Landen's trilogarithm functional equation and ℓ-adic Galois multiple polylogarithms

In memory of Toshie Takata

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Abstract. The Galois action on the pro-ℓ étale fundamental groupoid of the projective line minus three points with rational base points gives rise to a non-commutative formal power series in two variables with ℓ-adic coefficients, called the ℓ-adic Galois associator. In the present paper, we focus on how Landen's functional equation of trilogarithms and its ℓ-adic Galois analog can be derived algebraically from the S_3-symmetry of the projective line minus three points. Twofold proofs of the functional equation will be presented, one is based on Zagier's tensor criterion devised in the framework of graded Lie algebras and the other is based on the chain rule for the associator power series. In the course of the second proof, we are led to investigate ℓ-adic Galois multiple polylogarithms appearing as regular coefficients of the ℓ-adic Galois associator. As an application, we show an ℓ-adic Galois analog of Oi-Ueno's functional equation between $Li_{1,\dots,1,2}(1-z)$ and $Li_k(z)$'s $(k=1,2,\dots)$.

Contents

1. Introduction

The study of polylogarithms, especially their functional equations, originated in the late 18th century by Euler, Landen, and others. The classical polylogarithm they studied is a complex function defined by the following power series

$$Li_k(z) := \frac{z}{1^k} + \frac{z^2}{2^k} + \frac{z^3}{3^k} + \cdots \ (|z|<1).$$

For $k=2$, it is called the dilogarithm, and for $k=3$, it is called the trilogarithm. The multiple polylogarithm $Li_{\mathbf{k}}(z)$ for a multi-index $\mathbf{k}=(k_1\dots,k_d)\in\mathbb{N}^d$ generalizes $Li_k(z)$, which is defined by the power series

$$Li_{\mathbf{k}}(z) := \sum_{0<n_1<\cdots<n_d} \frac{z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}} \quad (|z|<1).$$

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Note that $Li_k(z) = Li_{(k)}(z)$. The functions $Li_{\mathbf{k}}(z)$ can be analytically continued to a holomorphic function on the universal covering space of the three punctured Riemann sphere $\mathbf{P}^1(\mathbb{C}) - \{0,1,\infty\}$. There are known a number of functional equations between these functions evaluated at points with suitably chosen tracking paths from the unit segment $(0,1)$ on $\mathbf{P}^1(\mathbb{C}) - \{0,1,\infty\}$. For example, the following formulas are typical:

$$Li_2(z) + Li_2(1-z) = \zeta(2) - \log(z)\log(1-z), \tag{1.1}$$

$$Li_2(z) + Li_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\log^2(1-z), \tag{1.2}$$

$$\begin{aligned} &Li_3(z) + Li_3(1-z) + Li_3\left(\frac{z}{z-1}\right) \\ &\quad = \zeta(3) + \zeta(2)\log(1-z) - \frac{1}{2}\log(x)\log^2(1-z) + \frac{1}{6}\log^3(1-z). \end{aligned} \tag{1.3}$$

The former (1.1) is due to Leonhard Euler [E1768] and the latter two (1.2)-(1.3) are due to John Landen [L1780]. See Lewin's book [L81] for many other functional equations for polylogarithms. As for multiple polylogarithms, in [Oi09]-[OU13], Shu Oi and Kimio Ueno showed the following functional equation:

$$\sum_{j=0}^{k-1} Li_{k-j}(z)\frac{(-\log z)^j}{j!} + Li_{\underbrace{1,\ldots,1}_{k-2 \text{ times}},2}(1-z) = \zeta(k) \qquad (k \geq 2). \tag{1.4}$$

Let ℓ be a fixed prime. The ℓ-adic Galois multiple polylogarithm

$$Li^{\ell}_{\mathbf{k}}(z) \left(= Li^{\ell}_{\mathbf{k}}(\gamma_z : \overrightarrow{01}\rightsquigarrow z)\right) : G_K \to \mathbb{Q}_\ell$$

is a function on the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ of a subfield K of $\mathbb{C}$ defined, for $\mathbf{k} = (k_1 \ldots, k_d) \in \mathbb{N}^d$ and an ℓ-adic étale path γ_z from $\overrightarrow{01}$ to a K-rational (tangential) point z on $\mathbb{P}^1 - \{0,1,\infty\}$, as a certain (signed) coefficient of the non-commutative formal power series

$$\mathfrak{f}^{\gamma_z}_\sigma(X,Y) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle \quad (\sigma \in G_K) \tag{1.5}$$

called the ℓ-adic Galois associator. The functions $Li^{\ell}_{\mathbf{k}}(z)$ were originally introduced and called *the ℓ-adic iterated integrals* in a series of papers by Zdzisław Wojtkowiak (cf. e.g., [W0]-[W3]). In particular,

$$\boldsymbol{\zeta}^{\ell}_{\mathbf{k}}(\sigma) := Li^{\ell}_{\mathbf{k}}(\delta : \overrightarrow{01}\rightsquigarrow\overrightarrow{10})(\sigma) \tag{1.6}$$

for the standard path δ along the unit interval $(0,1) \subset \mathbb{R}$. For $z \in K$ with a path $\gamma_z : \overrightarrow{01}\rightsquigarrow z$, we also write

$$\rho_z (= \rho_{\gamma_z}) : G_K \to \mathbb{Z}_\ell$$

for the Kummer 1-cocycle of the ℓ-th power roots $\{z^{1/\ell^n}\}_n$ determined by γ_z.

In [NW12], Wojtkowiak and the first named author of the present paper devised Zagier's tensor criterion for functional equations as a means to calculate exact forms of identities with lower degree terms for both complex and ℓ-adic Galois polylogarithms. Applying the method, we established a few examples of functional equations in both polylogarithms. In particular, the above (1.1) and (1.2) were shown to have the following ℓ-adic Galois counterparts:

$$Li^{\ell}_2(z)(\sigma) + Li^{\ell}_2(1-z)(\sigma) = \boldsymbol{\zeta}^{\ell}_2(\sigma) - \rho_z(\sigma)\rho_{1-z}(\sigma), \tag{1.7}$$

$$Li^{\ell}_2(z)(\sigma) + Li^{\ell}_2\left(\frac{z}{z-1}\right)(\sigma) = -\frac{\rho_{1-z}(\sigma)^2 + \rho_{1-z}(\sigma)}{2} \tag{1.8}$$

for $\sigma \in G_K$ (cf. [NW12]. See §below for some adjustment of notations.)

The purpose of this paper is to provide algebraic proofs of (1.3) and (1.4) which can be used to obtain their ℓ-adic Galois analogs reading as follows:

Theorem 1.1 (ℓ-adic Galois analog of the Landen trilogarithm functional equation)**.** *There are suitable paths* $\overrightarrow{01}\rightsquigarrow 1-z, \overrightarrow{01}\rightsquigarrow \frac{z}{1-z}$ *associated to a given path* $\gamma_z : \overrightarrow{01}\rightsquigarrow z$ *such that the following functional equation*

$$\begin{aligned} &Li_3^\ell(z)(\sigma) + Li_3^\ell(1-z)(\sigma) + Li_3^\ell\left(\frac{z}{z-1}\right)(\sigma) \\ &= \boldsymbol{\zeta}_3^\ell(\sigma) - \boldsymbol{\zeta}_2^\ell(\sigma)\rho_{1-z}(\sigma) + \frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)^2 - \frac{1}{6}\rho_{1-z}(\sigma)^3 - \frac{1}{2}Li_2^\ell(z)(\sigma) - \frac{1}{12}\rho_{1-z}(\sigma) - \frac{1}{4}\rho_z(\sigma)^2 \end{aligned}$$

holds for $\sigma \in G_K$.

Theorem 1.2 (ℓ-adic Galois analog of the Oi-Ueno functional equation)**.**

$$\sum_{j=0}^{k-1} Li_{k-j}^\ell(z)(\sigma)\frac{\rho_z(\sigma)^j}{j!} + Li^\ell_{\underbrace{1,\dots,1}_{k-2\ \ times},2}(1-z)(\sigma) = \boldsymbol{\zeta}_k^\ell(\sigma) \quad (\sigma \in G_K).$$

Remark 1.3. In [S21], the second named author showed that the functional equation (1.7) has an application to a reciprocity law of the triple mod-$\{2,3\}$ symbols of rational primes via Ihara-Morishita theory (cf. [HM19]). Theorem 1.2 was shortly announced in a talk by the first named author at online Oberwolfach meeting ([N21]).

The contents of this paper will be arranged as follows: After a quick set up in §2 on the notations of standard paths on $\mathbf{P}^1 - \{0, 1, \infty\}$, in §3 we discuss complex and ℓ-adic Galois associators as formal power series in two non-commuting variables, and define the multiple polylogarithms as their coefficients of certain monomials. We then review in the complex analytic context that (1.3) and (1.4) can be derived from algebraic relations (chain rules) of associators along simple compositions of paths. With this line in mind, we prove Theorems 1.1 and 1.2 in the ℓ-adic Galois case by tracing arguments in parallel ways to the complex case. In §4, after shortly recalling polylogarithmic characters introduced in a series of collaboration by Wojtkowiak and the first named author, we present $\mathbb{Z}_\ell$-integrality test for ℓ-adic Galois Landen's equation obtained in Theorem 1.1. Section 5 turns to an alternative approach to functional equations of polylogarithms based on a set of tools devised in [NW12] to enhance Zagier's tensor criterion for functional equations into a concrete form. Then we give alternative proofs of (1.3) and Theorem 1.1 with this method. Appendix will be devoted to exhibiting lower degree terms of the complex and ℓ-adic Galois associators as a convenient reference from the text.

2. Set up

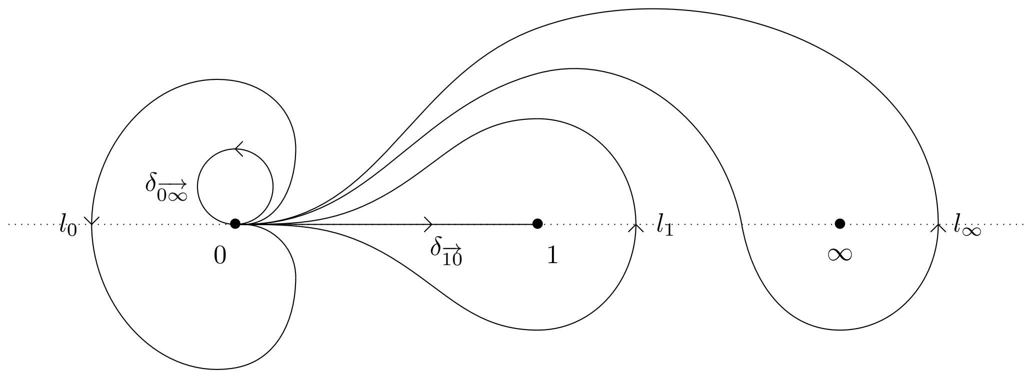
Fix a prime number ℓ. Let K be a subfield of the complex number field $\mathbb{C}$, $\overline{K}$ the algebraic closure of K in $\mathbb{C}$, and $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. Let $U := \mathbf{P}^1_K - \{0, 1, \infty\}$ be the projective line minus three points over K, $U_{\overline{K}}$ the base-change of U via the inclusion $K \hookrightarrow \overline{K}$, and $U^{\mathrm{an}} = \mathbf{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ the complex analytic space associated to the base-change of $U_{\overline{K}}$ via the inclusion $\overline{K} \hookrightarrow \mathbb{C}$.

In the following, we shall write $\overrightarrow{01}$ for the standard K-rational tangential base point on U. Let z be a K-rational point of U or a K-rational tangential base point on U. We consider $\overrightarrow{01}$, z also as points on $U_{\overline{K}}$ or U^{an} by inclusions $K \hookrightarrow \overline{K}$ and $\overline{K} \hookrightarrow \mathbb{C}$.

Let $\pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, z)$ be the set of homotopy classes of topological paths on U^{an} from $\overrightarrow{01}$ to z, and let $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}; \overrightarrow{01}, z)$ be the pro-ℓ-finite set of pro-ℓ étale paths on $U_{\overline{K}}$ from $\overrightarrow{01}$ to z. Note that there is a canonical comparison map

$$\pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, z) \to \pi_1^{\ell\text{-ét}}(U_{\overline{K}}; \overrightarrow{01}, z)$$

that allows us to consider topological paths on U^{an} as pro-ℓ étale paths on $U_{\overline{K}}$.



The dashed line represents $\mathbf{P}^1(\mathbb{R}) - \{0, 1, \infty\}$.
The upper half-plane is above the dashed line.

Let l_0, l_1, l_∞ be the topological paths on $U^{\rm an}$ with base point $\overrightarrow{01}$ circling counterclockwise around $0, 1, \infty$, respectively. Then, $\{l_0, l_1\}$ is a free generating system of the topological fundamental group $\pi_1^{\rm top}(U^{\rm an}, \overrightarrow{01}) := \pi_1^{\rm top}(U^{\rm an}; \overrightarrow{01}, \overrightarrow{01})$ or the pro-ℓ étale fundamental group $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01}) := \pi_1^{\ell\text{-ét}}(U_{\overline{K}}; \overrightarrow{01}, \overrightarrow{01})$. Then, $\pi_1^{\rm top}(X^{\rm an}, \overrightarrow{01})$ is a free group of rank 2 generated by $\{l_0, l_1\}$ and $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$ is a free pro-ℓ group of rank 2 topologically generated by $\{l_0, l_1\}$.

Fix a topological path $\gamma_z \in \pi_1^{\rm top}(U^{\rm an}; \overrightarrow{01}, z)$ on $U^{\rm an}$ from $\overrightarrow{01}$ to z. Moreover, let $\delta_{\overrightarrow{10}} \in \pi_1^{\rm top}(U^{\rm an}; \overrightarrow{01}, \overrightarrow{10})$ be the topological path on $U^{\rm an}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ along the real interval, and let $\delta_{\overrightarrow{0\infty}} \in \pi_1^{\rm top}(U^{\rm an}; \overrightarrow{01}, \overrightarrow{0\infty})$ be the topological path on the upper half-plane in $U^{\rm an}$ from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$.

Let $\phi, \psi \in {\rm Aut}(U^{\rm an})$ be automorphisms of $U^{\rm an}$ defined by

$$\phi(t) = 1 - t, \quad \psi(t) = \frac{t}{t-1}, \tag{2.1}$$

and introduce specific paths from $\overrightarrow{01}$ to $1 - z$ and to $\frac{z}{z-1}$ by

$$\begin{cases} \gamma_{1-z} & := \delta_{\overrightarrow{10}} \cdot \phi(\gamma_z) \in \pi_1^{\rm top}(U^{\rm an}; \overrightarrow{01}, 1-z), \\ \gamma_{\frac{z}{z-1}} & := \delta_{\overrightarrow{0\infty}} \cdot \psi(\gamma_z) \in \pi_1^{\rm top}\left(U^{\rm an}; \overrightarrow{01}, \dfrac{z}{z-1}\right). \end{cases} \tag{2.2}$$

Here, paths are composed from left to right.

3. **Associators and multiple polylogarithms**

Recall that the multiple polylogarithms appear as coefficients of the non-commutative formal power series in two variables with complex coefficients, determined as the basic solution of the KZ equation (Knizhnik-Zamolodchikov equation) on $\mathbf{P}^1(\mathbb{C}) - \{0, 1, \infty\}$. More precisely, let $G_0(X, Y)(z)$ be the fundamental solution of the formal KZ equation

$$\frac{d}{dz} G(X, Y)(z) = \left(\frac{X}{z} + \frac{Y}{z-1}\right) G(X, Y)(z)$$

on $\mathbf{P}^1(\mathbb{C}) - \{0, 1, \infty\}$, which is an analytic function with values in $\mathbb{C}\langle\langle X, Y\rangle\rangle$ characterized by the asymptotic behavior $G_0(X, Y)(z) \approx z^X \ (z \to 0)$ and analytically continued to the universal cover of

$\mathbf{P}^1(\mathbb{C})-\{0,1,\infty\}$. Let M be the non-commutative free monoid generated by the non-commuting indeterminates X,Y. One can expand $G_0(X,Y)(z)$ in the words $w\in \mathrm{M}$ in the form

$$G_0(X,Y)(z)=1+\sum_{w\in \mathrm{M}\setminus\{1\}} c_w(\gamma_z)\cdot w. \tag{3.1}$$

The multiple polylogarithm $Li_{\mathbf{k}}(\gamma_z)$ associated to a tuple $\mathbf{k}=(k_1\dots,k_d)\in\mathbb{N}^d$ and a topological path γ_z from $\overrightarrow{01}$ to z is equal to the coefficient of $G_0(X,Y)(z)$ at the 'regular' word $w(\mathbf{k}):=X^{k_d-1}Y\cdots X^{k_1-1}Y$ multiplied by $(-1)^d$ (where 'regular' means that the word ends in the letter Y). In summary, writing the length d of the tuple $\mathbf{k}=(k_1\dots,k_d)$ as $\mathrm{dep}(\mathbf{k})$, we have

$$Li_{\mathbf{k}}(\gamma_z)=(-1)^{\mathrm{dep}(\mathbf{k})}c_{w(\mathbf{k})}(\gamma_z). \tag{3.2}$$

To define the ℓ-adic Galois multiple polylogarithms, we make use of the G_K-action on the étale paths instead of the fundamental KZ-solution. Given a pro-ℓ étale path $\gamma_z\in\pi_1^{\ell\text{-ét}}(U_{\overline{K}};\overrightarrow{01},z)$, form a pro-$\ell$ étale loop $\mathfrak{f}_\sigma^{\gamma}:=\gamma\cdot\sigma(\gamma)^{-1}\in\pi_1^{\ell\text{-ét}}(U_{\overline{K}},\overrightarrow{01})$, and expand it via the Magnus embedding $\pi_1^{\ell\text{-ét}}(U_{\overline{K}},\overrightarrow{01})\hookrightarrow\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ defined by $l_0\mapsto\exp(X)$, $l_1\mapsto\exp(Y)$.

Notation 3.1. By abuse of notation, we shall write the above image of $\mathfrak{f}_\sigma^{\gamma_z}\in\pi_1^{\ell\text{-ét}}(U_{\overline{K}},\overrightarrow{01})$ in $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ as

$$\mathfrak{f}_\sigma^{\gamma_z}(X,Y)=1+\sum_{w\in \mathrm{M}\setminus\{1\}} c_w^\ell(\gamma_z)(\sigma)\cdot w \qquad (\sigma\in G_K). \tag{3.3}$$

This is what we described in 1.5. In the parallel way to the above (3.2), for any tuple $\mathbf{k}$ of positive integers, we define the ℓ-adic Galois multiple polylogarithm $Li_{\mathbf{k}}^\ell$ to be the function $G_K\to\mathbb{Q}_\ell$ determined by

$$Li_{\mathbf{k}}^\ell(\gamma_z)(\sigma)=(-1)^{\mathrm{dep}(\mathbf{k})}c^\ell_{w(\mathbf{k})}(\gamma_z)(\sigma) \tag{3.4}$$

for $\sigma\in G_K$. The ℓ-adic zeta function $\boldsymbol{\zeta}_{\mathbf{k}}^\ell: G_{\mathbb{Q}}\to\mathbb{Q}_\ell$ is a special case given with (1.6) in Introduction.

Remark 3.2. It is worth noting that the ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)\in\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ is the ℓ-adic Galois analog of the Drinfeld associator

$$\Phi(X,Y):=\Big(G_0(Y,X)(1-z)\Big)^{-1}\cdot G_0(X,Y)(z)\in\mathbb{C}\langle\langle X,Y\rangle\rangle.$$

We summarize analogy between ℓ-adic Galois and complex associators as Table 1, where the 3rd and 4th rows reflect the chain rules of associators in regards of the path compositions (2.2).

TABLE 1

ℓ-adic Galois side	complex side
$\mathfrak{f}_\sigma^{\gamma_z}(X,Y)\in\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$	$G_0(X,Y)(z)\in\mathbb{C}\langle\langle X,Y\rangle\rangle$
$\mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)\in\mathbb{Q}_\ell\langle\langle X,Y\rangle$	$\Phi(X,Y)\in\mathbb{C}\langle X,Y\rangle\rangle$
$\mathfrak{f}_\sigma^{\gamma_z}(X,Y)=\mathfrak{f}_\sigma^{\gamma_{1-z}}(Y,X)\cdot\mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)$	$G_0(X,Y)(z)=G_0(Y,X)(1-z)\cdot\Phi(X,Y)$
$\mathfrak{f}_\sigma^{\gamma_{\frac{z}{z-1}}}(X,Y)=\mathfrak{f}_\sigma^{\gamma_z}(X,Z)\cdot\mathfrak{f}_\sigma^{\delta_{\overrightarrow{0\infty}}}(X,Y)$, $Z:=\log(\exp(-Y)\exp(-X))$	$G_0(X,Y)\left(\frac{z}{z-1}\right)=G_0(X,Z)(z)\cdot\exp(\pi\mathtt{i}X)$, $Z:=-Y-X$
$Li_{\mathbf{k}}^\ell(\gamma_z)(\sigma)$: ℓ-adic Galois multiple polylog value	$Li_{\mathbf{k}}(z)$: multiple polylog value
$\boldsymbol{\zeta}_{\mathbf{k}}^\ell(\sigma)$: ℓ-adic Galois multiple zeta value	$\zeta(\mathbf{k})$: multiple zeta value

Algebraic proof of (1.3)-(1.4). The following arguments are motivated from an enlightening remark given in Appendix of Furusho's lecture note [F14, A.24]. By the explicit formula of Le-Murakami [LM96] type due to Furusho [F04, Theorem 3.15], the coefficient of YX^{k-1} in $G_0(X,Y)(z)$ is

$$\mathtt{Coeff}_{YX^{k-1}}(G_0(X,Y)(z)) = -\sum_{\substack{s+t=k-1\\ s,t\geq 0}} (-1)^s Li_{f'(B⧢A^s)}(z)\frac{\log^t z}{t!} = (-1)^k\sum_{t=0}^{k-1}(-1)^t Li_{k-t}(z)\frac{\log^t z}{t!},$$

where f' indicates the operation annihilating terms ending with the letter X. Applying this to the chain rule $G_0(Y,X)(1-z) = G_0(X,Y)(z)\cdot\Phi(Y,X)$ from Table 1, we see that

$$\mathtt{Coeff}_{YX^{k-1}}(G_0(Y,X)(1-z)) = \mathtt{Coeff}_{XY^{k-1}}(G_0(X,Y)(1-z)) = (-1)^{k-1}Li_{\underbrace{1,\dots,1}_{k-2\text{ times}},2}(1-z)$$

is equal to

$$\begin{aligned}&\mathtt{Coeff}_{YX^{k-1}}(G_0(X,Y)(z)) + \mathtt{Coeff}_{YX^{k-1}}(\Phi(Y,X))\\ &= \mathtt{Coeff}_{YX^{k-1}}(G_0(X,Y)(z)) + \mathtt{Coeff}_{XY^{k-1}}(\Phi(X,Y))\\ &= (-1)^k\sum_{t=0}^{k-1}(-1)^t Li_{k-t}(z)\frac{\log^t z}{t!} + (-1)^{k-1}\zeta(\underbrace{1,\dots,1}_{k-2\text{ times}},2)\end{aligned}$$

Here we used a tautological identity $\mathtt{Coeff}_{w(A,B)}(f(A,B)) = \mathtt{Coeff}_{w(B,A)}(f(B,A))$ and the fact that $\mathtt{Coeff}_{X^i}(\Phi(X,Y)) = \mathtt{Coeff}_{Y^i}(\Phi(X,Y)) = 0$ for all $i\geq 1$. This together with the well known identity $\zeta(\underbrace{1,\dots,1}_{k-2\text{ times}},2) = \zeta(k)$ (duality formula) derives (1.4).

Before going to prove (1.3), we compare the coefficients of YXY in the same identity $G_0(X,Y)(z) = G_0(Y,X)(1-z)\cdot\Phi(X,Y)$ from Table 1. By simple calculation, we obtain

$$\begin{aligned}c_{YXY}(\gamma_z) &= -\zeta(2)c_X(\gamma_{1-z}) - 2\zeta(3) + c_{XYX}(\gamma_{1-z})\\ &= -\zeta(2)c_X(\gamma_{1-z}) - 2\zeta(3) + \big(c_{XY}(\gamma_{1-z})c_X(\gamma_{1-z}) - 2c_{X^2Y}(\gamma_{1-z})\big)\end{aligned}\tag{3.5}$$

where, in the former equaltiy are used known identities (cf. Appendix) $\mathtt{Coeff}_{XY}(\Phi(X,Y)) = -\zeta(2)$, $\mathtt{Coeff}_{YXY}(\Phi(X,Y)) = -2\zeta(3)$, and in the last equality is used the shuffle relation according to $XY⧢X = XYX + 2X^2Y$. This leads to

$$Li_{2,1}(z) = -\frac{\pi^2}{6}\log(1-z) - 2\zeta(3) - Li_2(1-z)\log(1-z) + 2Li_3(1-z).\tag{3.6}$$

Now let us compare the coefficients of X^2Y in the both sides of the chain rule

$$G_0(X,Y)\left(\frac{z}{z-1}\right) = G_0(X,Z)(z)\cdot\exp(\pi\mathtt{i}X)$$

from Table 1. It follows easily that

$$c_{XXY}(\gamma_{\frac{z}{z-1}}) = -c_{XXY}(\gamma_z) - c_{YYY}(\gamma_z) + c_{XYY}(\gamma_z) + c_{YXY}(\gamma_z),\tag{3.7}$$

or equivalently,

$$-Li_3\left(\frac{z}{z-1}\right) = Li_3(z) + Li_{1,1,1}(z) + Li_{1,2}(z) + Li_{2,1}(z).\tag{3.8}$$

We know from the case $k=3$ of (1.4) with interchange $z\leftrightarrow 1-z$ that

$$Li_{1,2}(z) = \zeta(3) - \left(Li_3(1-z) - Li_2(1-z)\log(1-z) - \frac{1}{2}\log z\log^2(1-z)\right).\tag{3.9}$$

Putting (3.6) and (3.9) into the last two terms of (3.8) with noticing $Li_{1,1,1}(z) = -\frac{1}{6}\log^3(1-z)$ (cf. Appendix), we obtain a proof of Landen's trilogarithm functional equation (1.3). □

Proof of Theorem 1.2: In the ℓ-adic Galois setting, the argument for the assertion goes in almost parallel way to the above proof for (1.4). In fact, the formula of Le-Murakami and Furusho type is generalized to any group-like elements of $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ in [N21b], so that it holds that

$$\texttt{Coeff}_{YX^{k-1}}(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) = (-1)^k \sum_{t=0}^{k-1} (-1)^t Li_{k-t}^{\ell}(\gamma_z)\frac{(-\rho_z(\sigma))^t}{t!}. \tag{3.10}$$

Comparing the coefficients of YX^{k-1} of the identity

$$\mathfrak{f}_\sigma^{\gamma_z}(X,Y) = \mathfrak{f}_\sigma^{\gamma_{1-z}}(Y,X)\cdot \mathfrak{f}_\sigma^{\delta_{\overrightarrow{10}}}(X,Y)$$

in Table 1, we obtain

$$\sum_{j=0}^{k-1} Li_{k-j}^{\ell}(\gamma_z)(\sigma)\frac{\rho_z(\sigma)^j}{j!} + Li^{\ell}_{\underbrace{1,\dots,1}_{k-2\ \text{times}},2}(\gamma_{1-z})(\sigma) = \boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{k-2\ \text{times}},2}(\sigma) \quad (\sigma\in G_K). \tag{3.11}$$

Note here that, in the special case $z=\overrightarrow{10}$ with $\gamma_z=\delta_{\overrightarrow{10}}$, we should interpret that $Li^{\ell}_{1,\dots,1,2}(\gamma_{1-z})(\sigma)=0$ and that $\rho_z(\sigma)^j = 0,1$ according to whether $j>0$ or $j=0$, from which we obtain the duality formula:

$$\boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{k-2\ \text{times}},2}(\sigma) = \boldsymbol{\zeta}^{\ell}_{k}(\sigma) \qquad (\sigma\in G_K). \tag{3.12}$$

Putting this back to (3.11) settles the proof of Theorem 1.2. □

Proof of Theorem 1.1: We only have to examine the ℓ-adic Galois versions of the identities (3.6), (3.8) and (3.9) with replacing the role of $G_0(X,Y)(\gamma_*)$ by $\mathfrak{f}_\sigma^{\gamma_*}(X,Y)$. It turns out that the two identities (3.6), (3.9) have exactly the parallel counterparts:

$$Li^{\ell}_{2,1}(\gamma_z)(\sigma) = \boldsymbol{\zeta}^{\ell}_2(\sigma)\rho_{1-z}(\sigma) + \boldsymbol{\zeta}^{\ell}_{2,1}(\sigma) + Li^{\ell}_2(\gamma_{1-z})(\sigma)\rho_{1-z}(\sigma) + 2Li^{\ell}_3(\gamma_{1-z})(\sigma), \tag{3.13}$$

$$Li^{\ell}_{1,2}(\gamma_z)(\sigma) = \boldsymbol{\zeta}^{\ell}_3(\sigma) - \left(Li^{\ell}_3(\gamma_{1-z})(\sigma) + Li^{\ell}_2(\gamma_{1-z})(\sigma)\rho_{1-z}(\sigma) + \frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)^2\right) \tag{3.14}$$

with $\sigma\in G_K$. There occurs a small difference for (3.8) when evaluating the identity $\mathfrak{f}_\sigma^{\gamma_{\frac{z}{z-1}}}(X,Y) = \mathfrak{f}_\sigma^{\gamma_z}(X,Z)\cdot\mathfrak{f}_\sigma^{\delta_{\overrightarrow{0\infty}}}(X,Y)$ with $Z:=\log(\exp(-Y)\exp(-X))$ as the Campbell-Hausdorff sum. At the level of coefficients of ℓ-adic Galois associators, the complex case (3.7) turns to have extra additional terms as:

$$\begin{aligned} c^{\ell}_{XXY}(\gamma_{\frac{z}{z-1}})(\sigma) &= -c^{\ell}_{XXY}(\gamma_z)(\sigma) - c^{\ell}_{YYY}(\gamma_z)(\sigma) + c^{\ell}_{XYY}(\gamma_z)(\sigma) + c^{\ell}_{YXY}(\gamma_z)(\sigma)\\ &\quad - \left(\frac{1}{2}c^{\ell}_{XY}(\gamma_z)(\sigma) - \frac{1}{2}c^{\ell}_{YY}(\gamma_z)(\sigma) + \frac{1}{12}c^{\ell}_{Y}(\gamma_z)(\sigma)\right), \end{aligned} \tag{3.15}$$

from which follows that

$$\begin{aligned} Li^{\ell}_3(\gamma_{\frac{z}{z-1}})(\sigma) &= -Li^{\ell}_3(\gamma_z)(\sigma) - Li^{\ell}_{1,1,1}(\gamma_z)(\sigma) - Li^{\ell}_{1,2}(\gamma_z)(\sigma) - Li^{\ell}_{2,1}(\gamma_z)(\sigma)\\ &\quad - \left(\frac{1}{2}Li^{\ell}_2(\gamma_z)(\sigma) + \frac{1}{4}\rho_{1-z}(\sigma)^2 + \frac{1}{12}\rho_{1-z}(\sigma)\right) \end{aligned} \tag{3.16}$$

for $\sigma\in G_K$. The asserted formula follows from (3.16) after $Li^{\ell}_{1,2}(\gamma_z)(\sigma)$, $Li^{\ell}_{2,1}(\gamma_z)(\sigma)$ in the RHS are replaced by the equations (3.14), (3.13) respectively and from knowledge of a few coefficients of $\mathfrak{f}_\sigma^{\gamma_*}(X,Y)$ in lower degrees (cf. Appendix). □

4. Polylogarithmic characters and $\mathbb{Z}_\ell$-integrality test

There is a specific series of functions $\tilde{\chi}_m^z : G_K \to \mathbb{Z}_\ell$ (called the polylogarithmic characters) closely related to the ℓ-adic Galois polylogarithms $Li_k^\ell(z) : G_K \to \mathbb{Q}_\ell$.

Definition 4.1 ([NW99]: ℓ-adic Galois polylogarithmic character)**.** For each $m \in \mathbb{N}$ and $\sigma \in G_K$, we define $\tilde{\chi}_m^{\gamma_z}(\sigma)$ (often written shortly as $\tilde{\chi}_m^z(\sigma)$) by the (sequential) Kummer properties

$$\zeta_{\ell^n}^{\tilde{\chi}_m^z(\sigma)} = \sigma\left(\prod_{i=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{\chi(\sigma)^{-1}i}z^{1/\ell^n})^{\frac{i^{m-1}}{\ell^n}}\right)\Big/\prod_{i=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{i+\rho_z(\sigma)}z^{1/\ell^n})^{\frac{i^{m-1}}{\ell^n}}$$

over $n \in \mathbb{N}$, where the roots $z^{1/n}, (1-z)^{1/n}, (1-\zeta_n^a z^{1/n})^{1/m}$ $(n, m \in \mathbb{N}, a \in \mathbb{Z})$ are chosen along the path $\gamma_z \in \pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, z)$, $\rho_z(= \rho_{\gamma_z}) : G_K \to \mathbb{Z}_\ell$ is the Kummer 1-cocycle of the ℓ-th power roots $\{z^{1/\ell^n}\}_n$ along γ_z, and $\chi : G_K \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character. We call the function

$$\tilde{\chi}_m^z\,(= \tilde{\chi}_m^{\gamma_z}) : G_K \to \mathbb{Z}_\ell$$

the (ℓ-adic Galois) polylogarithmic character associated to $\gamma_z \in \pi_1^{\mathrm{top}}(U^{\mathrm{an}}; \overrightarrow{01}, z)$.

We first begin with summarizing the relations between the polylogarithmic characters and ℓ-adic Galois polylogarithms:

Proposition 4.2. *Let $\mathfrak{f}_\sigma^{\gamma_z}(X,Y)$ be the Magnus expansion of the ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\gamma_z}$ as in (3.3). Then, we have:*

$$\text{(i)}\qquad \mathtt{Coeff}_{YX^{m-1}}(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) = -\frac{\tilde{\chi}_m^z(\sigma)}{(m-1)!}\ \left(= (-1)^m\sum_{k=0}^{m-1} Li_{m-k}^\ell(\gamma_z)(\sigma)\frac{\rho_z(\sigma)^k}{k!}\right),$$

$$\text{(ii)}\qquad \mathtt{Coeff}_{X^{m-1}Y}(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)) = (-1)^m\sum_{k=0}^{m-1}\frac{\rho_z(\sigma)^k}{k!}\frac{\tilde{\chi}_{m-k}^z(\sigma)}{(m-1-k)!}\ \left(= -Li_m^\ell(\gamma_z)(\sigma)\right)$$

for $\sigma \in G_K$.

Proof. The first equality of (i) is proved in [NW20, Proposition 8 (ii)], where the symbol $\mathtt{Li}_w$ in loc.cit. differs from our Li_w by the sign corresponding to the parity of the number of appearances of letter Y in w. (ii) follows from (3.10), i.e., is a consequence of the special case of the formula of Le-Murakami and Furusho type generalized to group-like elements in [N21b]. Note that the equality in the bracket of (ii) is just due to our definition of $Li_{\mathbf{k}}^\ell$ (3.4). The equality in the bracket of (i) follows from (ii) by inductively reversing the sequence $\{Li_m^\ell\}_m$ to $\{\tilde{\chi}_m\}_m$. □

Often we prefer a functional equation of ℓ-adic Galois polylogarithms expressed in a form of the corresponding identity between polylogarithmic characters, because the latter enables us to check the $\mathbb{Z}_\ell$-integrality of both sides of the equation.

For example, the functional equations (1.7), (1.8) are equivalent to

$$\tilde{\chi}_2^z(\sigma) + \tilde{\chi}_2^{1-z}(\sigma) + \rho_z(\sigma)\rho_{1-z}(\sigma) = \frac{1}{24}(\chi(\sigma)^2-1), \tag{4.1}$$

$$\tilde{\chi}_2^z(\sigma) + \tilde{\chi}_2^{z/(1-z)}(\sigma) = -\frac{1}{2}\rho_{1-z}(\rho_{1-z}(\sigma)-\chi(\sigma)) \tag{4.2}$$

for each $\sigma \in G_K$ respectively. Noting that $\chi(\sigma) \equiv 1 \pmod 2$ and $\chi(\sigma)^2 \equiv 1 \pmod{24}$, we easily see that each of the RHSs has no denominator, i.e., $\in \mathbb{Z}_\ell$ for every prime ℓ. From this viewpoint, it is worth rewriting Landen's trilogarithm functional equation (Theorem 1.1) in terms of polylogarithmic characters. By simple computation, it results in:

$$\tilde{\chi}_3^{z}(\sigma)+\tilde{\chi}_3^{1-z}(\sigma)+\tilde{\chi}_3^{z/(z-1)}(\sigma)=\tilde{\chi}_3^{\overrightarrow{10}}(\sigma)+\chi(\sigma)\tilde{\chi}_2^{z}(\sigma)+\rho_z(\sigma)\rho_{1-z}(\sigma)^2-\frac{\rho_{1-z}(\sigma)}{12}(\chi(\sigma)^2-1)$$
$$-\frac{\rho_{1-z}(\sigma)}{6}\Big(\chi(\sigma)-\rho_{1-z}(\sigma)\Big)\Big(\chi(\sigma)-2\rho_{1-z}(\sigma)\Big). \tag{4.3}$$

It is not difficult to see that each term of the above right hand side has no denominator in $\mathbb{Z}_\ell$.

5. Tensor criterion for Landen's equation for Li_3

It would be worth giving alternative proofs of complex/ℓ-adic Galois Landen's trilogarithm functional equations (1.3) and Theorem 1.1 with the method of [NW12] not only for checking the validity of proofs given in §3 but also for providing a typical sample showing utility of Zagier's tensor criterion for functional equations (cf. e.g. [G13]).

Let $\mathcal{O}:=\overline{K}[t,\frac{1}{t},\frac{1}{1-t}]$ be the coordinate ring of $U_{\overline{K}}=\mathbf{P}^1_{\overline{K}}-\{0,1,\infty\}$ with unit group $\mathcal{O}^\times$, and let $f_1,f_2,f_3:U_{\overline{K}}\to U_{\overline{K}}$ be (auto)morphisms of $U_{\overline{K}}$ defined by

$$f_1(t)=t,\quad f_2(t)=1-t,\quad f_3(t)=\frac{t}{t-1}.$$

Considering $f_1,f_2,f_3:U\to\mathbf{G}_m$ as elements of $\mathcal{O}^\times$, we specialize Zagier's tensor criterion for Landen's functional equation of Li_3's in the following proposition:

Proposition 5.1 (Tensor criterion for Landen's functional equation for Li_3)**.** *In the tensor product $\mathcal{O}^\times\otimes(\mathcal{O}^\times\wedge\mathcal{O}^\times)$ of abelian groups, we have*

$$f_1\otimes(f_1\wedge(f_1-1))+f_2\otimes(f_2\wedge(f_2-1))+f_3\otimes(f_3\wedge(f_3-1))=0.$$

Proof. By simple calculations, we have:

$$\begin{aligned}&a\otimes(a\wedge b)+(b+c)\otimes((b+c)\wedge(a+c))+(a-b)\otimes((a-b)\wedge(-b))\\&=b\otimes(c\wedge a)+b\otimes(b\wedge c)+c\otimes(a\wedge b)+c\otimes(a\wedge c)+c\otimes(b\wedge c)=0\end{aligned}$$

in $\mathcal{O}^\times\otimes(\mathcal{O}^\times\wedge\mathcal{O}^\times)$. The assertion is a consequence of the special case $a:=t$, $b:=t-1$ $c:=-1$. □

To compute the functional equations in concrete forms, we shall plug the above Proposition 5.1 into [NW12, Theorem 5.7] (ii)$_\mathbb{C}$ → (iii)$_\mathbb{C}$ and (ii)$_\ell$ → (iii)$_\ell$. Fix a family of paths $\{\delta_1,\delta_2,\delta_3\}$ from $\overrightarrow{01}$ to $f_1(\overrightarrow{01})=\overrightarrow{01}$, $f_2(\overrightarrow{01})=\overrightarrow{10}$, $f_3(\overrightarrow{01})=\overrightarrow{0\infty}$, with $\delta_1:=1$(= trivial path), $\delta_2:=\delta_{\overrightarrow{10}}$, $\delta_3:=\delta_{\overrightarrow{0\infty}}$ respectively. Suppose we are given a topological path $\gamma_z:\overrightarrow{01}\rightsquigarrow z$ on $\mathbf{P}^1(\mathbb{C})-\{0,1,\infty\}$. Then, δ_i $(i=1,2,3)$ provides a natural path $\delta_i\cdot f_i(\gamma_z):\overrightarrow{01}\rightsquigarrow f_i(z)$. Below we always consider the three points $f_1(z)=z$, $f_2(z)=1-z$ and $f_3(z)=\frac{z}{z-1}$ to accompany those natural tracking paths from the base point $\overrightarrow{01}$ $(i=1,2,3)$ in this way.

5.1. Complex case.

With the notations being as above, [NW12, Theorem 5.7 (iii)$_\mathbb{C}$] asserts the existence of a functional equation of the form

$$\sum_{i=1}^{3}\mathcal{L}_{\mathbb{C}}^{\varphi_3}(f_i(z),f_i(\overrightarrow{01});f_i(\gamma_z))=0, \tag{5.1}$$

where each term can be calculated by a concrete algorithm [NW12, Proposition 5.11]. Below let us exhibit the calculation by enhancing [NW12, Examples 6.1-6.2] to their "Li_3" version, for which we start with the graded Lie-versions of complex polylogarithms, written $\mathrm{li}_k(z,\gamma_z)$ for any path $\overrightarrow{01}\rightsquigarrow z$. These can be converted to usual polylogarithms by [NW12, Proposition 5.2]; in particular, for $k=0,...,3$ we have:

$$(5.2)\qquad \begin{cases} \mathrm{li}_0(z) = -\dfrac{1}{2\pi\mathtt{i}}\log(z), \\ \mathrm{li}_1(z) = -\dfrac{1}{2\pi\mathtt{i}}\log(1-z), \\ \mathrm{li}_2(z) = \dfrac{1}{4\pi^2}\left(Li_2(z)+\dfrac{1}{2}\log(z)\log(1-z)\right), \\ \mathrm{li}_3(z) = \dfrac{1}{(2\pi\mathtt{i})^3}\left(Li_3(z)-\dfrac{1}{2}\log(z)Li_2(z)-\dfrac{1}{12}\log^2(z)\log(1-z)\right). \end{cases}$$

Each term of (5.1) relies only on the chain $f_i(\gamma_z)$ that does not start from $\overrightarrow{01}$ if $i\neq 1$, in which case we need to interpret the chain $f_i(\gamma_z)$ as the difference "$\delta_i\cdot f_i(\gamma_z)$ minus δ_i". At the level of graded Lie-version of polylogarithms, the difference can be evaluated by the polylog-BCH formula [NW12, Proposition 5.9]: In our case, a crucial role is played by the polynomial

$$(5.3)\qquad \mathsf{P}_3(\{a_j\}_{j=0}^3,\{b_j\}_{j=0}^3) = a_3+b_3+\frac{1}{2}(a_0b_2-b_0a_2)+\frac{1}{12}(a_0^2b_1-a_0a_1b_0-a_0b_0b_1+a_1b_0^2)$$

in 8 variables a_j, b_j $(j=0,\dots,3)$. Using this and applying [NW12, Proposition 5.11 (i)], we have

$$(5.4)\qquad \mathcal{L}_{\mathbb{C}}^{\varphi_3}(f_i(z),f_i(\overrightarrow{01});f_i(\gamma_z)) = \mathsf{P}_3\big(\{\mathrm{li}_j(f_i(z),\delta_i\cdot f_i(\gamma_z))\}_{j=0}^3,\{-\mathrm{li}_j(f_i(\overrightarrow{01}),\delta_i)\}_{j=0}^3\big)$$

for $i=1,2,3$. Noting then that

$$\left(-\mathrm{li}_j(\overrightarrow{01},\delta_1)\right)_{0\le j\le 3} = (0,0,0,0),$$
$$\left(-\mathrm{li}_j(\overrightarrow{10},\delta_2)\right)_{0\le j\le 3} = \left(0,0,-\mathrm{li}_2(\overrightarrow{10}),-\mathrm{li}_3(\overrightarrow{10})\right) = \left(0,0,-\frac{1}{4\pi^2}Li_2(1),-\frac{1}{(2\pi\mathtt{i})^3}Li_3(1)\right),$$
$$\left(-\mathrm{li}_j(\overrightarrow{0\infty},\delta_3)\right)_{0\le j\le 3} = \left(\frac{1}{2},0,0,0\right),$$

we compute (5.4) for $i=1,2,3$ as:

$$(5.5)\qquad \begin{cases} \mathcal{L}_{\mathbb{C}}^{\varphi_3}(z,\overrightarrow{01};\gamma_z) = \mathrm{li}_3(z), \\ \mathcal{L}_{\mathbb{C}}^{\varphi_3}(1-z,\overrightarrow{10};f_2(\gamma_z)) = \mathrm{li}_3(1-z)-\mathrm{li}_3(\overrightarrow{10})+\frac{1}{2}\mathrm{li}_0(1-z)(-\mathrm{li}_0(z)), \\ \mathcal{L}_{\mathbb{C}}^{\varphi_3)}\left(\frac{z}{z-1},\overrightarrow{0\infty};f_3(\gamma_z)\right) = \mathrm{li}_3(\frac{z}{z-1})+\frac{1}{2}\left(-\frac{1}{2}\mathrm{li}_2(\frac{z}{z-1})\right)+\frac{1}{12}\left(\frac{1}{4}\mathrm{li}_1(\frac{z}{z-1})-\frac{1}{2}\mathrm{li}_1(\frac{z}{z-1})\mathrm{li}_0(\frac{z}{z-1})\right). \end{cases}$$

Putting these together into (5.1) and applying (5.2), we obtain Landen's functional equation (1.3).

5.2. **ℓ-adic Galois case.** Let us apply [NW12, Theorem 5.7 (iii)$_\ell$] in the parallel order to our above discussion in the complex case. The ℓ-adic version of the functional equation (5.1) in loc.cit. relies on the choice of our free generator system $\vec{x}:=(l_0,l_1)$ of $\pi_1^{\ell\text{-ét}}(U_{\overline{K}},\overrightarrow{01})$ which plays an indispensable role to specify a splitting of the pro-unipotent Lie algebra of $\pi_1^{\ell\text{-ét}}$ into the weight gradation over $\mathbb{Q}_\ell$ (cf. [NW12, §4.2]). Then, the functional equation turns out in the form

$$(5.6)\qquad \sum_{i=1}^{3}\mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_i),\vec{x}}(f_i(z),f_i(\overrightarrow{01});f_i(\gamma_z))(\sigma) = E(\sigma,\gamma_z) \qquad (\sigma\in G_K)$$

where $E(\sigma,\gamma_z)$ is called the ℓ-adic error term ([NW12, §4.3]). The graded Lie-version of ℓ-adic Galois polylogarithm $\ell i_k(z,\gamma_z,\vec{x})$ (for $k\ge 1$) is then defined as the coefficient of $\mathrm{ad}(X)^{k-1}(Y)=[X,[X,[\cdots[X,Y]..]$ in $\log(\mathfrak{f}_\sigma^{\gamma_z}(X,Y)^{-1})$ as an element of Lie formal series $\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X,Y\rangle\rangle$. Recall that the variables X,Y are determined by $\vec{x}:=\{l_0,l_1\}$ by the the Magnus embedding $\pi_1^{\ell\text{-ét}}(U_{\overline{K}},\overrightarrow{01})\hookrightarrow\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ defined by $l_0\mapsto\exp(X)$, $l_1\mapsto\exp(Y)$. For brevity below, let us often omit references to the loop system $\vec{x}=(l_0,l_1)$

and/or tracking paths $\delta_i \cdot f_i(\gamma_z) : \overrightarrow{01} \rightsquigarrow f_i(z)$ in our notations as long as no confusions occur. The list corresponding to (5.2) reads then:

$$\tag{5.7} \begin{cases} \ell i_0(z)(\sigma) = \rho_z(\sigma), \\ \ell i_1(z)(\sigma) = \rho_{1-z}(\sigma), \\ \ell i_2(z)(\sigma) = -\tilde{\chi}_2^z(\sigma) - \dfrac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma), \\ \ell i_3(z)(\sigma) = \dfrac{1}{2}\tilde{\chi}_3^z(\sigma) + \dfrac{1}{2}\rho_z(\sigma)\tilde{\chi}_2^z(\sigma) + \dfrac{1}{12}\rho_z(\sigma)^2\rho_{1-z}(\sigma) \end{cases}$$

with $\sigma \in G_K$. Each term of the above (5.6) for $i = 1, 2, 3$ can be expressed by the graded Lie-version of polylogarithms ℓi_k $(k = 0, \ldots, 3)$ along "$\delta_i \cdot f_i(\gamma_z)$ minus δ_i" by the polylog-BCH formula ([NW12, Proposition 5.11 (ii)]) in the following way:

$$\mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_i),\vec{x}}(f_i(z), f_i(\overrightarrow{01}); f_i(\gamma_z)) = \mathsf{P}_3\big(\{-\ell i_j(f_i(\overrightarrow{01}), \delta_i, \vec{x})\}_{j=0}^3, \{\ell i_j(f_i(z), \delta_i \cdot f_i(\gamma_z), \vec{x})\}_{j=0}^3\big).$$

Noting that

$$\begin{aligned} &\Big(-\ell i_j(\overrightarrow{01}, \delta_1)\Big)_{0\le j\le 3} = (0,0,0,0), \\ &\Big(-\ell i_j(\overrightarrow{10}, \delta_2)\Big)_{0\le j\le 3} = \Big(0, 0, -\ell i_2(\overrightarrow{10}), -\ell i_3(\overrightarrow{10})\Big) = \left(0, 0, \tilde{\chi}_2^{\overrightarrow{10}}(\sigma), -\frac{1}{2}\tilde{\chi}_3^{\overrightarrow{10}}(\sigma)\right), \\ &\Big(-\ell i_j(\overrightarrow{0\infty}, \delta_3)\Big)_{0\le j\le 3} = \left(\frac{1-\chi(\sigma)}{2}, 0, 0, 0\right), \end{aligned}$$

we obtain for $\sigma \in G_K$:

$$\tag{5.8} \begin{cases} \mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_1)}(z, \overrightarrow{01}; \gamma_z)(\sigma) & = \frac{1}{2}\tilde{\chi}_3^z(\sigma) + \frac{1}{2}\rho_z(\sigma)\tilde{\chi}_2^z(\sigma) + \frac{1}{12}\rho_z(\sigma)^2\rho_{1-z}(\sigma), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_2)}(1-z, \overrightarrow{10}; f_2(\gamma_z))(\sigma) & = \frac{1}{2}\tilde{\chi}_3^{1-z}(\sigma) + \frac{1}{2}\rho_{1-z}(\sigma)\tilde{\chi}_2^{1-z}(\sigma) + \frac{1}{12}\rho_{1-z}(\sigma)^2\rho_z(\sigma) \\ & \quad -\frac{1}{2}\tilde{\chi}_3^{\overrightarrow{10}}(\sigma) - \frac{1}{2}\rho_{1-z}(\sigma)\tilde{\chi}_2^{\overrightarrow{10}}(\sigma), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_3(f_3)}\left(\frac{z}{z-1}, \overrightarrow{0\infty}; f_3(\gamma_z)\right)(\sigma) & = \frac{1}{2}\tilde{\chi}_3^{\frac{z}{z-1}}(\sigma) + \frac{1}{2}\rho_{\frac{z}{z-1}}(\sigma)\tilde{\chi}_2^{\frac{z}{z-1}}(\sigma) + \frac{1}{12}\rho_{\frac{z}{z-1}}(\sigma)^2\rho_{\frac{1}{1-z}}(\sigma) \\ & \quad +\frac{1}{2}\left(\frac{1-\chi(\sigma)}{2}\right)\left(-\tilde{\chi}_2^{\frac{z}{z-1}}(\sigma) - \dfrac{1}{2}\rho_{\frac{z}{z-1}}(\sigma)\rho_{\frac{1}{1-z}}(\sigma)\right) \\ & \quad +\frac{1}{12}\left(\frac{1-\chi(\sigma)}{2}\right)^2\rho_{\frac{1}{1-z}}(\sigma) - \frac{1}{12}\left(\frac{1-\chi(\sigma)}{2}\right)\rho_{\frac{z}{z-1}}(\sigma)\rho_{\frac{1}{1-z}}(\sigma). \end{cases}$$

Combining the identities in (5.8) enables us to rewrite the LHS of (5.6) in terms of ℓ-adic Galois polylogarithmic characters. It remains to compute the error term $E(\sigma, \gamma_z)$ in the right hand side of (5.6).

Lemma 5.2. *Notations begin as above, we have*

$$E(\sigma, \gamma_z) = -\frac{1}{12}\rho_{1-z}(\sigma) + \frac{1}{2}\tilde{\chi}_2^z(\sigma) + \frac{1}{4}\rho_z(\sigma)\rho_{1-z}(\sigma).$$

Proof. We shall apply the formula [NW12, Corollary 5.8] to compute the error term. Let $[\log(\mathfrak{f}_\sigma^{\gamma_z})^{-1}]_{<3}$ be the part of degree < 3 cut out from the Lie formal series $\log(\mathfrak{f}_\sigma^{\gamma_z})^{-1} \in \mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X, Y\rangle\rangle$ with respect obtained by the Magnus embedding $l_0 \to e^X$, $l_1 \to e^Y$ with respect to the fixed free generator system $\vec{x} = (l_0, l_1)$ of $\pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$. We also write $\varphi_3 : \mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X, Y\rangle\rangle \to \mathbb{Q}_\ell$ for the $\mathbb{Q}_\ell$-linear form that picks up the coefficient of $[X, [X, Y]]$ (that is uniquely determined) for any Lie series of $\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X, Y\rangle\rangle$. Introduce the variable Z so that $e^X e^Y e^Z = 1$ in $\mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$. By the Campbell-Baker-Hausdorff formula, we have

$$\tag{5.9} Z = -X - Y - \frac{1}{2}[X, Y] - \frac{1}{12}[X, [X, Y]] + \cdots .$$

According to [NW12, Corollary 5.8], it follows then that

$$\begin{aligned} E(\sigma,\gamma_z) &= \sum_{i=1}^{3}\varphi_3\left(\delta_i\cdot f_i\left([\log(\mathfrak{f}_\sigma^{\gamma_z})^{-1}]_{<3}\right)\cdot {\delta_i}^{-1}\right) \\ &= \sum_{i=1}^{3}\varphi_3\Big(\delta_i\cdot f_i\left(\rho_z(\sigma)X+\rho_{1-z}(\sigma)Y+\ell i_2(z,\gamma_z)(\sigma)[X,Y]\right)\cdot {\delta_i}^{-1}\Big) \\ &= \varphi_3\Big(\rho_z(\sigma)X+\rho_{1-z}(\sigma)Y+\ell i_2(z,\gamma_z)(\sigma)[X,Y]\Big) \\ &\quad +\varphi_3\Big(\rho_z(\sigma)Y+\rho_{1-z}(\sigma)X+\ell i_2(z,\gamma_z)(\sigma)[Y,X]\Big) \\ &\quad +\varphi_3\Big(\rho_z(\sigma)X+\rho_{1-z}(\sigma)Z+\ell i_2(z,\gamma_z)(\sigma)[X,Z]\Big). \end{aligned}$$

Since φ_3 annihilates those terms $X, Y, [X,Y], [Y,X]$, we continue the above computation after (5.9) as:

$$\begin{aligned} E(\sigma,\gamma_z) &= \varphi_3\left(-\frac{1}{12}\rho_{1-z}(\sigma)[X,[X,Y]]-\frac{1}{2}\ell i_2(z,\gamma_z)(\sigma)[X,[X,Y]]\right) \\ &= -\frac{1}{12}\rho_{1-z}(\sigma)-\frac{1}{2}\ell i_2(z,\gamma_z)(\sigma) \\ &= -\frac{1}{12}\rho_{1-z}(\sigma)-\frac{1}{2}\left(-\tilde{\chi}_2^z(\sigma)-\frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)\right) \\ &= -\frac{1}{12}\rho_{1-z}(\sigma)+\frac{1}{2}\tilde{\chi}_2^z(\sigma)+\frac{1}{4}\rho_z(\sigma)\rho_{1-z}(\sigma). \end{aligned}$$

This concludes the assertion of the lemma. □

Alternative proof of Theorem 1.1. As discussed in §4, the ℓ-adic Galois Landen's trilogarithm functional equation in Theorem 1.1 is equivalent to the identity (4.3) between polylogarithmic characters. The latter follows from (5.6) with replacements of the terms of both sides by (5.8) and Lemma (5.2) by simple computations. □

Appendix A. Low degree terms of associators

Presentation of lower degree terms of $G_0(X,Y)(z)$ and $\mathfrak{f}_\sigma^{\gamma_z}(X,Y)$ are often useful as references. The former one presented below reconfirms Furusho's preceding computations found in [F04, 3.25]-[F14, A.16] (where the sign of $\log(z)Li_2(z)$ had an unfortunate misprint in the coefficient of XYX).

(A.1)
$$\begin{aligned} G_0(X,Y)(z) &= 1+\log(z)X+\log(1-z)Y+\frac{\log^2(z)}{2}X^2-Li_2(z)XY \\ &+\Big(Li_2(z)+\log(z)\log(1-z)\Big)YX+\frac{\log^2(1-z)}{2}Y^2+\frac{\log^3(z)}{6}X^3-Li_3(z)X^2Y \\ &+\Big(2Li_3(z)-\log(z)Li_2(z)\Big)XYX+Li_{1,2}(z)XY^2 \\ &-\left(Li_3(z)-\log(z)Li_2(z)-\frac{\log^2(z)\log(1-z)}{2}\right)YX^2+Li_{2,1}(z)YXY \\ &-\left(Li_{1,2}(z)+Li_{2,1}(z)-\frac{\log(z)\log^2(1-z)}{2}\right)Y^2X+\frac{\log^3(1-z)}{6}Y^3 \\ &+\cdots(\text{higher degree terms}). \end{aligned}$$

This is a group-like element of $\mathbb{C}\langle\langle X,Y\rangle\rangle$ whose coefficients satisfy what are called the shuffle relations ([Ree58]). The regular coefficients (viz. those coefficients of monomials ending with letter Y) are given by iterated integrals of a sequence of dz/z, $dz/(1-z)$. This immediately shows $G_0(0,Y)(z) = \sum_{k=0} \frac{\log^k(1-z)}{k!} Y^k$ and say, $Li_{1,1,1}(z) = -\frac{1}{6}\log^3(1-z)$. Furusho gave an explicit formula that expresses arbitrary coefficients of $G_0(X,Y)$ in terms only of the regular coefficients ([F04, Theorem 3.15]). The specialization $z \to \overrightarrow{10}$ (cf. [W97] p.239 for a naive account) interprets $\log z \to 0, \log(1-z) \to 0$ so as to produce the Drinfeld's associator:

$$\begin{aligned}\Phi(X,Y) \quad & \left(= G_0(X,Y)(\overrightarrow{10})\right) \\ = 1 & - \zeta(2)XY + \zeta(2)YX - \zeta(3)X^2Y + 2\zeta(3)XYX \\ & + \zeta(1,2)XY^2 - \zeta(3)YX^2 - 2\zeta(1,2)YXY + \zeta(1,2)Y^2X \\ & + \cdots \text{(higher degree terms)}\end{aligned} \tag{A.2}$$

which forms the primary component of the Grothendieck-Teichmüller group ([D90], [Ih90]). Among other symmetric relations of Φ, the 2-cyclic relation $\Phi(X,Y)\cdot\Phi(Y,X) = 1$ (which may be derived at $z = \overrightarrow{10}$ in the chain rule $G_0(Y,X)(1-z) = G_0(X,Y)(z)\cdot\Phi(Y,X)$ from Table 1) implies Euler's celebrated relation $\zeta(3) = \zeta(1,2)$ (due to $\mathtt{Coeff}_{X^2Y} + \mathtt{Coeff}_{Y^2X} = 0$). One also observes that the shuffle relation corresponding to $XY \sqcup\!\sqcup Y = YXY + 2XYY$ implies $\mathtt{Coeff}_{YXY} = -2\zeta(1,2)$.

The expansion in $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ of the ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\gamma_z} \in \pi_1^{\ell\text{-ét}}(U_{\overline{K}}, \overrightarrow{01})$ via the Magnus embedding $l_0 \mapsto e^X$, $l_1 \mapsto e^Y$ over $\mathbb{Q}_\ell$ reads as follows:

$$\begin{aligned}\mathfrak{f}_\sigma^{\gamma_z}(X,Y) = 1 & - \rho_{\gamma_z}(\sigma)X - \rho_{\gamma_{1-z}}(\sigma)Y + \frac{\rho_{\gamma_z}(\sigma)^2}{2}X^2 - Li_2^\ell(\gamma_z)(\sigma)XY \\ & + \left(Li_2^\ell(\gamma_z)(\sigma) + \rho_{\gamma_z}(\sigma)\rho_{\gamma_{1-z}}(\sigma)\right)YX + \frac{\rho_{\gamma_{1-z}}(\sigma)^2}{2}Y^2 - \frac{\rho_{\gamma_z}(\sigma)^3}{6}X^3 - Li_3^\ell(\gamma_z)(\sigma)X^2Y \\ & + \left(2Li_3^\ell(\gamma_z)(\sigma) + \rho_{\gamma_z}(\sigma)Li_2^\ell(\gamma_z)(\sigma)\right)XYX + Li_{1,2}^\ell(\gamma_z)(\sigma)XY^2 \\ & - \left(Li_3^\ell(\gamma_z)(\sigma) + \rho_{\gamma_z}(\sigma)Li_2^\ell(\gamma_z)(\sigma) + \frac{\rho_{\gamma_z}(\sigma)^2\rho_{\gamma_{1-z}}(\sigma)}{2}\right)YX^2 + Li_{2,1}^\ell(\gamma_z)(\sigma)YXY \\ & - \left(Li_{1,2}^\ell(\gamma_z)(\sigma) + Li_{2,1}^\ell(\gamma_z)(\sigma) + \frac{\rho_{\gamma_z}(\sigma)\rho_{\gamma_{1-z}}(\sigma)^2}{2}\right)Y^2X - \frac{\rho_{\gamma_{1-z}}(\sigma)^3}{6}Y^3 \\ & + \cdots \text{(higher degree terms)} \qquad (\sigma \in G_K).\end{aligned} \tag{A.3}$$

The coefficients of X, Y, YX^k $(k = 1,2,...)$ were calculated in terms of polylogarithmic characters explicitly in [NW99]. A formula of Le-Murakami, Furusho type for arbitrary group-like power series was shown in [N21b]. As illustrated in Proposition 4.2, the family of polylogarithmic characters and that of ℓ-adic Galois polylogarithms are converted to each other. The terms appearing in the above (A.3) can be derived from them.

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