal to the coefficient of $G_0(X,Y)(\gamma_z)$ at the 'regular' word $w(\mathbf{k}) := X^{k_d - 1} Y \cdots X^{k_1 - 1} Y$ multiplied by $(-1)^d$ (where 'regular' means that the word ends in the letter Y). In summary, writing the length d of the tuple $\mathbf{k} = (k_1 \ldots, k_d)$ as $\mathrm{dep}(\mathbf{k})$, we have

$$Li_{\mathbf{k}}(\gamma_z) = (-1)^{\mathrm{dep}(\mathbf{k})} \mathtt{Coeff}_{w(\mathbf{k})}(G_0(X,Y)(\gamma_z)). \tag{13}$$

To define the ℓ-adic Galois multiple polylogarithms, we make use of the G_K-action on the étale paths instead of the fundamental KZ-solution. Given a pro-ℓ étale path $\gamma_z \in \pi_1^{\ell\text{-ét}}(U_{\overline{K}}; \overrightarrow{01}, z)$, form a pro-$\ell$ étale loop $\mathfrak{f}_\sigma^\gamma := \gamma \cdot \sigma(\gamma)^{-1} \in \pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$, and expand it via the Magnus embedding $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01}) \hookrightarrow \mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$ defined by $l_0 \mapsto \exp(X)$, $l_1 \mapsto \exp(Y)$.

Notation 3.1. We shall often identify $\mathfrak{f}_\sigma^{\gamma_z} \in \pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$ with the above image in $\mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$. Then, following notation of (11), let us write

$$\mathfrak{f}_\sigma^{\gamma_z}(X,Y) = 1 + \sum_{w \in \mathrm{M}\setminus\{1\}} \mathtt{Coeff}_w(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) \cdot w \qquad (\sigma \in G_K). \tag{14}$$

This is what we described in (5). In the parallel way to the above (13), for any tuple $\mathbf{k}$ of positive integers, we define the ℓ-adic Galois multiple polylogarithm $Li_{\mathbf{k}}^\ell$ to be the function $G_K \to \mathbb{Q}_\ell$ determined by

$$Li_{\mathbf{k}}^\ell(\gamma_z)(\sigma) = (-1)^{\mathrm{dep}(\mathbf{k})} \mathtt{Coeff}_{w(\mathbf{k})}(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) \qquad (\sigma \in G_K). \tag{15}$$

The ℓ-adic zeta function $\boldsymbol{\zeta}^{\ell}_{\mathbf{k}} : G_K \to \mathbb{Q}_\ell$ is a special case when γ_z is the unit interval path $\delta_{\overrightarrow{10}}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ as mentioned in Introduction (6).

Remark 3.2. It is worth noting that the ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ is the ℓ-adic Galois analog of the Drinfeld associator

$$\Phi(X,Y) := \Big(G_0(Y,X)(\gamma_{1-z})\Big)^{-1} \cdot G_0(X,Y)(\gamma_z) \in \mathbb{C}\langle\langle X,Y\rangle\rangle.$$

See also Appendix A for some basic properties and explicit coefficients of low degree terms of $\Phi(X,Y)$ and $\mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)$.

Lemma 3.3 (Key identities)**.** The notations being as above, the following identities hold.

(1) $G_0(X,Y)(\gamma_z) = G_0(Y,X)(\gamma_{1-z}) \cdot \Phi(X,Y)$.
(2) $G_0(X,Y)(\gamma_{\frac{z}{z-1}}) = G_0(X,Z)(\gamma_z) \cdot \exp(\pi \mathtt{i} X)$, where $Z := -X-Y$.
(3) $\mathfrak{f}_\sigma^{\gamma_z}(X,Y) = \mathfrak{f}_\sigma^{\gamma_{1-z}}(Y,X) \cdot \mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)$.
(4) $\mathfrak{f}_\sigma^{\gamma_{\frac{z}{z-1}}}(X,Y) = \mathfrak{f}_\sigma^{\gamma_z}(X,Z) \cdot \exp(\frac{1-\chi(\sigma)}{2}X)$, where $Z := \log(\exp(-Y)\exp(-X))$.

Proof. The identity (1) was remarked in [F14, A.24]. We shall prove (1) and (2) as consequences of the chain rule of iterated integrals along composition of paths: $\Lambda(\alpha\beta) = \Lambda(\alpha)\Lambda(\beta)$ for $\alpha : x \rightsquigarrow y$ and $\beta : y \rightsquigarrow z$, where $\Lambda(\gamma) = 1 + \sum_{m=1}^\infty \int_\gamma \underbrace{\omega \cdots \omega}_{m} \in \mathbb{C}\langle\langle X,Y\rangle\rangle$ is Chen's transport of the formal connection associated to $\omega = \frac{dt}{t}X + \frac{dt}{t-1}Y \in V_1 \otimes \Omega^1$. (Here Ω^1 is the space of meromorphic 1-forms with log-singularities on $(\mathbf{P}^1, \{0,1,\infty\})$ and V_1 is the dual of Ω^1. For extension to cases of tangential base points, see [De89], [W97]). Below, for simplicity, write $\delta := \delta_{\overrightarrow{10}}$, $\varepsilon := \delta_{\overrightarrow{0\infty}}$. It is easy to see from (10) and $\phi(\delta) = \delta^{-1}$ that $\gamma_z = \delta \cdot \phi(\gamma_{1-z})$. This identity and another $\gamma_{\frac{z}{z-1}} = \varepsilon \cdot \psi(\gamma_z)$ from (10) imply $\Lambda(\gamma_z) = \Lambda(\delta)\Lambda(\phi(\gamma_{1-z})) = \Lambda(\delta)\phi_*^{-1}(\Lambda(\gamma_{1-z}))$ and $\Lambda(\gamma_{\frac{z}{1-z}}) = \Lambda(\varepsilon)\Lambda(\psi(\gamma_z)) = \Lambda(\varepsilon)\psi_*^{-1}(\Lambda(\gamma_z))$ respectively (cf. [NW12, (4.5)]). The assertions (1), (2) follow from them together with a normalization of convention: $G_0(X,Y)(\gamma_z) = \overline{\Lambda(\gamma_z)(X,Y)}$, where $\overline{f(X,Y)}$ denotes the non-commutative formal power series in X,Y obtained from $f(X,Y)$ by reversing the order of letters in each word (i.e., $\overline{x_1 \cdots x_m} = x_m \cdots x_1$ for $x_i \in \{X,Y\}$, $i = 1,\dots,m$, $m \geq 1$). Note that $\overline{\Lambda_1\Lambda_2} = \overline{\Lambda_2} \cdot \overline{\Lambda_1}$.

(3): From (10) again, we have $\gamma_z = \delta \cdot \phi(\gamma_{1-z})$ as above. Noting that the automorphism ϕ is defined over K and that $\delta\phi(l_i)\delta^{-1} = l_{1-i}$ $(i = 0,1)$, we compute

$$\begin{aligned}\mathfrak{f}_\sigma^{\gamma_z} &= \gamma_z \cdot \sigma(\gamma_z)^{-1} = \delta \cdot \phi(\gamma_{1-z}) \cdot \sigma(\phi(\gamma_{1-z})^{-1}) \cdot \sigma(\delta)^{-1} \\ &= \delta \cdot \phi(\mathfrak{f}_\sigma^{\gamma_{1-z}}) \cdot \delta^{-1} \cdot \mathfrak{f}_\sigma^{\overrightarrow{10}} = \mathfrak{f}_\sigma^{\gamma_{1-z}}(Y,X) \cdot \mathfrak{f}_\sigma^{\overrightarrow{10}}(X,Y).\end{aligned}$$

(4): Recall that $l_\infty = \exp(Z)$ represents a loop in $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$ such that $l_0 l_1 l_\infty = 1$. Noting that ψ is defined over K and that $\varepsilon\psi(l_0)\varepsilon^{-1} = l_0$, $\varepsilon\psi(l_1)\varepsilon^{-1} = l_\infty$, we compute from $\gamma_{\frac{z}{z-1}} = \varepsilon \cdot \psi(\gamma_z)$ (10):

$$\begin{aligned}\mathfrak{f}_\sigma^{\gamma_{\frac{z}{z-1}}} &= \varepsilon \cdot \psi(\gamma_z) \cdot \sigma(\varepsilon \cdot \psi(\gamma_z))^{-1} = \varepsilon \cdot \psi(\mathfrak{f}_\sigma^{\gamma_z}) \cdot \varepsilon^{-1} \cdot \varepsilon\sigma(\varepsilon)^{-1} \\ &= \mathfrak{f}_\sigma^{\gamma_z}(X,Z) \cdot \mathfrak{f}_\sigma^{\varepsilon}(X,Y) = \mathfrak{f}_\sigma^{\gamma_z}(X,Z) \cdot l_0^{-\frac{\chi(\sigma)-1}{2}}.\end{aligned}$$

In the last equality, we used a formula $\sigma(\varepsilon) = l_0^{\frac{\chi(\sigma)-1}{2}}\varepsilon$ $(\sigma \in G_K)$. The proof of Lemma is completed. $\square$

We summarize analogy between ℓ-adic Galois and complex associators as Table 1, where the 3rd and 4th rows reflect the key identities of Lemma 3.3.

Algebraic proof of (3)-(4). The following arguments are motivated from an enlightening remark given in Appendix of Furusho's lecture note [F14, A.24]. By the explicit formula of Le-Murakami [LM96] type

Table 1.

ℓ-adic Galois side	complex side
$\mathfrak{f}_\sigma^{\gamma_z}(X,Y) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$	$G_0(X,Y)(\gamma_z) \in \mathbb{C}\langle\langle X,Y\rangle\rangle$
$\mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$	$\Phi(X,Y) \in \mathbb{C}\langle\langle X,Y\rangle\rangle$
$\mathfrak{f}_\sigma^{\gamma_z}(X,Y) = \mathfrak{f}_\sigma^{\gamma_{1-z}}(Y,X)\cdot \mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)$	$G_0(X,Y)(\gamma_z) = G_0(Y,X)(\gamma_{1-z})\cdot\Phi(X,Y)$
$\mathfrak{f}_\sigma^{\gamma_{\frac{z}{z-1}}}(X,Y) = \mathfrak{f}_\sigma^{\gamma_z}(X,Z)\cdot\exp(\frac{1-\chi(\sigma)}{2}X)$ $Z := \log(\exp(-Y)\exp(-X))$	$G_0(X,Y)(\gamma_{\frac{z}{z-1}}) = G_0(X,Z)(\gamma_z)\cdot\exp(\pi\mathtt{i}X)$, $Z := -Y-X$
$Li^\ell_{\mathbf{k}}(\gamma_z)(\sigma)$: ℓ-adic Galois multiple polylog value	$Li_{\mathbf{k}}(\gamma_z)$: multiple polylog value
$\boldsymbol{\zeta}^\ell_{\mathbf{k}}(\sigma)$: ℓ-adic Galois multiple zeta value	$\zeta(\mathbf{k})$: multiple zeta value

due to Furusho [F04, Theorem 3.15], the coefficient of YX^{k-1} in $G_0(X,Y)(\gamma_z)$ is

$$\begin{aligned}\mathtt{Coeff}_{YX^{k-1}}(G_0(X,Y)(\gamma_z)) &= -\sum_{\substack{s+t=k-1\\ s,t\geq 0}}(-1)^s Li_{f'(Y\sqcup\!\sqcup X^s)}(z)\frac{\log^t z}{t!}\\ &= (-1)^k\sum_{t=0}^{k-1}(-1)^t Li_{k-t}(z)\frac{\log^t z}{t!},\end{aligned}$$

where f' indicates the operation annihilating terms ending with the letter X. Applying this to the key identity $G_0(Y,X)(\gamma_{1-z}) = G_0(X,Y)(\gamma_z)\cdot\Phi(Y,X)$ from Lemma 3.3 (1) (cf. also [F14, A.24]), we see that

$$\begin{aligned}\mathtt{Coeff}_{YX^{k-1}}(G_0(Y,X)(\gamma_{1-z})) &= \mathtt{Coeff}_{XY^{k-1}}(G_0(X,Y)(\gamma_{1-z}))\\ &= (-1)^{k-1}Li_{\underbrace{1,\dots,1}_{k-2\ \text{times}},2}(1-z)\end{aligned}$$

is equal to

$$\begin{aligned}&\mathtt{Coeff}_{YX^{k-1}}(G_0(X,Y)(\gamma_z)) + \mathtt{Coeff}_{YX^{k-1}}(\Phi(Y,X))\\ &= \mathtt{Coeff}_{YX^{k-1}}(G_0(X,Y)(\gamma_z)) + \mathtt{Coeff}_{XY^{k-1}}(\Phi(X,Y))\\ &= (-1)^k\sum_{t=0}^{k-1}(-1)^t Li_{k-t}(z)\frac{\log^t z}{t!} + (-1)^{k-1}\zeta(\underbrace{1,\dots,1}_{k-2\ \text{times}},2)\end{aligned}$$

Here we used a tautological identity

$$\mathtt{Coeff}_{w(X,Y)}(\Phi(X,Y)) = \mathtt{Coeff}_{w(Y,X)}(\Phi(Y,X))$$

and the fact that $\mathtt{Coeff}_{X^i}(\Phi(X,Y)) = \mathtt{Coeff}_{Y^i}(\Phi(X,Y)) = 0$ for all $i \geq 1$. This together with the well–known identity $\zeta(\underbrace{1,\dots,1}_{k-2\ \text{times}},2) = \zeta(k)$ (duality formula) derives the identity (4).

Remark 3.4. The duality formula is known to be a consequence of the 2-cycle relation and the shuffle product formula. Cf. [F22, Lemma 2.2], [Sou13, p.12 (5)] or Appendix A (45).

Before going to prove (3), we compare the coefficients of YXY in the same identity $G_0(X,Y)(\gamma_z) = G_0(Y,X)(\gamma_{1-z})\cdot\Phi(X,Y)$ of Lemma 3.3 (1). For simplicity, we shall use the following abbreviated notation:

$$c_w(\gamma_z) := \mathtt{Coeff}_w(G_0(X,Y)(\gamma_z))$$

for any word $w \in \mathrm{M}$ and a path γ_z from $\overrightarrow{01}$ to a point z. By simple calculation, we then obtain

$$\begin{aligned} c_{YXY}(\gamma_z) &= -\zeta(2)c_X(\gamma_{1-z}) - 2\zeta(3) + c_{XYX}(\gamma_{1-z}) \\ &= -\zeta(2)c_X(\gamma_{1-z}) - 2\zeta(3) + \big(c_{XY}(\gamma_{1-z})c_X(\gamma_{1-z}) - 2c_{X^2Y}(\gamma_{1-z})\big) \end{aligned} \tag{16}$$

where, in the former equality are used known identities $\mathtt{Coeff}_{XY}(\Phi(X,Y)) = -\zeta(2)$, $\mathtt{Coeff}_{YXY}(\Phi(X,Y)) = -2\zeta(3)$ (see Appendix A (43),(47)), and in the last equality is used the shuffle relation according to $XY ⧢ X = XYX + 2X^2Y$ (cf. Appendix A (48)). This leads to

$$Li_{2,1}(z) = -\frac{\pi^2}{6}\log(1-z) - 2\zeta(3) - Li_2(1-z)\log(1-z) + 2Li_3(1-z). \tag{17}$$

Now let us compare the coefficients of X^2Y on both sides of the key identity

$$G_0(X,Y)(\gamma_{\frac{z}{z-1}}) = G_0(X,Z)(\gamma_z)\cdot\exp(\pi\mathtt{i}X)$$

from Lemma 3.3 (2). It follows easily that

$$c_{XXY}(\gamma_{\frac{z}{z-1}}) = c_{XXY}(\gamma_z) - c_{YYY}(\gamma_z) + c_{XYY}(\gamma_z) + c_{YXY}(\gamma_z), \tag{18}$$

or equivalently,

$$-Li_3\left(\frac{z}{z-1}\right) = Li_3(z) + Li_{1,1,1}(z) + Li_{1,2}(z) + Li_{2,1}(z). \tag{19}$$

We know from the case $k=3$ of (4) with interchange $z \leftrightarrow 1-z$ that

$$Li_{1,2}(z) = \zeta(3) - \left(Li_3(1-z) - Li_2(1-z)\log(1-z) - \frac{1}{2}\log z\log^2(1-z)\right). \tag{20}$$

Putting (17) and (20) into the last two terms of (19) with noticing $Li_{1,1,1}(z) = -\frac{1}{6}\log^3(1-z)$ (cf. Appendix A), we obtain a proof of Landen's trilogarithm functional equation (3). □

Proof of Theorem 1.2: In the ℓ-adic Galois setting, the argument for the assertion goes in an almost parallel way to the above proof for (4). In fact, the formula of Le-Murakami and Furusho type is generalized to any group-like elements of $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ in [N23], so that it holds that

$$\mathtt{Coeff}_{YX^{k-1}}(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) = (-1)^k\sum_{t=0}^{k-1} Li_{k-t}^{\ell}(\gamma_z)\frac{\rho_z(\sigma)^t}{t!}. \tag{21}$$

Comparing the coefficients of YX^{k-1} in the key identity

$$\mathfrak{f}_\sigma^{\gamma_z}(X,Y) = \mathfrak{f}_\sigma^{\gamma_{1-z}}(Y,X)\cdot\mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)$$

of Lemma 3.3 (3), we obtain

$$\sum_{j=0}^{k-1} Li_{k-j}^{\ell}(\gamma_z)(\sigma)\frac{\rho_z(\sigma)^j}{j!} + Li^{\ell}_{\underbrace{1,\dots,1}_{k-2\ \text{times}},2}(\gamma_{1-z})(\sigma) = \boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{k-2\ \text{times}},2}(\sigma) \quad (\sigma\in G_K). \tag{22}$$

Note here that, in the special case $z = \overrightarrow{10}$ with $\gamma_z = \delta_{\overrightarrow{10}}$, we should interpret that $Li^{\ell}_{1,\dots,1,2}(\gamma_{1-z})(\sigma) = 0$ and that $\rho_z(\sigma)^j = 0, 1$ according to whether $j>0$ or $j=0$, from which we obtain the duality formula:

$$\boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{k-2\ \text{times}},2}(\sigma) = \boldsymbol{\zeta}^{\ell}_k(\sigma) \qquad (\sigma\in G_K). \tag{23}$$

Putting this back to (22) settles the proof of Theorem 1.2. □

Proof of Theorem 1.1: We only have to examine the ℓ-adic Galois versions of the identities (17), (19) and (20) with replacing the role of $G_0(X,Y)(\gamma_*)$ by $\mathfrak{f}_\sigma^{\gamma_*}(X,Y)$. It turns out that the two identities (17), (20) have exactly the parallel counterparts:

$$\begin{aligned}(24)\qquad & Li^\ell_{2,1}(\gamma_z)(\sigma)\\ &= \boldsymbol{\zeta}^\ell_2(\sigma)\rho_{1-z}(\sigma)+\boldsymbol{\zeta}^\ell_{2,1}(\sigma)+Li^\ell_2(\gamma_{1-z})(\sigma)\rho_{1-z}(\sigma)+2Li^\ell_3(\gamma_{1-z})(\sigma),\end{aligned}$$

$$\begin{aligned}(25)\qquad & Li^\ell_{1,2}(\gamma_z)(\sigma)\\ &= \boldsymbol{\zeta}^\ell_3(\sigma)-\left(Li^\ell_3(\gamma_{1-z})(\sigma)+Li^\ell_2(\gamma_{1-z})(\sigma)\rho_{1-z}(\sigma)+\frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)^2\right)\end{aligned}$$

with $\sigma\in G_K$. There occurs a small difference for (19) when evaluating the key identity

$$\mathfrak{f}_\sigma^{\gamma_{\frac{z}{z-1}}}(X,Y)=\mathfrak{f}_\sigma^{\gamma_z}(X,Z)\cdot\exp\left(\frac{1-\chi(\sigma)}{2}X\right)$$

from Lemma 3.3 (4) with taking into accounts the Campbell-Hausdorff sum

$$\begin{aligned}(26)\qquad Z:=&\log(\exp(-Y)\exp(-X))\\ =&-Y-X\quad\underbrace{+\frac{1}{2}(YX-XY)-\frac{1}{12}XXY+\cdots}_{(*)},\end{aligned}$$

where do exist nontrivial nonlinear terms $(*)$ in the ℓ-adic Galois case. To simplify our presentation, we shall use the following abbreviated notation:

$$c^\ell_w(\gamma_z)(\sigma):=\mathtt{Coeff}_w(\mathfrak{f}_\sigma^{\gamma_z}(X,Y))$$

for any word $w\in\mathrm{M}$, a path γ_z from $\overrightarrow{01}$ to a point z and $\sigma\in G_K$. Then $\mathfrak{f}_\sigma^{\gamma_z}(X,Z)$ is calculated as follows:

$$\begin{aligned}(27)\qquad & \mathfrak{f}_\sigma^{\gamma_z}(X,Z)\\
=&1+c^\ell_X(\gamma_z)(\sigma)X+c^\ell_Y(\gamma_z)(\sigma)Z+c^\ell_{XY}(\gamma_z)(\sigma)XZ+c^\ell_{Y^2}(\gamma_z)(\sigma)Z^2+\cdots\\
=&1+\left(c^\ell_X(\gamma_z)(\sigma)-c^\ell_Y(\gamma_z)(\sigma)\right)X\\
&-c^\ell_Y(\gamma_z)(\sigma)Y\quad\underbrace{+c^\ell_Y(\gamma_z)(\sigma)\left(\frac{1}{2}YX-\frac{1}{2}XY-\frac{1}{12}XXY+\cdots\right)}_{\ell\text{-adic extra terms in } c^\ell_Y(\gamma_z)(\sigma)Z}\\
&+\left(c^\ell_{Y^2}(\gamma_z)(\sigma)-c^\ell_{XY}(\gamma_z)(\sigma)\right)XX\quad\underbrace{+c^\ell_{XY}(\gamma_z)(\sigma)\left(\frac{1}{2}XYX-\frac{1}{2}XXY+\cdots\right)}_{\ell\text{-adic extra terms in } c^\ell_{XY}(\gamma_z)(\sigma)XZ}\\
&+\left(c^\ell_{Y^2}(\gamma_z)(\sigma)-c^\ell_{XY}(\gamma_z)(\sigma)\right)XY\quad\underbrace{+c^\ell_{Y^2}(\gamma_z)(\sigma)\left(\frac{1}{2}XXY-\frac{1}{2}YYX+\cdots\right)}_{\ell\text{-adic extra terms in } \mathtt{Coeff}^\ell_{Y^2}(\gamma_z)(\sigma)Z^2}\\
&+c^\ell_{Y^2}(\gamma_z)(\sigma)YX+c^\ell_{Y^2}(\gamma_z)(\sigma)YY+\cdots\qquad(\sigma\in G_K).\end{aligned}$$

Summing up, we find that the ℓ-adic Galois analog to identity (18) turns to get extra additional terms as:

$$\begin{aligned}(28)\qquad c^\ell_{XXY}(\gamma_{\frac{z}{z-1}})(\sigma)=&-c^\ell_{XXY}(\gamma_z)(\sigma)-c^\ell_{YYY}(\gamma_z)(\sigma)+c^\ell_{XYY}(\gamma_z)(\sigma)+c^\ell_{YXY}(\gamma_z)(\sigma)\\
&-\left(\frac{1}{2}c^\ell_{XY}(\gamma_z)(\sigma)-\frac{1}{2}c^\ell_{YY}(\gamma_z)(\sigma)+\frac{1}{12}c^\ell_Y(\gamma_z)(\sigma)\right),\end{aligned}$$

from which follows that

$$Li_3^\ell(\gamma_{\frac{z}{z-1}})(\sigma) = -Li_3^\ell(\gamma_z)(\sigma) - Li_{1,1,1}^\ell(\gamma_z)(\sigma) - Li_{1,2}^\ell(\gamma_z)(\sigma) - Li_{2,1}^\ell(\gamma_z)(\sigma) - \left(\frac{1}{2}Li_2^\ell(\gamma_z)(\sigma) + \frac{1}{4}\rho_{1-z}(\sigma)^2 + \frac{1}{12}\rho_{1-z}(\sigma)\right) \tag{29}$$

for $\sigma \in G_K$. The asserted formula follows from (29) after $Li_{1,2}^\ell(\gamma_z)(\sigma)$, $Li_{2,1}^\ell(\gamma_z)(\sigma)$ in the RHS are replaced by the equations (25), (24) respectively and from knowledge of the lower degree coefficients of $\mathfrak{f}_\sigma^{\gamma_*}(X,Y)$ illustrated in (44), (46) and (48) of Appendix A. □

4. Polylogarithmic characters and $\mathbb{Z}_\ell$-integrality test

There is a specific series of functions $\tilde{\chi}_m^z : G_K \to \mathbb{Z}_\ell$ (called the polylogarithmic characters) closely related to the ℓ-adic Galois polylogarithms $Li_k^\ell(z) : G_K \to \mathbb{Q}_\ell$. Let γ_z be an ℓ-adic étale path from $\overrightarrow{01}$ to a K-rational (tangential) point z on $\mathbb{P}^1 - \{0, 1, \infty\}$.

Definition 4.1 ([NW99]: ℓ-adic Galois polylogarithmic character). For each $m \in \mathbb{N}$ and $\sigma \in G_K$, we define $\tilde{\chi}_m^{\gamma_z}(\sigma)$ (often written shortly as $\tilde{\chi}_m^z(\sigma)$) by the (sequential) Kummer properties

$$\zeta_{\ell^n}^{\tilde{\chi}_m^z(\sigma)} = \sigma\left(\prod_{i=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{\chi(\sigma)^{-1}i}z^{1/\ell^n})^{\frac{i^{m-1}}{\ell^n}}\right) \Big/ \prod_{i=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{i+\rho_z(\sigma)}z^{1/\ell^n})^{\frac{i^{m-1}}{\ell^n}}$$

over $n \in \mathbb{N}$, where the roots $z^{1/n}, (1-z)^{1/n}, (1-\zeta_n^a z^{1/n})^{1/m}$ $(n, m \in \mathbb{N}, a \in \mathbb{Z})$ are chosen along the path $\gamma_z \in \pi_1^{\rm top}(U^{\rm an}; \overrightarrow{01}, z)$, $\rho_z (= \rho_{\gamma_z}) : G_K \to \mathbb{Z}_\ell$ is the Kummer 1-cocycle of the ℓ-th power roots $\{z^{1/\ell^n}\}_n$ along γ_z, and $\chi : G_K \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character. We call the function

$$\tilde{\chi}_m^z (= \tilde{\chi}_m^{\gamma_z}) : G_K \to \mathbb{Z}_\ell$$

the (ℓ-adic Galois) polylogarithmic character associated to $\gamma_z \in \pi_1^{\rm top}(U^{\rm an}; \overrightarrow{01}, z)$. When $z = \overrightarrow{10}$ and $\gamma_z = \delta_{\overrightarrow{10}}$, it gives the Soulé character (cf. [NW99, REMARK 2]).

We first begin with summarizing the relations between the polylogarithmic characters and ℓ-adic Galois polylogarithms:

Proposition 4.2. Let $\mathfrak{f}_\sigma^{\gamma_z}(X,Y)$ be the Magnus expansion of the ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\gamma_z}$ as in (14). Then, we have:

(i)
$$\mathtt{Coeff}_{YX^{m-1}}(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) = -\frac{\tilde{\chi}_m^z(\sigma)}{(m-1)!} \left(= (-1)^m \sum_{k=0}^{m-1} Li_{m-k}^\ell(\gamma_z)(\sigma)\frac{\rho_z(\sigma)^k}{k!}\right),$$

(ii)
$$\mathtt{Coeff}_{X^{m-1}Y}(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) = -Li_m^\ell(\gamma_z)(\sigma) \left(= (-1)^m \sum_{k=0}^{m-1} \frac{\rho_z(\sigma)^k}{k!}\frac{\tilde{\chi}_{m-k}^z(\sigma)}{(m-1-k)!}\right)$$

for $\sigma \in G_K$.

Proof. The first equality of (i) is proved in [NW20, Proposition 8 (ii)], where the symbol $\mathtt{Li}_w$ in loc.cit. differs from our Li_w by the sign corresponding to the parity of the number of appearances of letter Y in w. The first equality of (ii) is just due to our definition of $Li_{\mathbf{k}}^\ell$ in (15). The equality in the bracket of (i) is nothing but (21), i.e., is a consequence of a formula of Le-Murakami and Furusho type ([N23]). The equality in the bracket of (ii) follows from (i) by inductively reversing the sequence $\{\tilde{\chi}_m\}_m$ to $\{Li_m^\ell\}_m$. □

Often we prefer a functional equation of ℓ-adic Galois polylogarithms converted to a form of the corresponding identity between polylogarithmic characters by Proposition 4.2, because the latter enables us to check the $\mathbb{Z}_\ell$-integrality of both sides of the equation.

For example, the functional equations (7), (8) are respectively equivalent to

$$\tilde{\chi}_2^z(\sigma)+\tilde{\chi}_2^{1-z}(\sigma)+\rho_z(\sigma)\rho_{1-z}(\sigma)=\frac{1}{24}(\chi(\sigma)^2-1), \tag{30}$$

$$\tilde{\chi}_2^z(\sigma)+\tilde{\chi}_2^{z/(1-z)}(\sigma)=-\frac{1}{2}\rho_{1-z}(\sigma)(\rho_{1-z}(\sigma)-\chi(\sigma)) \tag{31}$$

for $\sigma\in G_K$. Noting that $\chi(\sigma)\equiv 1 \pmod 2$ and $\chi(\sigma)^2\equiv 1 \pmod{24}$, we easily see that each of the RHSs have no denominator, i.e., $\in\mathbb{Z}_\ell$ for every prime ℓ. From this viewpoint, it is worth rewriting Landen's trilogarithm functional equation (Theorem 1.1) in terms of polylogarithmic characters. By simple computation, it results in:

$$\begin{aligned}
&\tilde{\chi}_3^z(\sigma)+\tilde{\chi}_3^{1-z}(\sigma)+\tilde{\chi}_3^{z/(z-1)}(\sigma)\\
&=\tilde{\chi}_3^{\overrightarrow{10}}(\sigma)+\chi(\sigma)\tilde{\chi}_2^z(\sigma)+\rho_z(\sigma)\rho_{1-z}(\sigma)^2-\frac{\rho_{1-z}(\sigma)}{12}(\chi(\sigma)^2-1)\\
&\quad-\frac{\rho_{1-z}(\sigma)}{6}\Big(\chi(\sigma)-\rho_{1-z}(\sigma)\Big)\Big(\chi(\sigma)-2\rho_{1-z}(\sigma)\Big).
\end{aligned} \tag{32}$$

It is not difficult to see that each term of the above right–and side has no denominator in $\mathbb{Q}_\ell$.

5. Tensor criterion for Landen's equation for Li_3

It would be worth giving alternative proofs of complex/ℓ-adic Galois Landen's trilogarithm functional equations (3) and Theorem 1.1 with the method of [NW12] not only for checking the validity of proofs given in §3 but also for providing a typical sample showing the utility of Zagier's tensor criterion for functional equations (cf. e.g. [G13]).

Let $\mathcal{O}:=\mathbb{C}[t,\frac{1}{t},\frac{1}{1-t}]$ be the coordinate ring of $U_\mathbb{C}=\mathbf{P}^1_\mathbb{C}-\{0,1,\infty\}$ with unit group $\mathcal{O}^\times$, and let $f_1,f_2,f_3:U_\mathbb{C}\to U_\mathbb{C}$ be (auto)morphisms of $U_\mathbb{C}$ defined by

$$f_1(t)=t,\quad f_2(t)=1-t,\quad f_3(t)=\frac{t}{t-1}.$$

Considering $f_1,f_2,f_3:U\to\mathbf{G}_m$ as elements of $\mathcal{O}^\times$, we specialize Zagier's tensor criterion for Landen's functional equation of Li_3's in the following proposition:

Proposition 5.1 (Tensor criterion for Landen's functional equation for Li_3)**.** Let $\overline{\mathcal{O}}^\times:=\mathcal{O}^\times/\mathbb{C}^\times$ and denote the image of f_i $(i=1,2,3)$ in $\overline{\mathcal{O}}^\times$ by the same symbol. Then, in the tensor product $\overline{\mathcal{O}}^\times\otimes\big(\overline{\mathcal{O}}^\times\wedge\overline{\mathcal{O}}^\times\big)$ of multiplicative groups (where $\otimes$ and $\wedge$ are taken over $\mathbb{Z}$), we have

$$f_1\otimes(f_1\wedge(f_1-1))+f_2\otimes(f_2\wedge(f_2-1))+f_3\otimes(f_3\wedge(f_3-1))\equiv 0.$$

Proof. Set $a:=t$, $b:=t-1$ $c:=-1$ and write the multiplication of $\mathcal{O}^\times$ in additive form. Then, we find $f_1=a$, $f_1-1=b$, $f_2=b+c$, $f_2-1=a+c$, $f_3=a-b$, $f_3-1=-b$ so that

$$\begin{aligned}
&f_1\otimes(f_1\wedge(f_1-1))+f_2\otimes(f_2\wedge(f_2-1))+f_3\otimes(f_3\wedge(f_3-1))\\
&=a\otimes(a\wedge b)+(b+c)\otimes((b+c)\wedge(a+c))+(a-b)\otimes((a-b)\wedge(-b))\\
&=b\otimes(c\wedge a)+b\otimes(b\wedge c)+c\otimes(a\wedge b)+c\otimes(a\wedge c)+c\otimes(b\wedge c).
\end{aligned}$$

Obviously, this last side is annihilated in $\overline{\mathcal{O}}^\times\otimes\big(\overline{\mathcal{O}}^\times\wedge\overline{\mathcal{O}}^\times\big)$, since $c\equiv 0$ in $\overline{\mathcal{O}}^\times$. The assertion of proposition is proved. □

To compute the functional equations in concrete forms, we shall plug the above Proposition 5.1 into [NW12, Remark 2.3 and Theorem 5.7: $\text{(ii)}_\mathbb{C}\to\text{(iii)}_\mathbb{C}$ and $\text{(ii)}_\ell\to\text{(iii)}_\ell$]. For a quick review on the method of [NW12], we also refer the reader to Appendix B. Fix a family of paths $\{\delta_1,\delta_2,\delta_3\}$ from $\overrightarrow{01}$ to $f_1(\overrightarrow{01})=\overrightarrow{01}$, $f_2(\overrightarrow{01})=\overrightarrow{10}$, $f_3(\overrightarrow{01})=\overrightarrow{0\infty}$, with $\delta_1:=1$(= trivial path), $\delta_2:=\delta_{\overrightarrow{10}}$, $\delta_3:=\delta_{\overrightarrow{0\infty}}$ respectively. Suppose we are given a topological path $\gamma_z:\overrightarrow{01}\rightsquigarrow z$ on $\mathbf{P}^1(\mathbb{C})-\{0,1,\infty\}$. Then, δ_i $(i=1,2,3)$ provides a natural path $\delta_i \cdot f_i(\gamma_z) : \overrightarrow{01} \rightsquigarrow f_i(z)$. Below we always consider the three points $f_1(z) = z$, $f_2(z) = 1 - z$ and $f_3(z) = \frac{z}{z-1}$ to accompany those natural tracking paths from the base point $\overrightarrow{01}$ $(i = 1, 2, 3)$ in this way.

5.1. **Complex case.** With the notations being as above, [NW12, Theorem 5.7 (iii)$_{\mathbb{C}}$] asserts the existence of a functional equation of the form

$$\sum_{i=1}^{3} \mathcal{L}_{\mathbb{C}}^{\varphi_3}(f_i(z), f_i(\overrightarrow{01}); f_i(\gamma_z)) = 0, \tag{33}$$

where each term can be calculated by a concrete algorithm [NW12, Proposition 5.11]. Below let us exhibit the calculation by enhancing [NW12, Examples 6.1-6.2] to their "Li_3" version, for which we start with the graded Lie-versions of complex polylogarithms, written $\mathrm{li}_k(z, \gamma_z)$ for any path $\overrightarrow{01} \rightsquigarrow z$. These can be converted to usual polylogarithms by [NW12, Proposition 5.2]; in particular, for $k = 0, ..., 3$ we have:

$$\begin{cases} \mathrm{li}_0(z, \gamma_z) = -\dfrac{1}{2\pi \mathtt{i}} \log(z), \\ \mathrm{li}_1(z, \gamma_z) = -\dfrac{1}{2\pi \mathtt{i}} \log(1 - z), \\ \mathrm{li}_2(z, \gamma_z) = \dfrac{1}{4\pi^2} \left(Li_2(z) + \dfrac{1}{2} \log(z) \log(1 - z) \right), \\ \mathrm{li}_3(z, \gamma_z) = \dfrac{1}{(2\pi \mathtt{i})^3} \left(Li_3(z) - \dfrac{1}{2} \log(z) Li_2(z) - \dfrac{1}{12} \log^2(z) \log(1 - z) \right). \end{cases} \tag{34}$$

Each term of (33) relies only on the chain $f_i(\gamma_z)$ that does not start from $\overrightarrow{01}$ if $i \neq 1$, in which case we need to interpret the chain $f_i(\gamma_z)$ as the difference "$\delta_i \cdot f_i(\gamma_z)$ minus δ_i". At the level of graded Lie-version of polylogarithms, the difference can be evaluated by the polylog-BCH formula [NW12, Proposition 5.9]: In our case, a crucial role is played by the polynomial

$$\mathsf{P}_3(\{a_j\}_{j=0}^3, \{b_j\}_{j=0}^3) = a_3 + b_3 + \frac{1}{2}(a_0 b_2 - b_0 a_2) + \frac{1}{12}(a_0^2 b_1 - a_0 a_1 b_0 - a_0 b_0 b_1 + a_1 b_0^2) \tag{35}$$

in 8 variables a_j, b_j $(j = 0, \ldots, 3)$. Using this and applying [NW12, Proposition 5.11 (i)], we have

$$\mathcal{L}_{\mathbb{C}}^{\varphi_3}(f_i(z), f_i(\overrightarrow{01}); f_i(\gamma_z)) = \mathsf{P}_3\big(\{\mathrm{li}_j(f_i(z), \delta_i \cdot f_i(\gamma_z))\}_{j=0}^3, \{-\mathrm{li}_j(f_i(\overrightarrow{01}), \delta_i)\}_{j=0}^3\big) \tag{36}$$

for $i = 1, 2, 3$. Noting then that

$$\begin{aligned} \left(-\mathrm{li}_j(\overrightarrow{01}, \delta_1)\right)_{0 \le j \le 3} &= (0, 0, 0, 0), \\ \left(-\mathrm{li}_j(\overrightarrow{10}, \delta_2)\right)_{0 \le j \le 3} &= \left(0, 0, -\mathrm{li}_2(\overrightarrow{10}), -\mathrm{li}_3(\overrightarrow{10})\right) \\ &= \left(0, 0, -\frac{1}{4\pi^2} Li_2(1), -\frac{1}{(2\pi \mathtt{i})^3} Li_3(1)\right), \\ \left(-\mathrm{li}_j(\overrightarrow{0\infty}, \delta_3)\right)_{0 \le j \le 3} &= \left(\frac{1}{2}, 0, 0, 0\right), \end{aligned}$$

we compute (36) for $i = 1, 2, 3$ as:

$$
\begin{cases}
\mathcal{L}_{\mathbb{C}}^{\varphi_3}(z, \overrightarrow{01}; \gamma_z) & = \mathrm{li}_3(z, \gamma_z), \\
\mathcal{L}_{\mathbb{C}}^{\varphi_3}(1-z, \overrightarrow{10}; f_2(\gamma_z)) & = \mathrm{li}_3(1-z, \gamma_{1-z}) - \mathrm{li}_3(\overrightarrow{10}, \delta_{\overrightarrow{01}}) \\
& \quad + \frac{1}{2}\mathrm{li}_0(1-z, \gamma_{1-z})(-\mathrm{li}_0(z, \gamma_z)), \\
\mathcal{L}_{\mathbb{C}}^{\varphi_3}\left(\frac{z}{z-1}, \overrightarrow{0\infty}; f_3(\gamma_z)\right) & = \mathrm{li}_3(\frac{z}{z-1}, \gamma_{\frac{z}{z-1}}) + \frac{1}{2}\left(-\frac{1}{2}\mathrm{li}_2(\frac{z}{z-1}, \gamma_{\frac{z}{z-1}})\right) \\
& \quad + \frac{1}{12}\left(\frac{1}{4}\mathrm{li}_1(\frac{z}{z-1}, \gamma_{\frac{z}{z-1}}) - \frac{1}{2}\mathrm{li}_1(\frac{z}{z-1}, \gamma_{\frac{z}{z-1}})\mathrm{li}_0(\frac{z}{z-1}, \gamma_{\frac{z}{z-1}})\right).
\end{cases}
\tag{37}
$$

Putting these together into (33) and applying (34), we obtain Landen's functional equation (3).

5.2. **ℓ-adic Galois case.** Let us apply [NW12, Theorem 5.7 (iii)$_\ell$] in the parallel order to our above discussion in the complex case. The ℓ-adic version of the functional equation (33) in loc.cit. relies on the choice of our free generator system $\vec{x} := (l_0, l_1)$ of $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$ which plays an indispensable role to specify a splitting of the pro-unipotent Lie algebra of $\pi_1^{\ell\text{-ét}}$ into the weight gradation over $\mathbb{Q}_\ell$ (cf. [NW12, §4.2]). Then, the functional equation turns out in the form

$$
\sum_{i=1}^{3} \mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_i),\vec{x}}(f_i(z), f_i(\overrightarrow{01}); f_i(\gamma_z))(\sigma) = E(\sigma, \gamma_z) \qquad (\sigma \in G_K)
\tag{38}
$$

where $E(\sigma, \gamma_z)$ is called the ℓ-adic error term ([NW12, §4.3]). The graded Lie-version of ℓ-adic Galois polylogarithm $\ell i_k(z, \gamma_z, \vec{x})$ (for $k \geq 1$) is then defined as the coefficient of $\mathrm{ad}(X)^{k-1}(Y) = [X, [X, [\cdots [X, Y]..]$ in $\log(\mathfrak{f}_\sigma^{\gamma_z}(X, Y)^{-1})$ as an element of Lie formal series $\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X, Y\rangle\rangle$. Recall that the variables X, Y are determined by $\vec{x} := \{l_0, l_1\}$ by the the Magnus embedding $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01}) \hookrightarrow \mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$ defined by $l_0 \mapsto \exp(X)$, $l_1 \mapsto \exp(Y)$. For brevity below, let us often omit references to the loop system $\vec{x} = (l_0, l_1)$ and/or tracking paths $\delta_i \cdot f_i(\gamma_z) : \overrightarrow{01} \rightsquigarrow f_i(z)$ in our notations as long as no confusions occur. The list corresponding to (34) reads then:

$$
\begin{cases}
\ell i_0(z, \gamma_z)(\sigma) = \rho_z(\sigma), \\
\ell i_1(z, \gamma_z)(\sigma) = \rho_{1-z}(\sigma), \\
\ell i_2(z, \gamma_z)(\sigma) = -\tilde{\chi}_2^z(\sigma) - \dfrac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma), \\
\ell i_3(z, \gamma_z)(\sigma) = \dfrac{1}{2}\tilde{\chi}_3^z(\sigma) + \dfrac{1}{2}\rho_z(\sigma)\tilde{\chi}_2^z(\sigma) + \dfrac{1}{12}\rho_z(\sigma)^2\rho_{1-z}(\sigma)
\end{cases}
\tag{39}
$$

with $\sigma \in G_K$. Each term of the above (38) for $i = 1, 2, 3$ can be expressed by the graded Lie-version of polylogarithms ℓi_k $(k = 0, \ldots, 3)$ along "$\delta_i \cdot f_i(\gamma_z)$ minus δ_i" by the polylog-BCH formula ([NW12, Proposition 5.11 (ii)]) in the following way:

$$
\begin{aligned}
&\mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_i),\vec{x}}(f_i(z), f_i(\overrightarrow{01}); f_i(\gamma_z)) \\
&= \mathsf{P}_3\big(\{-\ell i_j(f_i(\overrightarrow{01}), \delta_i, \vec{x})\}_{j=0}^3, \{\ell i_j(f_i(z), \delta_i \cdot f_i(\gamma_z), \vec{x})\}_{j=0}^3\big).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\left(-\ell i_j(\overrightarrow{01}, \delta_1)\right)_{0 \leq j \leq 3} = (0, 0, 0, 0), \\
&\left(-\ell i_j(\overrightarrow{10}, \delta_2)\right)_{0 \leq j \leq 3} = \left(0, 0, -\ell i_2(\overrightarrow{10}), -\ell i_3(\overrightarrow{10})\right) = \left(0, 0, \tilde{\chi}_2^{\overrightarrow{10}}(\sigma), -\frac{1}{2}\tilde{\chi}_3^{\overrightarrow{10}}(\sigma)\right), \\
&\left(-\ell i_j(\overrightarrow{0\infty}, \delta_3)\right)_{0 \leq j \leq 3} = \left(\frac{1 - \chi(\sigma)}{2}, 0, 0, 0\right),
\end{aligned}
$$

we obtain for $\sigma \in G_K$:

$$(40)\quad \begin{cases} \mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_1)}(z,\overrightarrow{01};\gamma_z)(\sigma)=\frac{1}{2}\tilde{\chi}_3^{z}(\sigma)+\frac{1}{2}\rho_z(\sigma)\tilde{\chi}_2^{z}(\sigma)+\frac{1}{12}\rho_z(\sigma)^2\rho_{1-z}(\sigma), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_2)}(1-z,\overrightarrow{10};f_2(\gamma_z))(\sigma) \\ \qquad =\frac{1}{2}\tilde{\chi}_3^{1-z}(\sigma)+\frac{1}{2}\rho_{1-z}(\sigma)\tilde{\chi}_2^{1-z}(\sigma)+\frac{1}{12}\rho_{1-z}(\sigma)^2\rho_z(\sigma) \\ \qquad\quad -\frac{1}{2}\tilde{\chi}_3^{\overrightarrow{10}}(\sigma)-\frac{1}{2}\rho_{1-z}(\sigma)\tilde{\chi}_2^{\overrightarrow{10}}(\sigma), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_3)}\left(\frac{z}{z-1},\overrightarrow{0\infty};f_3(\gamma_z)\right)(\sigma) \\ \qquad =\frac{1}{2}\tilde{\chi}_3^{\frac{z}{z-1}}(\sigma)+\frac{1}{2}\rho_{\frac{z}{z-1}}(\sigma)\tilde{\chi}_2^{\frac{z}{z-1}}(\sigma)+\frac{1}{12}\rho_{\frac{z}{z-1}}(\sigma)^2\rho_{\frac{1}{1-z}}(\sigma) \\ \qquad\quad +\frac{1}{2}\left(\frac{1-\chi(\sigma)}{2}\right)\left(-\tilde{\chi}_2^{\frac{z}{z-1}}(\sigma)-\frac{1}{2}\rho_{\frac{z}{z-1}}(\sigma)\rho_{\frac{1}{1-z}}(\sigma)\right) \\ \qquad\quad +\frac{1}{12}\left(\frac{1-\chi(\sigma)}{2}\right)^2\rho_{\frac{1}{1-z}}(\sigma)-\frac{1}{12}\left(\frac{1-\chi(\sigma)}{2}\right)\rho_{\frac{z}{z-1}}(\sigma)\rho_{\frac{1}{1-z}}(\sigma). \end{cases}$$

Combining the identities in (40) enables us to rewrite the LHS of (38) in terms of ℓ-adic Galois polylogarithmic characters. It remains to compute the error term $E(\sigma,\gamma_z)$ in the right–and side of (38).

Lemma 5.2. Notations being as above, we have

$$E(\sigma,\gamma_z)=-\frac{1}{12}\rho_{1-z}(\sigma)+\frac{1}{2}\tilde{\chi}_2^{z}(\sigma)+\frac{1}{4}\rho_z(\sigma)\rho_{1-z}(\sigma).$$

Proof. We shall apply the formula [NW12, Corollary 5.8] to compute the error term. Let $[\log(\mathfrak{f}_\sigma^{\gamma_z})^{-1}]_{<3}$ be the part of degree <3 cut out from the Lie formal series $\log(\mathfrak{f}_\sigma^{\gamma_z})^{-1}\in \mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X,Y\rangle\rangle$ with respect obtained by the Magnus embedding $l_0\to e^X$, $l_1\to e^Y$ with respect to the fixed free generator system $\vec{x}=(l_0,l_1)$ of $\pi_1^{\ell\text{-ét}}(U_{\overline{K}},\overrightarrow{01})$. We also write $\varphi_3:\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X,Y\rangle\rangle\to\mathbb{Q}_\ell$ for the $\mathbb{Q}_\ell$-linear form that picks up the coefficient of $[X,[X,Y]]$ (that is uniquely determined) for any Lie series of $\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X,Y\rangle\rangle$. Introduce the variable Z so that $e^Xe^Ye^Z=1$ in $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$. By the Campbell-Baker-Hausdorff formula, we have

$$(41)\qquad Z=-X-Y-\frac{1}{2}[X,Y]-\frac{1}{12}[X,[X,Y]]+\cdots.$$

According to [NW12, Corollary 5.8], it follows then that

$$\begin{aligned} E(\sigma,\gamma_z) &= \sum_{i=1}^{3}\varphi_3\left(\delta_i\cdot f_i\left([\log(\mathfrak{f}_\sigma^{\gamma_z})^{-1}]_{<3}\right)\cdot{\delta_i}^{-1}\right) \\ &= \sum_{i=1}^{3}\varphi_3\Big(\delta_i\cdot f_i\left(\rho_z(\sigma)X+\rho_{1-z}(\sigma)Y+\ell i_2(z,\gamma_z)(\sigma)[X,Y]\right)\cdot{\delta_i}^{-1}\Big) \\ &= \varphi_3\Big(\rho_z(\sigma)X+\rho_{1-z}(\sigma)Y+\ell i_2(z,\gamma_z)(\sigma)[X,Y]\Big) \\ &\quad +\varphi_3\Big(\rho_z(\sigma)Y+\rho_{1-z}(\sigma)X+\ell i_2(z,\gamma_z)(\sigma)[Y,X]\Big) \\ &\quad +\varphi_3\Big(\rho_z(\sigma)X+\rho_{1-z}(\sigma)Z+\ell i_2(z,\gamma_z)(\sigma)[X,Z]\Big). \end{aligned}$$

Here in the last equality, we applied the following table where $\delta_i\cdot f_i(\#)\cdot\delta_i^{-1}$ for $i\in\{1,2,3\}$, $\#\in\{X,Y\}$ are summarized.

i	1	2	3
$\delta_i \cdot f_i(X) \cdot \delta_i^{-1}$	X	Y	X
$\delta_i \cdot f_i(Y) \cdot \delta_i^{-1}$	Y	X	Z

Since φ_3 annihilates those terms $X, Y, [X,Y], [Y,X]$, we continue the above computation after (41) as:

$$\begin{aligned} E(\sigma,\gamma_z) &= \varphi_3\left(-\frac{1}{12}\rho_{1-z}(\sigma)[X,[X,Y]] - \frac{1}{2}\ell i_2(z,\gamma_z)(\sigma)[X,[X,Y]]\right) \\ &= -\frac{1}{12}\rho_{1-z}(\sigma) - \frac{1}{2}\ell i_2(z,\gamma_z)(\sigma) \\ &= -\frac{1}{12}\rho_{1-z}(\sigma) - \frac{1}{2}\left(-\tilde{\chi}_2^z(\sigma) - \frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)\right) \\ &= -\frac{1}{12}\rho_{1-z}(\sigma) + \frac{1}{2}\tilde{\chi}_2^z(\sigma) + \frac{1}{4}\rho_z(\sigma)\rho_{1-z}(\sigma). \end{aligned}$$

This concludes the assertion of the lemma. □

Alternative proof of Theorem 1.1. As discussed in §4, the ℓ-adic Galois Landen's trilogarithm functional equation in Theorem 1.1 is equivalent to the identity (32) between polylogarithmic characters. The latter follows from (38) with replacements of the terms of both sides by (40) and Lemma 5.2 by simple computations. □

Appendix A: Low degree terms of associators

Presentation of lower degree terms of $G_0(X,Y)(z)$ and $\mathfrak{f}_\sigma^{\gamma_z}(X,Y)$ are often useful as references. The former one presented below reconfirms Furusho's preceding computations found in [F04, 3.25]-[F14, A.16] (where the sign of $\log(z)Li_2(z)$ had an unfortunate misprint in the coefficient of XYX).

(42)
$$\begin{aligned} G_0(X,Y)(z) = {} & 1 + \log(z)X + \log(1-z)Y + \frac{\log^2(z)}{2}X^2 - Li_2(z)XY \\ & + \Big(Li_2(z) + \log(z)\log(1-z)\Big)YX + \frac{\log^2(1-z)}{2}Y^2 + \frac{\log^3(z)}{6}X^3 - Li_3(z)X^2Y \\ & + \Big(2Li_3(z) - \log(z)Li_2(z)\Big)XYX + Li_{1,2}(z)XY^2 \\ & - \left(Li_3(z) - \log(z)Li_2(z) - \frac{\log^2(z)\log(1-z)}{2}\right)YX^2 + Li_{2,1}(z)YXY \\ & - \left(Li_{1,2}(z) + Li_{2,1}(z) - \frac{\log(z)\log^2(1-z)}{2}\right)Y^2X + \frac{\log^3(1-z)}{6}Y^3 \\ & + \cdots \text{(higher degree terms)}. \end{aligned}$$

This is a group-like element of $\mathbb{C}\langle\langle X,Y\rangle\rangle$ whose coefficients satisfy what are called the shuffle relations ([Ree58]). The regular coefficients (viz. those coefficients of monomials ending with the letter Y) are given by iterated integrals of a sequence of dz/z, $dz/(1-z)$. This immediately shows $G_0(0,Y)(z) = \sum_{k=0} \frac{\log^k(1-z)}{k!}Y^k$ and say, $Li_{1,1,1}(z) = -\frac{1}{6}\log^3(1-z)$. Furusho gave an explicit formula that expresses arbitrary coefficients of $G_0(X,Y)$ in terms only of the regular coefficients ([F04, Theorem 3.15]). The specialization $z \to \overrightarrow{10}$ with $\gamma_z = \delta_{\overrightarrow{10}}$ (cf. [W97] p.239 for a naive account) interprets $\log z \to 0, \log(1-z) \to$

0 so as to produce the Drinfeld's associator:

$$\begin{aligned}(43)\qquad &\Phi(X,Y)\quad \Big(= G_0(X,Y)(\overrightarrow{10})\Big)\\ &= 1-\zeta(2)XY+\zeta(2)YX-\zeta(3)X^2Y+2\zeta(3)XYX\\ &\quad +\zeta(1,2)XY^2-\zeta(3)YX^2-2\zeta(1,2)YXY+\zeta(1,2)Y^2X\\ &\quad +\cdots\text{(higher degree terms)}\end{aligned}$$

which plays a primary role to define the Grothendieck-Teichmüller group ([Dr90], [Ih90]).

The expansion in $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ of the ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\gamma_z}\in\pi_1^{\ell\text{-ét}}(U_{\overline{K}},\overrightarrow{01})$ via the Magnus embedding $l_0\mapsto e^X$, $l_1\mapsto e^Y$ over $\mathbb{Q}_\ell$ reads as follows:

$$\begin{aligned}(44)\qquad \mathfrak{f}_\sigma^{\gamma_z}(X,Y) &= 1-\rho_{\gamma_z}(\sigma)X-\rho_{\gamma_{1-z}}(\sigma)Y+\frac{\rho_{\gamma_z}(\sigma)^2}{2}X^2-Li_2^\ell(\gamma_z)(\sigma)XY\\ &+\Big(Li_2^\ell(\gamma_z)(\sigma)+\rho_{\gamma_z}(\sigma)\rho_{\gamma_{1-z}}(\sigma)\Big)YX+\frac{\rho_{\gamma_{1-z}}(\sigma)^2}{2}Y^2-\frac{\rho_{\gamma_z}(\sigma)^3}{6}X^3-Li_3^\ell(\gamma_z)(\sigma)X^2Y\\ &+\Big(2Li_3^\ell(\gamma_z)(\sigma)+\rho_{\gamma_z}(\sigma)Li_2^\ell(\gamma_z)(\sigma)\Big)XYX+Li_{1,2}^\ell(\gamma_z)(\sigma)XY^2\\ &-\left(Li_3^\ell(\gamma_z)(\sigma)+\rho_{\gamma_z}(\sigma)Li_2^\ell(\gamma_z)(\sigma)+\frac{\rho_{\gamma_z}(\sigma)^2\rho_{\gamma_{1-z}}(\sigma)}{2}\right)YX^2+Li_{2,1}^\ell(\gamma_z)(\sigma)YXY\\ &-\left(Li_{1,2}^\ell(\gamma_z)(\sigma)+Li_{2,1}^\ell(\gamma_z)(\sigma)+\frac{\rho_{\gamma_z}(\sigma)\rho_{\gamma_{1-z}}(\sigma)^2}{2}\right)Y^2X-\frac{\rho_{\gamma_{1-z}}(\sigma)^3}{6}Y^3\\ &+\cdots\text{(higher degree terms)}\qquad (\sigma\in G_K).\end{aligned}$$

The coefficients of X,Y,YX^k $(k=1,2,...)$ were calculated in terms of polylogarithmic characters explicitly in [NW99]. A formula of Le-Murakami, Furusho type for arbitrary group-like power series was shown in [N23]. As illustrated in Proposition 4.2, the family of polylogarithmic characters and that of ℓ-adic Galois polylogarithms are converted to each other. The terms appearing in the above (44) can be derived from them.

The ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\overrightarrow{10}}(X,Y)$ specialized at $z=\overrightarrow{10}$ with $\gamma_z=\delta_{\overrightarrow{10}}$ plays an important role to define the pro-ℓ version of the Grothendieck-Teichmüller group similarly to the Drinfeld associator $\Phi(X,Y)$.

Let $f(X,Y)\in F\langle\langle X,Y\rangle\rangle$ be one of the power series either of $\Phi(X,Y)\in\mathbb{C}\langle\langle X.Y\rangle\rangle$ or $\mathfrak{f}_\sigma^{\overrightarrow{10}}(X,Y)\in\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ for some $\sigma\in G_\mathbb{Q}$ with coefficients in $F=\mathbb{C},\mathbb{Q}_\ell$ respectively. The following properties are well known to be held by $f(X,Y)$:

(i) $f(X,Y)$ is a group-like element, i.e., the coefficients satisfy the shuffle relations ([Ree58]);
(ii) $f(X,0)=f(0,Y)=f(X,X)=1$ and $f(X,Y)\equiv 1 \bmod (X,Y)^2$;
(iii) $f(X,Y)f(Y,X)=1$ (2-cyclic relation).

The condition (iii) combined with (i) is also known to be equivalent to the following

Duality-relation: If a power series $f(X,Y)=1+\sum_{w\in\mathrm{M}}c_w w\in F\langle\langle X,Y\rangle\rangle$ $(c_w\in F)$ satisfies the above conditions (i) and (iii), then

$$(45)\qquad c_w=(-1)^{|w|}c_{\overline{w'}}.$$

Here, for a word $w=x_1\cdots x_m$ $(x_i=X,Y)$, we write $|w|=m$, and designate $\overline{w'}:=\overline{x'_m\cdots x'_1}$ to mean the word obtained by applying the substitutions $X'=Y$, $Y'=X$ after reversing the order of letters in w. See, e.g., discussions around [Sou13, p.12 (5)] for an algebraic proof of the duality relation under (i), (iii).

The duality relation is already useful in degree 3, for example, to obtain

$$c_{XXY} = -c_{XYY}, \quad c_{YXY} = -c_{XYX}, \quad c_{YXX} = -c_{YYX}. \tag{46}$$

The first equation implies Euler's celebrated relation

$$\zeta(3) = \zeta(1,2) \tag{47}$$

when $F = \mathbb{C}$ and its ℓ-adic Galois analog $\boldsymbol{\zeta}^{\ell}(3)(\sigma) = \boldsymbol{\zeta}^{\ell}(1,2)(\sigma)$ when $F = \mathbb{Q}_\ell$ (recall (6) for the latter notation). One also observes that the shuffle relations corresponding to $XY ⧢ Y = YXY + 2XYY$, $Y ⧢ XX = YXX + XYX + XXY$ imply

$$c_{YXY} + 2c_{XYY} = 0 = c_{YXX} + c_{XYX} + c_{XXY}. \tag{48}$$

These equalities in (46), (48) enable us to find the above description of the degree 3 part of (43) and its obvious ℓ-adic Galois analog.

Appendix B: A quick review of [NW12] with remarks

Let $U = \mathbf{P}^1 - \{0,1,\infty\}$ and let $\pi = \pi_1(U_{\mathbb{C}}, \overrightarrow{01})$ be the discrete fundamental group. Let $m \geq 2$ be an integer, and suppose we are given a $\mathbb{Z}$-homomorphism $\varphi : \mathrm{gr}^m(\pi) \to \mathbb{Z}$ from the m-th graded quotient of the lower central series of π. Let H be the abelianization $\pi/[\pi,\pi]$ of π, and define the standard projection

$$\mathrm{st} : H^{\otimes m-2} \otimes \wedge^2 H \twoheadrightarrow \mathrm{gr}^m(\pi) \tag{49}$$

by $x_1 \otimes \cdots \otimes x_{m-2} \otimes (x_{m-1} \wedge x_m) \mapsto [x_1, [x_2, \ldots, [x_{m-1}, x_m]..]]$. Let $\mathcal{O} = \mathbb{C}[t, \frac{1}{t}, \frac{1}{t-1}]$ be the affine ring defining U_C. In [NW12, Corollary 3.7], it is shown that there is an element

$$\widehat{\kappa_{\otimes m}}(\varphi) \in \mathcal{O}^{\times \otimes n-2} \otimes \left(\mathcal{O}^\times \wedge \mathcal{O}^\times\right) \tag{50}$$

whose multi-Kummer dual $\kappa^{\otimes m}(\widehat{\kappa_{\otimes m}}(\varphi)) : H^{\otimes m-2} \otimes \wedge^2(H) \to \mathbb{Z}$ coincides with the composite $\varphi \circ \mathrm{st}$. Note here that $\wedge^2 H$ is understood as the quotient wedge tensor (the maximal quotient of $H \otimes H$ satisfying $x \wedge y + y \wedge x = 0$), while $\mathcal{O}^\times \wedge \mathcal{O}^\times$ is understood as the submodule wedge tensor (the submodule generated by the elements of the form $a \otimes b - b \otimes a$ in $(\mathcal{O}^\times)^{\otimes 2}$) (cf. [NW12, Notation 3.4]).

Now, let $\mathfrak{X}$ be a normal affine variety defined by a ring $\mathcal{O}_{\mathfrak{X}}$, and suppose that a collection of morphisms $f_i : \mathfrak{X} \to \mathbf{P}^1 - \{0,1,\infty\}$ and $c_i \in \mathbb{Z}$ ($i = 1, \ldots, n$) satisfy a multi-linear relation (called the tensor criterion for a functional equation)

$$\sum_{i=1}^n c_i\, f_i^*(\widehat{\kappa_{\otimes m}}(\varphi)) \equiv 0 \qquad \text{in } \ \overline{\mathcal{O}}_{\mathfrak{X}}^{\times \otimes n-2} \otimes \left(\overline{\mathcal{O}}_{\mathfrak{X}}^\times \wedge \overline{\mathcal{O}}_{\mathfrak{X}}^\times\right), \tag{51}$$

where $f_i^* : \mathcal{O} \to \mathcal{O}_{\mathfrak{X}}$ is the pull-back of functions on U and $\overline{\mathcal{O}}_{\mathfrak{X}}^\times := \mathcal{O}_{\mathfrak{X}}^\times / \mathbb{C}^\times$. For each $i = 1, \ldots, n$ and a topological path $\gamma : v \rightsquigarrow \xi$ on $\mathfrak{X}(\mathbb{C})$, the image $f_i(\gamma)$ forms a path from $f_i(v)$ to $f_i(\xi)$ on $\mathbf{P}^1(\mathbb{C}) - \{0,1,\infty\}$. Let $\Lambda_{f_i(\gamma)}$ (or written also $\Lambda(f_i(\gamma))$) denote Chen's transport formal series (discussed in the proof of Lemma 3.3) whose coefficients are iterated integrals along the path $f_i(\gamma)$. This is known as a group-like element of $\mathbb{C}\langle\langle X, Y\rangle\rangle$ so that $\log(\Lambda_{f_i(\gamma)}^{-1})$ lies in the space of Lie formal series $\mathrm{Lie}_{\mathbb{C}}\langle\langle X, Y\rangle\rangle$ inside $\mathbb{C}\langle\langle X, Y\rangle\rangle$. The homogeneous degree m part $\mathrm{Lie}_m(\mathbb{C})$ of $\mathrm{Lie}_{\mathbb{C}}\langle\langle X, Y\rangle\rangle$ is naturally isomorphic to $\mathrm{gr}^m(\pi) \otimes \mathbb{C}$. Define $\mathcal{L}_{\mathbb{C}}^{\varphi}(f_i(\xi), f_i(v); f_i(\gamma))$ (called the complex iterated integrals in [NW12, Definition 4.4]) to be the image of $\log(\Lambda_{f_i(\gamma)}^{-1})$ under the composition

$$\mathrm{Lie}_{\mathbb{C}}\langle\langle X, Y\rangle\rangle \twoheadrightarrow \mathrm{Lie}_m(\mathbb{C}) \xrightarrow{\sim} \mathrm{gr}^m(\pi) \otimes \mathbb{C} \xrightarrow{\varphi_{\mathbb{C}}} \mathbb{C} \tag{52}$$

induced from $\varphi_{\mathbb{C}} := \varphi \otimes \mathbb{C}$. Then it follows from [NW12, Theorem 4.13] that

$$\sum_{i=1}^n c_i\, \mathcal{L}_{\mathbb{C}}^{\varphi}(f_i(\xi), f_i(v); f_i(\gamma)) = 0. \tag{53}$$

Next, to exhibit the ℓ-adic Galois version, suppose that $\mathfrak{X}$ and points v, ξ on it together with $f_i : \mathfrak{X} \to U$ $(i = 1, \dots, n)$ are defined over a subfield $K \subset \mathbb{C}$, and assume that the tensor condition (51) is satisfied by them. We fix a path system $\delta_i : \overrightarrow{01} \rightsquigarrow f_i(v)$ $(i = 1, \dots, n)$. For each pro-ℓ path $\gamma : v \rightsquigarrow \xi$ on $\mathfrak{X}_{\overline{K}}$ and $\sigma \in G_K$, we define $\mathcal{L}_{\mathrm{nv}}^{\varphi(f_i),\vec{x}}\big(f_i(\xi), f_i(v); f_i(\gamma)\big)(\sigma)$ (which is called the naive ℓ-adic iterated integral in [NW12, Definition 4.7]) to be the image of the Lie formal series $\log\big((\delta_i \cdot f_i(\gamma) \cdot \sigma(f_i(\gamma)^{-1}) \cdot \delta_i^{-1})^{-1}\big)$ under the composition

$$\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X, Y\rangle\rangle \twoheadrightarrow \mathrm{Lie}_m(\mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{gr}^m(\pi) \otimes \mathbb{Q}_\ell \xrightarrow{\varphi_{\mathbb{Q}_\ell}} \mathbb{Q}_\ell \tag{54}$$

induced from $\varphi_{\mathbb{Q}_\ell} := \varphi \otimes \mathbb{Q}_\ell$. (The prefix $\vec{x}$ of $\mathcal{L}_{\mathrm{nv}}^{\varphi(f_i),\vec{x}}$ is to designate dependency on $\vec{x} = (l_0, l_1)$, the initially fixed free generator of π; note that $X = \log(l_0), Y = \log(l_1)$ in the ℓ-adic Galois case). Then, [NW12, Theorem 4.14] implies

$$\sum_{i=1}^{n} c_i\, \mathcal{L}_{\mathrm{nv}}^{\varphi(f_i),\vec{x}}\big(f_i(\xi), f_i(v); f_i(\gamma)\big)(\sigma) = E(\sigma, \gamma) \qquad (\sigma \in G_K), \tag{55}$$

where the error term $E(\sigma, \gamma)$ is a function of $\sigma \in G_K$ and of $\gamma : v \rightsquigarrow \xi$ satisfying a certain condition on small variation over (σ, γ) ([NW12, p.276]).

The map φ_3 employed in (33), (38) corresponds to the trilogaritm, and is the special case $m = 3$ of $\varphi_m : \mathrm{gr}^m(\pi) \to \mathbb{Z}$ defined as the dual element to $\mathrm{ad}_X^{m-1}(Y) = [X[...[X[X, Y]]..]]$ with respect to the Hall basis (for the order $X < Y$) of the free Lie algebra generated by X, Y. To obtain a polylogarithmic identity in the case $\varphi = \varphi_m$, we first translate the "iterated integrals along $f_i(v) \rightsquigarrow f_i(\xi)$" appearing in the left hand sides of (53) and (55) into the terms of those along $\overrightarrow{01} \rightsquigarrow f_i(v)$ and $\overrightarrow{01} \rightsquigarrow f_i(\xi)$ under the natural network of paths composed of $\{f_i(\gamma)\}_{i=1}^n \cup \{\delta_i \cdot f_i(\gamma)\}_{i=1}^n$. This is achieved by what is called a "polylog BCH formula" elaborated in [NW12, Proposition 5.9]. The second task is to evaluate the error term $E(\sigma, \gamma)$ of the right hand side of (55), which is figured out in [NW12, Corollary 5.8]. We refer the reader to [NW12, Sect.5] for more details of these procedures when φ is of the form φ_m $(m \in \mathbb{N}_{\geq 2})$.

On the other hand, when φ is chosen to be a general character looking at non-polylogarithmic (or to say, multiple polylogarithmic) coefficients, then there occur more complicated computational procedures. We expect future studies for them.

Finally, we would like to present the following list of typos in the previous papers [NW12], [NW20], many pieces of which have been found during the course of our present collaboration. Since these papers are not only crucial but also indispensable to test computations of our above investigation, we hope to be allowed to append the list here below:

▶ `Misprints in [NW12]:`

p.284, line −11: The LHS of the displayed formula should read: $1 - \frac{1}{2}(e^{(\log z)X} - 1) + \frac{1}{3}(e^{(\log z)X} - 1)^2 - + \cdots$

p.286, line 6: The two Galois groups G_* should respectively have subscripts $* = K(\mu_{\ell^\infty}, z^{1/\ell^\infty})$ and $* = K(\mu_{\ell^\infty}, z^{1/\ell^\infty}, (1-z)^{1/\ell^\infty})$.

p.287, in (iii)$_{\mathbb{C}}$: $f_i(x)$ should read $f_i(v)$, and $x \rightsquigarrow z$ should read $v \rightsquigarrow z$.

p.288, in the last line of (iii)$_\ell$, $p : x \rightsquigarrow z$ should read $p : v \rightsquigarrow z$

p.294: In the formula (5.13), the denominator of RHS should read: $\prod_{a=0}^{\ell^n - 1}(1 - \zeta_{\ell^n}^a)^{\frac{a^{2k-1}}{\ell^n}}$.

p.300: The last line of calculation of $E(\sigma, \gamma)$ (displayed in the middle of page) should read:

$$= \varphi_{2,\vec{x}}\left(-\frac{1}{2}\rho_{1-z}(\sigma)[\log x, \log y])\right) = -\frac{1}{2}\rho_{1-z}(\sigma).$$

(*Note*: The subsequent displayed formula reflects (6.20) with both sides multiplied by −1 so that the formula (6.22) is itself correct.)

p.300, line −1: $f_1(v) = \overrightarrow{\infty 1}$ should read $f_2(v) = \overrightarrow{\infty 1}$.

p.303, line −7: The latter equality should read $\delta f_2(y)\delta^{-1} = y$.

p.304, line 2: The RHS of the displayed identity should read $= -\left(\frac{-\chi}{e^{-\chi t} - 1}\right) - \left(\frac{e^{\mathrm{L}_0 t}}{e^{-t} - 1}\right)$.

p.305, line 4: The second identity should read $f_2(x_{14}) = x_{13} = (yx)^{-1}$.
(*Note*: The forgetful map $f_2 : M_{0,5} \to M_{0,4} = \mathbf{P}^1 - \{0, 1, \infty\}$ sends braid-like generators in such a way that x_{ij} $(1 \le i < j \le 5)$ of $\pi_1(M_{0,5}(\mathbb{C}), \overrightarrow{v})$ is mapped to x_{kl} $(1 \le k < l \le 4)$ of $\pi_1(M_{0,4}(\mathbb{C}), \overrightarrow{v})$ by $k := i - 1$, $l := j - 1$, except the cases with $f_2(x_{ij}) = 1$ $(2 \in \{i, j\})$ or with $f_2(x_{1j}) = x_{1,j-1}$ $(j = 3, 4, 5)$. We have $x_{12}x_{13}x_{23} = 1$ in $\pi_1(M_{0,4}(\mathbb{C}), \overrightarrow{v})$ freely generated by $x := x_{12}$ and $y := x_{23}$.)
p.305, line 9: The first line of the table should read:

$\sharp$	$\mathrm{gr}^2_\Gamma f_{5*}(\sharp)$	$-\mathrm{gr}^2_\Gamma f_{4*}(\sharp)$	$\mathrm{gr}^2_\Gamma f_{3*}(\sharp)$	$-\mathrm{gr}^2_\Gamma f_{2*}(\sharp)$	$\mathrm{gr}^2_\Gamma f_{1*}(\sharp)$

▶ Misprints in [NW20]:
p.603, line −11: "$(-1)^{m-1}$-multiple" should read "$(-1)^m$-multiple".
p.603, line −10: The LHS of $(**)$ should read $(-1)^m \ell i_m(z, \gamma)$.
p.603, line −8: The equality should read $d_{i+1} = \tilde{\chi}^z_{i+1}(\sigma)/i!$ $(i \ge 0)$.

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