GENERALIZED RADEMACHER FUNCTIONS AND SOME CONGRUENCE PROPERTIES

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Abstract In §1, we introduce the classical Dedekind sum and the Rademacher function. Then, in §2, certain generalized Rademacher functions are introduced as the Eichler-Shimura type period integrals of Eisenstein series. In §3, we present a version of continued fraction algorithm which computes efficiently the generalized Rademacher functions. In §4, we show a congruence formula connecting values of the generalized Rademacher functions of weight 2 and weight $k > 2$. This formula will be applied in a forthcoming paper [N4].

1. The classical Dedekind sums

The classical Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \qquad (q = e^{2\pi i \tau},\ \tau \in \mathfrak{H}) \tag{1.1}$$

on the upper half plane $\mathfrak{H}$ is one of the most beautiful objects in number theory. Since $\Delta(\tau) = (2\pi i)^{12}\eta^{24}(\tau)$ is a cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$, it is clear that, for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the ratio $\eta(A\tau)/\sqrt{\frac{c\tau+d}{i}}\eta(\tau)$ is a 24-th root of unity. The behavior of this little ratio with respect to $A \in \mathrm{SL}_2(\mathbb{Z})$ looks delicate and mysteri-

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ous. R. Dedekind, in his note [D] included in *Gesammelte Mathematische Werke* of B. Riemann, studied the behavior with Riemann's method. Outstanding is that he found an explicit *integer valued* function $\varphi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}$ controlling this ratio in the form

$$\eta(A\tau) = \begin{cases} e^{\frac{2\pi i}{24}\varphi(A)}\eta(\tau), & (c=0); \\ e^{\frac{2\pi i}{24}\varphi(A)}\sqrt{\frac{c\tau+d}{i}}\eta(\tau), & (c>0). \end{cases} \tag{1.2}$$

Dedekind's formula for the function φ (which, by definition, factors through $\mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$) reads as follows:

$$\varphi(A) = \begin{cases} \frac{b}{d}, & (c=0), \\ \frac{a+d}{c} - 12s(a,c), & (c>0). \end{cases} \tag{1.3}$$

Here $s(a,c)$, called now the Dedekind sum, is defined by

$$s(a,c) = \sum_{i=0}^{c-1} P_1(\frac{i}{c})P_1(\frac{ai}{c})$$

with $P_1(*)$ the "sawtooth" function:

$$P_1(x) = \begin{cases} 0 & (x \in \mathbb{Z}); \\ x - [x] - \frac{1}{2} & (x \notin \mathbb{Z}). \end{cases}$$

($[x]$: the greatest integer not exceeding $x \in \mathbb{R}$.)

In [R], H. Rademacher intensively studied algebraic properties of the function φ. He derived, for example, the composition formula

$$\varphi(AB) = \varphi(A) + \varphi(B) - 3\mathrm{sgn}(c_A c_B c_{AB}) \qquad (A, B \in \mathrm{SL}_2(\mathbb{Z})), \tag{1.4}$$

where, for any matrix $S \in \mathrm{SL}_2(\mathbb{Z})$, c_S denotes the lower left entry of S. Following [KM], we shall call φ the Rademacher (φ-)function.

One of the key ingredients that enabled Dedekind to find the above result was to consider the modular transformation of not only $\eta(\tau)$ but also of $\log \eta(\tau)$. In fact, Dedekind's key formula (for the case $c > 0$) reads:

$$\log \eta(A\tau) - \log \eta(\tau) = -\frac{1}{2}\log\frac{i}{c\tau+d} + \pi i\big(\frac{a+d}{12c} - s(a,c)\big). \tag{1.5}$$

This left hand side may also be written in the form:

$$\log \eta(A\tau) - \log \eta(\tau) = -\pi i \int_{\tau}^{A\tau} E_2(z)dz, \tag{1.6}$$

where

$$E_2(\tau) = \frac{1}{(2\pi i)^2} \sum_{m,n}{}' \frac{1}{(m\tau+n)^2}$$

is the Eisenstein series of weight 2 (with specified conditional convergence). In other words, the Rademacher function φ can be viewed almost as the periods of the Eisenstein series. Unfortunately, $E_2(\tau)$ is only a quasi-modular form and not a usual modular form. This causes the need of the adjustment term $\frac{1}{2}\log\frac{i}{c\tau+d}$ of (1.5) which connects $\varphi(A)$ and the period. To gain a more satisfactory viewpoint, we would need a more subtle treatment by using a non-holomorphic modification of E_2. However, we do not enter this subtlety any more here, because this sort of adjustment is rather exceptional phenomenon occurring in this original case among generalizations of E_2 discussed in the next section.

2. Generalized Rademacher functions

We wish to consider a generalization of the Rademacher function φ introduced in §1 by considering φ as the case of weight 2 and level 1. With regard to the appearance of E_2 in (1.6), we shall begin by introducing the generalized Eisenstein series $E_k^{(x_1,x_2)}$'s ($k \geq 2$, $(x_1,x_2) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$) given by

$$\begin{aligned} &E_k^{(x_1,x_2)}(\tau) \\ &:= \frac{(k-1)!}{(2\pi i)^k} \sum_{\mathbf{a}\in(\mathbb{Z}/N\mathbb{Z})^2} e^{2\pi i(x_1a_2-x_2a_1)} \sum_{\mathbf{m}\equiv\mathbf{a}\,(N)}{}' \frac{1}{(m_1\tau+m_2)^k} \\ &= -\frac{P_k(x_1)}{k} + \sum_{\substack{s\in x_1+\mathbb{Z}\\ s>0}} \sum_{l=1}^{\infty} s^{k-1} e^{2\pi i l(x_2+s\tau)} + \sum_{\substack{s\in -x_1+\mathbb{Z}\\ s>0}} \sum_{l=1}^{\infty} s^{k-1} e^{2\pi i l(-x_2+s\tau)}, \end{aligned} \tag{2.1}$$

where $P_k(x) : \mathbb{R} \to \mathbb{R}$ is the k-th periodic Bernoulli function (whose definition will be recalled in (3.4)). It is well known that $E_k^{(\mathbf{x})}$ is a holomorphic modular form of level N and weight k, except for the original case $E_2 = E_2^{(\mathbf{0})}$ (cf. [St]).

Let $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be the principal congruence subgroup of level $N \geq 1$. For any pair of $k \geq 2$ and $\mathbf{x} = (x_1, x_2) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$ with

$(k,\mathbf{x}) \neq (2,\mathbf{0})$, we then consider the Eichler integral of $E_k^{(\mathbf{x})}$:

$$F_k^{(\mathbf{x})}(\tau) = -\frac{1}{(k-2)!}\int_\tau^{i\infty}\left(E_k^{(\mathbf{x})}(u) + \frac{P_k(x_1)}{k}\right)(\tau-u)^{k-2}du \\ - \frac{P_k(x_1)}{k}\frac{\tau^{k-1}}{(k-1)!}. \tag{2.2}$$

This (indefinite) integral satisfies $(\frac{d}{d\tau})^{k-1}F_k^{(\mathbf{x})} = E_k^{(\mathbf{x})}$. Writing $j(A,\tau) := (c\tau+d)$ for $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma(N)$, we then see that the difference

$$\phi_k^{(\mathbf{x})}(A)(\tau) := j(A,\tau)^{k-2}F_k^{(\mathbf{x})}(A\tau) - F_k^{(\mathbf{x})}(\tau) \tag{2.3}$$

is a polynomial in τ of degree $k-2$. What we wish to look closely at here is its "real part" $Re\phi_k^{(\mathbf{x})}(A)(\tau)$ which is, by definition, the polynomial in the "complex variable" τ whose coefficients consist of the real parts of the corresponding coefficients of $\phi_k^{(\mathbf{x})}(A)(\tau)$. It is known that $Re\phi_k^{(\mathbf{x})}(A)(\tau)$ has only rational coefficients.

The assignment $A \mapsto Re\phi_k^{(\mathbf{x})}(A)(\tau)$ enjoys the 1-cocycle property of the form:

$$Re\phi_k^{(\mathbf{x})}(AB)(\tau) = j(B,\tau)^{k-2}Re\phi_k^{(\mathbf{x})}(A)(B\tau) + Re\phi_k^{(\mathbf{x})}(B)(\tau) \tag{2.4}$$

on $A, B \in \Gamma(N)$.

To switch $Re\phi_k^{(\mathbf{x})}(A)(\tau)$ into the form of 1-cocycle for a left module, let us introduce the space $\mathrm{Sym}^{k-2}(\mathbb{Q}^2) = \mathbb{Q}[X,Y]_{deg=k-2}$ of the homogeneous polynomials in X, Y of degree $k-2$ on which $\mathrm{SL}_2(\mathbb{Z})$ acts on the left (written ρ) by

$$\rho\begin{pmatrix} a & b \\ c & d\end{pmatrix}.\ \phi(X,Y) := \phi(aX+cY, bX+dY).$$

Definition (2.5). Let $(k,\mathbf{x}) \in \mathbb{Z}_{\geq 2} \times (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$ with $(k,\mathbf{x}) \neq (2,\mathbf{0})$. We define the generalized Rademacher function

$$\Phi_{\mathbf{x}}^{(k)}:\ \Gamma(N) \longrightarrow \mathrm{Sym}^{k-2}(\mathbb{Q}^2) = \mathbb{Q}[X,Y]_{\mathrm{deg}=k-2}$$
$$\left(A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \longmapsto \sum_{r=0}^{k-2}\Phi_{\mathbf{x}}^{(r+1,k-1-r)}(A)X^rY^{k-2-r}\right)$$

by

$$\Phi_{\mathbf{x}}^{(k)}(A) := (-1)^{k-1}(k-2)!\,X^{k-2}\cdot Re\phi_k^{(\mathbf{x})}(A^{-1})\left(-\frac{Y}{X}\right).$$

As expected from the above definition, (2.4) can be converted into the following 1-cocycle property of $\Phi_{\mathbf{x}}^{(k)}$:

$$\Phi_{\mathbf{x}}^{(k)}(AB) = \Phi_{\mathbf{x}}^{(k)}(A) + \rho(A).\Phi_{\mathbf{x}}^{(k)}(B) \tag{2.6}$$

on $A, B \in \Gamma(N)$. But one can get a nicer result. In fact, G. Stevens [St] showed how to interpret $\Phi_{\mathbf{x}}^{(k)}$ as the periods of a certain real differential form extended to the Borel-Serre compactification $\bar{\mathfrak{H}}$ of the upper half plane $\mathfrak{H}$. With his method, we can extend $\Phi_{\mathbf{x}}^{(k)}$ canonically to a function from $\mathrm{SL}_2(\mathbb{Z})$ (or even from $\mathrm{GL}_2(\mathbb{Q})^+$) to $\mathrm{Sym}^{k-2}(\mathbb{Q}^2)$. Moreover the case of $(k, \mathbf{x}) = (2, \mathbf{0})$ may naturally be included in this unified construction of $\Phi_{\mathbf{x}}^{(k)}$ with regarding $\Phi_{\mathbf{0}}^{(2)} = -\varphi(A)/12$. Then, the above properties (1.4) and (2.6) can be generalized to

$$\Phi_{\mathbf{x}}^{(k)}(AB) = \Phi_{\mathbf{x}}^{(k)}(A) + \rho(A).\Phi_{\mathbf{x}A}^{(k)}(B) + \frac{1}{4}\delta_{\mathbf{x}}^{k=2}\mathrm{sgn}(c_A c_B c_{AB}). \tag{2.7}$$

Here, $\mathrm{sgn}(*) \in \{\pm 1, 0\}$ denotes the signature of $*$, and the Kronecker symbol $\delta_{\mathbf{x}}^{k=2}$ is defined by $\delta_{\mathbf{x}}^{k=2} = 1$ when $(k, \mathbf{x}) = (2, \mathbf{0})$ and $\delta_{\mathbf{x}}^{k=2} = 0$ otherwise.

It will also be nice to have generalizations of the beautiful formula (1.3) to the cases of our generalized Rademacher functions $\Phi_{\mathbf{x}}^{(k)}$. Although the author could not find a literature describing the explicit form, by careful calculations one obtains the following formula of $\Phi_{\mathbf{x}}^{(k)}$: $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{Sym}^{k-2}(\mathbb{Q}^2) = \mathbb{Q}[X, Y]_{\deg = k-2}$;

$$\Phi_{(x_1,x_2)}^{(k)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} -\frac{P_k(x_1)}{k}\int_0^{\frac{b}{d}}(tX+Y)^{k-2}dt, & (c=0); \\ -\frac{P_k(x_1)}{k}\int_0^{\frac{a}{c}}(tX+Y)^{k-2}dt \\ \quad -\frac{P_k(ax_1+cx_2)}{k}\int_{-\frac{d}{c}}^{0}(t(aX+cY)+bX+dY)^{k-2}dt \\ \quad +\sum_{r=0}^{k-2}(-1)^r\binom{k-2}{r}X^r(aX+cY)^{k-2-r}s_{(x_1,x_2)}^{(k-1-r,r+1)}(a,c), & (c>0). \end{cases} \tag{2.8}$$

where the last factor (called the generalized Dedekind sum) is defined by

$$s_{(x_1,x_2)}^{(k-1-r,r+1)}(a,c) = \sum_{i=0}^{c-1}\frac{P_{k-1-r}(\frac{x_1+i}{c})}{k-1-r}\frac{P_{r+1}(x_2+a\frac{x_1+i}{c})}{r+1}. \tag{2.9}$$

The following distribution relation is an important property which connects values of $\Phi_{\mathbf{x}}^{(k)}(A)$ of different $\mathbf{x}$'s:

$$\Phi_{\mathbf{x}}^{(k)}(A) = n^{k-2} \sum_{\mathbf{y}\in\frac{1}{n}\mathbf{x}} \Phi_{\mathbf{y}}^{(k)}(A) \qquad (n \geq 1). \tag{2.10}$$

3. Continued fraction algorithm for $\Phi_{\mathbf{x}}^{(k)}$

The calculation of $\Phi_{\mathbf{x}}^{(k)}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ by means of (2.8) involve those of generalized Dedekind sums $s_{\mathbf{x}}^{(k-1-r,r+1)}(a,c)$ of (2.9) which need lots of times when c is big. Variants of continued fraction algorithms have been known to improve this sort of inefficiency on computers. We shall exploit a version of it well suited to calculations of our generalized Rademacher functions. The implementation and various numerical tests have been given by Y. Morimoto [Mo] by using PARI-GP and Maple softwares.

Given a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, choose an expression of a/c by a continued fraction of the form:

$$\begin{aligned} \frac{a}{c} &= k_0 - \frac{1}{k_1-}\,\frac{1}{k_2-}\cdots\frac{1}{-k_n} \\ &= k_0 - \cfrac{1}{k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots\, \cfrac{1}{k_{n-1} - \frac{1}{k_n}}}}} \end{aligned} \tag{3.1}$$

with an integer sequence $(k_0, k_1, \ldots, k_n)$. This expression may be obtained by first putting $r_0 = a$, $r_1 = c > 0$ and then defining integers $\{k_i\}_i$ with the Euclidean method:

$$\begin{cases} r_0 = k_0 r_1 - r_2, & |r_1| > |r_2| > 0, \\ r_1 = k_1 r_2 - r_3, & |r_2| > |r_3| > 0, \\ \cdots & \cdots \\ r_{n-1} = k_{n-1} r_n - r_{n+1}, & |r_n| > |r_{n+1}| = 1, \\ r_n = k_n r_{n+1} - 0, & |r_{n+1}| = 1,\ r_{n+2} = 0, \end{cases} \tag{3.2}$$

where each k_i $(i = 0, \ldots, n)$ is chosen to be either $[r_i/r_{i+1}]$ (the greatest integer not exceeding r_i/r_{i+1}) or $\lceil r_i/r_{i+1} \rceil$ (the least integer greater than or equal to r_i/r_{i+1}). Let us write $\varepsilon = r_{n+1} \in \{\pm 1\}$.

Define also a sequence of matrices $\{A_i\}_i$ in $\mathrm{SL}_2(\mathbb{Z})$ by

$$A_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_0 = \begin{pmatrix} k_0 & -1 \\ 1 & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} k_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\dots$$

$$A_n = \begin{pmatrix} k_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_n & -1 \\ 1 & 0 \end{pmatrix}.$$

and put

$$A_i = \begin{pmatrix} -p_i & p_{i-1} \\ -q_i & q_{i-1} \end{pmatrix} \qquad (-1 \le i \le n).$$

Then,

Lemma (3.3). *Notations being as above, we have*

(i) $a = -\varepsilon p_n, \; c = -\varepsilon q_n$;

(ii) $\delta := (d - \varepsilon q_{n-1})/c \in \mathbb{Z}$;

(iii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon \begin{pmatrix} k_0 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}.$

Proof. The Euclidean sequence (3.2) may be rephrased as

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} k_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{n+1} \\ 0 \end{pmatrix}.$$

The claim (i) follows immediately from this. Both A and εA_n are in $\mathrm{SL}_2(\mathbb{Z})$ and have the same left column. Therefore, there exists $\delta \in \mathbb{Z}$ with $A = \varepsilon A_n \left(\begin{smallmatrix} 1 & \delta \\ 0 & 1 \end{smallmatrix}\right)$, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon \begin{pmatrix} -p_n & p_{n-1} \\ -q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}.$$

The claims (ii) and (iii) follow from this simultaneously. □

Definition (3.4). The Bernoulli polynomials $B_k(X)$ $(k = 0, 1, 2, ...)$ are defined by the generating function

$$\frac{te^{tX}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(X) \frac{t^k}{k!},$$

and the Bernoulli number B_k is by definition $B_k(1)$. We define the periodic Bernoulli function $P_k : \mathbb{R} \to \mathbb{R}$ by $P_k(x) = B_k(x - [x])$ except for the case $k=1$ and $x \in \mathbb{Z}$ where we set $P_1(n)=0$ $(n \in \mathbb{Z})$. For $\mathbf{x}=(x_1,x_2) \in \mathbb{Q}^2$ and $(a,b) \in \mathbb{Z}^2_{\geq 0}$, define

$$\boldsymbol{\beta}_{a,b}(\mathbf{x}) := \frac{P_a(x_1)}{\max\{a,1\}} \cdot \frac{P_b(x_2)}{\max\{b,1\}}.$$

Proposition (3.5) (Continued fraction algorithm). *Notations being as above, put $k_{n+1} := \delta$. Then, for $\mathbf{x}=(x_1,x_2) \in \mathbb{Q}^2$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$\begin{aligned}\Phi^{(k)}_{\mathbf{x}}(A) = &- \sum_{i=0}^{n}\sum_{j=0}^{k-2} \boldsymbol{\beta}_{j+1,k-1-j}(\mathbf{x}A_i)\binom{k-2}{j} \\ &\times (p_{i-1}X+q_{i-1}Y)^j(p_iX+q_iY)^{k-2-j} \\ &- \sum_{i=0}^{n+1}\boldsymbol{\beta}_{k,0}(\mathbf{x}A_{i-1}) \\ &\times \int_0^{k_i}\big((-p_{i-1}t+p_{i-2})X+(-q_{i-1}t+q_{i-2})Y\big)^{k-2}dt \\ &+ \frac{\delta^{k=2}_{\mathbf{x}}}{4}(\mathrm{sgn}(q_{-1}q_0)+\mathrm{sgn}(q_0q_1)+\cdots+\mathrm{sgn}(q_{n-1}q_n)).\end{aligned}$$

Proof. Applying (2.7) iteratedly to (3.3)(iii), we obtain

$$\begin{aligned}\Phi^{(k)}_{\mathbf{x}} = &\sum_{i=0}^{n}\rho(A_{i-1}).\Phi^{(k)}_{\mathbf{x}A_{i-1}}(\begin{pmatrix} k_i & -1 \\ 1 & 0 \end{pmatrix}) \\ &+ \rho(A_n).\Phi^{(k)}_{\mathbf{x}A_n}(\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}) + \frac{\delta^{k=2}_{\mathbf{x}}}{4}\sum_{i=0}^{n}\mathrm{sgn}(q_{i-1}q_i).\end{aligned}$$

Then, the formula (2.8) gives

$$\Phi^{(k)}_{\mathbf{x}A_n}(\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}) = -\boldsymbol{\beta}_{k,0}(\mathbf{x}A_n)\int_0^{k_{n+1}}(tX+Y)^{k-2}dt,$$

$$\begin{aligned}\Phi^{(k)}_{\mathbf{x}A_{i-1}}(\begin{pmatrix} k_i & -1 \\ 1 & 0 \end{pmatrix}) = &-\boldsymbol{\beta}_{k,0}(\mathbf{x}A_{i-1})\int_0^{k_i}(tX+Y)^{k-2}dt \\ &+ \sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j} \\ &\times X^j(k_iX+Y)^{k-2-j}s^{(k-1-j,j+1)}_{\mathbf{x}A_{i-1}}(k_i,1).\end{aligned}$$

Writing here $(y_i, z_i) := \mathbf{x}A_{i-1}$ $(i = 0, \dots, n)$ so that

$$(y_i, z_i)\begin{pmatrix} k_i & -1 \\ 1 & 0 \end{pmatrix} = (y_{i+1}, z_{i+1}) \ (0 \le i < n),$$

we see that

$$\begin{aligned} s^{(k-1-j,j+1)}_{\mathbf{x}A_{i-1}}(k_i, 1) &= \frac{P_{k-1-j}(y_i)}{k-1-j}\frac{P_{j+1}(z_i + k_i y_i)}{j+1} \\ &= (-1)^{k-1-j}\boldsymbol{\beta}_{j+1,k-1-j}(\mathbf{x}A_i). \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi^{(k)}_{\mathbf{x}} = &- \sum_{i=0}^{n+1} \rho(A_{i-1}).\boldsymbol{\beta}_{k,0}(\mathbf{x}A_{i-1}) \int_0^{k_i} (tX+Y)^{k-2} dt \\ &+ (-1)^{k-1} \sum_{i=0}^{n}\sum_{j=0}^{k-2} \rho(A_{i-1}).\binom{k-2}{j} \\ &\times X^j (k_i X + Y)^{k-2-j} \boldsymbol{\beta}_{j+1,k-1-j}(\mathbf{x}A_i) \\ &+ \frac{\delta^{k=2}_{\mathbf{x}}}{4} \sum_{i=0}^{n} \operatorname{sgn}(q_{i-1}q_i). \end{aligned}$$

Noticing that $\rho(A_{i-1})$ maps

$$\begin{cases} X & \longmapsto \ -p_{i-1}X - q_{i-1}Y, \\ Y & \longmapsto \ p_{i-2}X + q_{i-2}Y, \\ k_i X + Y & \longmapsto \ -p_i X - q_i Y, \end{cases}$$

we conclude the proof of Proposition (3.5). □

4. Some congruence properties

In this section, we shall present a congruence formula (4.3) which connects N^2 values of $\Phi^{(2)}_{\mathbf{x}}(A)$ $(\mathbf{x} \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2)$ and the coefficients of $\Phi^{(k)}_{\mathbf{0}}(A)$. The congruence formula will be applied to the study of a certain measure function on the congruence kernel of SL_2 which governs the meta-abelian monodromy representation in the fundamental group of the universal elliptic curve minus the origin section ([N1-4]). We refer to the forthcoming paper [N4] as a main reference for details of such an application.

Let $\mathbb{Z}'_N \subset \mathbb{Q}$ denote the ring of fractional numbers with denominators prime to N. For $n \ge 1$, write $B^*_n(X) := B_n(X) - B_n(0)$, and define

d_n (resp. d_n^*) to be the least common multiple of the denominators of coefficients of the polynomial $\frac{1}{n}B_n(X)$ (resp. $\frac{1}{n}B_n^*(X)$). In the proof of Proposition (4.3) below, we shall frequently make use of the following well known properties:

$$\sum_{x=0}^{N-1} x^n = \frac{1}{n+1}B_{n+1}^*(N) \equiv 0 \ (\mathrm{mod}\ \frac{N}{d_{n+1}^*}\mathbb{Z}'_N), \tag{4.1}$$

$$\frac{N^{n-1}}{n}\Big(P_n(\frac{x}{N}+u)-P_n(\frac{x}{N})\Big) \equiv x^{n-1}\Big(P_1(\frac{x}{N}+u)-P_1(\frac{x}{N})\Big) \quad (\mathrm{mod}\ \frac{N}{d_n}\mathbb{Z}'_N) \tag{4.2}$$

for $x \in \mathbb{Z}$, $N \in \mathbb{N}$, $u \in \mathbb{Z}'_N$.

Proof. The formula (4.1) is quite popular, and its proof may be left to readers. To prove (4.2), let $\{t\} := t - [t]$ denote the fractional part of $t \in \mathbb{R}$, and put $v := \{\frac{x}{N}+u\} - \{\frac{x}{N}\}$ so that $\{u\} = \{v\}$. Then, $v \in \mathbb{Z}'_N$ and we compute

$$\begin{aligned}
&\frac{N^{n-1}}{n}(P_n(\frac{x}{N}+u)-P_n(\frac{x}{N}))\\
&= \frac{N^{n-1}}{n}(B_n(\{\frac{x}{N}\}+v)-B_n(\{\frac{x}{N}\}))\\
&\equiv \frac{N^{n-1}}{n}\left[\Big((\{\frac{x}{N}\}+v)^n-\frac{n}{2}(\{\frac{x}{N}\}+v)^{n-1}\Big)-\Big(\{\frac{x}{N}\}^n-\frac{n}{2}\{\frac{x}{N}\}^{n-1}\Big)\right]\\
&= \frac{N^{n-1}}{n}\left[n\{\frac{x}{N}\}^{n-1}v+P_{n-2}(\{\frac{x}{N}\})+\frac{n}{2}Q_{n-2}(\{\frac{x}{N}\})\right].
\end{aligned}$$

Here, the congruence is taken modulo $\frac{N}{d_n}\mathbb{Z}'_N$, and P_{n-2}, Q_{n-2} are polynomials of degree $n-2$ with coefficients in $\mathbb{Z}$. Since $\frac{1}{n}B_n(X)$ is a polynomial of the form $\frac{1}{n}x^n - \frac{1}{2}x^{n-1} + ...$, it follows that d_n is a multiple of $LCM\{2,n\}$. Therefore, the second and third terms of the above last side vanish modulo $\frac{N}{d_n}\mathbb{Z}'_N$. Moreover the integer $x' := \{x/N\}N$ is congruent to x (mod N). Thus, the above last side continues to

$$\equiv (x')^{n-1}v \equiv x^{n-1}v = x^{n-1}\Big(P_1(\frac{x}{N}+u)-P_1(\frac{x}{N})\Big).$$

□

Proposition (4.3). *Let N, r, k be integers satisfying $N \geq 1$, $k \geq 2$ and $0 \leq r \leq k-2$, and, for $s = 1, ..., k-2$, let e_s be the denominator of $\frac{B_k}{k(k-s)}$. Define $D_{k,r}$ to be the least common multiple of the set*

$$\left\{\begin{matrix} d_{j+1}d_{k-1-j}\ (r \leq j \leq k-2),\ e_1,\dots,e_{k-2} \\ d_{r+1}^*d_{k-r}^*,\ d_{r+2}^*d_{k-1-r}^*,\ d_{r+1}^*d_{k-r+1}^*,\ d_{r+2}^*d_{k-r}^*,\ d_{r+3}^*d_{k-1-r}^* \end{matrix}\right\}.$$

Then, for $A \in \Gamma(N)$, we have the following congruence:

$$12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\binom{k-2}{r}x^{k-2-r}(-y)^r\Phi^{(2)}_{(\frac{x}{N},\frac{y}{N})}(A)$$
$$\equiv 12\Phi_{\mathbf{0}}^{(r+1,k-1-r)}(A)\ (\mathrm{mod}\ \binom{k-2}{r}\frac{GCD\{6,N\}N}{D_{k,r}}\mathbb{Z}'_N).$$

This congruence formula was first proved by the author in the special case of $r=0$. The general case of the formula was then conjectured by Y. Morimoto after numerical computations and evidences [Mo].

(4.4) Proof of Proposition 4.3. Let $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma(N)$. Since $\left(\begin{smallmatrix}1&gN^3\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}a+cgN^3&b+dgN^3\\c&d\end{smallmatrix}\right)$, it is sufficient to prove the formula when $a\gg c>0$. In what follows, we shall assume this. Applying (2.7) to the equation $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}c&d\\-a&-b\end{smallmatrix}\right)=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, we find

$$\begin{aligned}\Phi^{(k)}_{(x_1,x_2)}(A) &= -\frac{P_k(x_2)}{k}\int_0^{-\frac{c}{a}}(tY-X)^{k-2}dt\\
&\quad-\frac{P_k(ax_1+cx_2)}{k}\int_{-\frac{b}{a}}^{0}((aX+cY)t+bX+dY)^{k-2}dt\\
&\quad+\sum_{r=0}^{k-2}X^rY^{k-2-r}\binom{k-2}{r}(-1)^r\frac{P_{k-1-r}(x_1)}{k-1-r}\frac{P_{r+1}(x_2)}{r+1}\\
&\quad-\sum_{r=0}^{k-2}(aX+cY)^rY^{k-2-r}\binom{k-2}{r}\\
&\quad\times(-1)^r\sum_{i=0}^{a-1}\frac{P_{k-1-r}(x_1+c\frac{x_2+i}{a})}{k-1-r}\frac{P_{r+1}(\frac{x_2+i}{a})}{r+1}. \qquad (4.4.0)\end{aligned}$$

Let us first evaluate the left hand side of the congruence formula (4.3). When $k=2$, $r=0$, the above (4.4.0) gives

$$\begin{aligned}\Phi^{(2)}_{\mathbf{x}}(A) &= \frac{c}{2a}P_2(x_2)-\frac{b}{2a}P_2(ax_1+cx_2)\\
&\quad+\left\{P_1(x_1)P_1(x_2)-\sum_{i=0}^{a-1}P_1(x_1+c\frac{x_2+i}{a})P_1(\frac{x_2+i}{a})\right\}.\end{aligned}$$

Since $a \equiv 1$, $b \equiv c \equiv 0 \pmod N$, for any integers x, y, $P_2(\frac{ax+cy}{N}) = P_2(\frac{x}{N})$. Thus,

$$
\begin{aligned}
12 \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \binom{k-2}{r} x^{k-2-r} (-y)^r \Phi^{(2)}_{(\frac{x}{N}, \frac{y}{N})}(A) \\
= \binom{k-2}{r} (-1)^r (S_0 + S_1 + S_2), \quad (4.4.1)
\end{aligned}
$$

where

$$
\begin{aligned}
S_0 &:= 12 \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} x^{k-2-r} y^r \big(\frac{c}{2a} P_2(\frac{y}{N}) - \frac{b}{2a} P_2(\frac{x}{N})\big), \\
S_1 &:= 12 \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} x^{k-2-r} y^r P_1(\frac{x}{N}) P_1(\frac{y}{N}), \\
S_2 &:= -12 \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \sum_{i=0}^{a-1} x^{k-2-r} y^r P_1(\frac{x}{N} + c\frac{\frac{y}{N}+i}{a}) P_1(\frac{\frac{y}{N}+i}{a}).
\end{aligned}
$$

Recalling the notation $B_s^*(X) := B_s(X) - B_s(0)$, by (4.1), we immediately see

$$
\begin{aligned}
S_0 = & \frac{6c}{a} \frac{B^*_{k-r-1}(N)}{k-r-1} \left(\frac{B^*_{r+3}(N)}{N^2(r+3)} - \frac{B^*_{r+2}(N)}{N(r+2)} + \frac{B^*_{r+1}(N)}{6(r+1)} \right) \\
& - \frac{6b}{a} \left(\frac{B^*_{k-r+1}(N)}{N^2(k-r+1)} - \frac{B^*_{k-r}(N)}{N(k-r)} + \frac{B^*_{k-r-1}(N)}{6(k-r-1)} \right) \frac{B^*_{r+1}(N)}{r+1} \\
\equiv & \ 0 \ (\mathrm{mod}\ \frac{GCD\{6,N\}N}{D_{k,r}} \mathbb{Z}'_N).
\end{aligned}
$$

(Note also here that $b \equiv c \equiv 0 \pmod N$.) Let us rewrite S_2. According to the decomposition of $P_1(\frac{\frac{y}{N}+i}{a})$ as $\frac{y}{aN} + (\frac{i}{a} - \frac{1}{2})$, we may write

$$
S_2 = S_2' + S_2'', \qquad (4.4.2)
$$

where

$$\begin{aligned}
S_2' &= -12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{i=0}^{a-1} x^{k-2-r}y^r P_1(\frac{x}{N}+c\frac{y+Ni}{aN})(\frac{y}{aN}) \\
&= -12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} x^{k-2-r}y^r P_1(\frac{ax+cy}{N})(\frac{y}{aN}) \\
&= -12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} x^{k-2-r}y^r P_1(\frac{x}{N})(\frac{y}{aN}), \\
S_2'' &= -12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{i=0}^{a-1} x^{k-2-r}y^r P_1(\frac{x}{N}+c\frac{y+Ni}{aN})(\frac{i}{a}-\frac{1}{2}).
\end{aligned}$$

Noticing that $\frac{i}{a}-\frac{1}{2}\in\frac{1}{2}\mathbb{Z}_N'$ and that $c\frac{y+iN}{aN}\in\mathbb{Z}_N'$ as $a\equiv 1$, $c\equiv 0 \pmod N$, we may apply (4.2) to the part "$x^{k-2-r}P_1(\frac{x}{N}+c\frac{y+iN}{aN})$" of S_2'' to get the congruence

$$S_2''\equiv T_1+T_2 \ (\mathrm{mod}\ \frac{12N}{2d_{k-r-1}}\mathbb{Z}_N') \tag{4.4.3}$$

where

$$\begin{aligned}
T_1 :=& -12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{i=0}^{a-1}\frac{N^{k-2-r}}{k-1-r} \\
&\times\left(P_{k-1-r}(\frac{x}{N}+c\frac{y+Ni}{aN})-P_{k-1-r}(\frac{x}{N})\right)y^r(\frac{i}{a}-\frac{1}{2}) \\
=& -12\sum_{y=0}^{N-1}\sum_{i=0}^{a-1}\frac{1}{k-1-r} \\
&\times\left(P_{k-1-r}(c\frac{y+Ni}{a})-P_{k-1-r}(0)\right)y^r(\frac{i}{a}-\frac{1}{2}), \\
T_2 :=& -12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{i=0}^{a-1} x^{k-2-r}P_1(\frac{x}{N})y^r(\frac{i}{a}-\frac{1}{2}) \\
=& -12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} x^{k-2-r}P_1(\frac{x}{N})y^r(-\frac{1}{2}).
\end{aligned}$$

Combining the above S_1, S_2', T_2, we obtain

$$\begin{aligned} &S_1 + S_2' + T_2 \\ &= 12(\sum_x x^{k-2-r} P_1(\frac{x}{N})) \cdot (\sum_y y^{r+1})(\frac{1}{N} - \frac{1}{aN}) \\ &= 12\frac{a-1}{aN}\frac{B^*_{r+2}(N)}{r+2}(\frac{B^*_{k-r}(N)}{N(k-r)} - \frac{B^*_{k-r-1}(N)}{2(k-r-1)}) \equiv 0 \pmod{\frac{6N}{D_{k,r}}\mathbb{Z}'_N}. \end{aligned}$$

Meanwhile, since

$$-12\sum_{y=0}^{N-1}\sum_{i=0}^{a-1}\frac{1}{k-1-r}P_{k-1-r}(c\frac{y+Ni}{a})y^r(-\frac{1}{2}) = \frac{6}{a^{k-r}}\frac{P_{k-1-r}(0)}{k-1-r}\frac{B^*_{r+1}(N)}{r+1}$$

and

$$-12\sum_{y=0}^{N-1}\sum_{i=0}^{a-1} -\frac{P_{k-1-r}(0)}{k-1-r}y^r(\frac{i}{a}-\frac{1}{2}) = -6\frac{P_{k-1-r}(0)}{k-1-r}\frac{B^*_{r+1}(N)}{r+1},$$

it follows that

$$T_1 \equiv -12\sum_{y=0}^{N-1}\sum_{i=0}^{a-1}\frac{1}{k-1-r}P_{k-1-r}(c\frac{y+Ni}{a})y^r\frac{i}{a} \pmod{\frac{6N}{D_{k,r}}\mathbb{Z}'_N}.$$

Summing up the above discussions with (4.4.2-3), we may rewrite (4.4.1) as

$$\begin{aligned} &12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\binom{k-2}{r}x^{k-2-r}(-y)^r\Phi^{(2)}_{(\frac{x}{N},\frac{y}{N})}(A) \\ &\equiv -12\binom{k-2}{r}(-1)^r\sum_{y=0}^{N-1}\sum_{i=0}^{a-1}\frac{1}{k-1-r}P_{k-1-r}(c\frac{y+Ni}{a})y^r\frac{i}{a} \quad (4.4.4) \end{aligned}$$

modulo $\binom{k-2}{r}\frac{GCD\{6,N\}N}{D_{k,r}}\mathbb{Z}'_N$. Next, we shall consider the right hand side of the statement formula. Reading the coefficients of X^rY^{k-2-r} of (4.4.0) for $(x_1, x_2) = (0,0)$, we get

$$12\Phi_{\mathbf{0}}^{(r+1,k-1-r)}(A) = U_1 + U_2 + U_3,$$

where

$$U_1 = -12\binom{k-2}{r}\frac{P_k(0)}{k} \times \left\{\int_0^{-c/a}(-1)^r t^{k-2-r}dt + \int_{-b/a}^{0}(at+b)^r(ct+d)^{k-2-r}dt\right\},$$

$$U_2 = \frac{12\binom{k-2}{r}(-1)^r}{(k-1-r)(r+1)} \times \left\{P_{k-1-r}(0)P_{r+1}(0) - a^r\sum_{i=0}^{a-1}P_{k-1-r}(c\frac{i}{a})P_{r+1}(\frac{i}{a})\right\},$$

$$U_3 = -12\sum_{j=r+1}^{k-2}a^r c^{j-r}\binom{j}{r}\binom{k-2}{j}(-1)^j\sum_{i=0}^{a-1}\frac{P_{k-1-j}(ci/a)P_{j+1}(i/a)}{(k-1-j)(j+1)}.$$

As $b \equiv c \equiv 0 \pmod N$, it is easy to see $U_1 \equiv 0 \pmod{\frac{12\binom{k-2}{r}N}{LCM\{e_1,\ldots,e_{k-2}\}}\mathbb{Z}'_N}$. Moreover, taking the equality $\binom{j}{r}\binom{k-2}{j} = \binom{k-2}{r}\binom{k-2-r}{j-r}$ into accounts, we easily see that U_3 vanishes modulo $\frac{12\binom{k-2}{r}N}{LCM\{d_{j+1}d_{k-1-j}\,|\,r+1\le j\le k-2\}}\mathbb{Z}'_N$ (again as $c \equiv 0 \pmod N$). Now, comparing the distribution relations

$$\begin{cases} \Phi_{\mathbf{0}}^{(r+1,k-1-r)}(A) = N^{k-2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\Phi_{(\frac{x}{N},\frac{y}{N})}^{(r+1,k-1-r)}(A), \\ P_k(0) = N^{k-2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}P_k(\frac{y}{N}) = N^{k-2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}P_k(\frac{ax+cy}{N}), \end{cases}$$

we obtain

$$\begin{aligned} U_2 = {} & \frac{12N^{k-2}\binom{k-2}{r}(-1)^r}{(k-1-r)(r+1)}\left\{\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}P_{k-1-r}(\frac{x}{N})P_{r+1}(\frac{y}{N})\right. \\ & \left. - \sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{i=0}^{a-1}a^r P_{k-1-r}(\frac{x}{N}+\frac{cy}{aN}+\frac{ci}{a})P_{r+1}(\frac{i+\frac{y}{N}}{a})\right\} \\ = {} & 12\binom{k-2}{r}(-1)^r\left\{\frac{P_{k-1-r}(0)}{k-1-r}\frac{P_{r+1}(0)}{r+1}\right. \\ & \left. - \sum_{y=0}^{N-1}\sum_{i=0}^{a-1}a^r\frac{P_{k-1-r}(\frac{cy}{a}+\frac{ciN}{a})}{k-1-r}N^r\frac{P_{r+1}(\frac{i+\frac{y}{N}}{a})}{r+1}\right\}. \end{aligned} \tag{4.4.5}$$

We shall decompose the second term of the above last side according to

$$\frac{(aN)^r}{r+1}P_{r+1}(\frac{y}{aN}+\frac{i}{a}) \equiv \frac{(aN)^r}{r+1}P_{r+1}(\frac{y}{aN})+y^r\frac{i}{a} \pmod{\frac{N}{d_{r+1}}\mathbb{Z}'_N},$$

which follows from (4.2) and $0 \le \frac{y}{aN}+\frac{i}{a} < 1$. We compute:

$$\begin{aligned}
&\sum_{i=0}^{a-1}\sum_{y=0}^{N-1}\frac{P_{k-1-r}(\frac{y+iN}{a}c)}{k-1-r}(aN)^r\frac{P_{r+1}(\frac{y}{aN})}{r+1}\\
&\quad=\sum_{y=0}^{N-1}a^{-(k-2-r)}\frac{P_{k-1-r}(0)}{k-1-r}(aN)^r\frac{P_{r+1}(\frac{y}{aN})}{r+1}\\
&\quad\equiv a^{-(k-2-r)}\frac{P_{k-1-r}(0)}{(k-1-r)(r+1)}\sum_{y=0}^{N-1}(\frac{y^{r+1}}{aN}-\frac{r+1}{2}y^r)\\
&\quad\quad \pmod{\frac{N}{d_{r+1}d_{k-r-1}}\mathbb{Z}'_N}.
\end{aligned}$$

Noticing that the coefficient of X in $B_{r+2}(X)$ is $(-1)^{r+1}(r+2)B_{r+1}$ and that $P_{r+1}(0)=0$ if $r+1$ is odd, the above last side continues to

$$\begin{aligned}
&= a^{-(k-2-r)}\frac{P_{k-1-r}(0)}{k-1-r}(\frac{B^*_{r+2}(N)}{aN(r+1)(r+2)}-\frac{B^*_{r+1}(N)}{2(r+1)})\\
&\equiv \frac{P_{k-1-r}(0)}{k-1-r}\frac{(-1)^{r+1}}{r+1}P_{r+1}(0) \pmod{\frac{N}{2D_{k,r}}\mathbb{Z}'_N}\\
&= \frac{P_{k-1-r}(0)}{k-1-r}\frac{P_{r+1}(0)}{r+1}.
\end{aligned}$$

Here we also used $a \equiv 1 \pmod N$. Therefore, we conclude

$$U_2 \equiv -\frac{12\binom{k-2}{r}(-1)^r}{k-1-r}\sum_{i=0}^{a-1}\sum_{y=0}^{N-1}P_{k-1-r}(\frac{y+iN}{a}c)y^r\frac{i}{a} \pmod{\frac{6\binom{k-2}{r}N}{D_{k,r}}\mathbb{Z}'_N},$$

which coincides with (4.4.4). This completes the proof of Proposition (4.3). □

Corollary (4.5). *Let $k \ge 2$. Then, there exists an integer $D(k)$ (depending only on k) such that for every $N \ge 1$,*

$$12\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}(xY-yX)^{k-2}\Phi^{(2)}_{(\frac{x}{N},\frac{y}{N})}(A) \equiv 12\Phi^{(k)}_{\mathbf{0}}(A) \pmod{\frac{N}{D(k)}\mathbb{Z}'_N}.$$

Proof. This is an immediate consequence of Proposition (4.2). We may take $D(k)$ so that

$$\frac{N}{D(k)}\mathbb{Z}'_N = \sum_{r=0}^{k-2}\binom{k-2}{r}\frac{GCD\{6,N\}N}{D_{k,r}}\mathbb{Z}'_N \subset \mathbb{Q}.$$

□

Example (4.6). Take a matrix $A = \begin{pmatrix} 122 & -4961 \\ 363 & -14761 \end{pmatrix} \in \Gamma(11^2)$, and let us examine our congruence formula (4.5) for $k = 6$, $N = 11$. In this case, for any $0 \le r \le 4$, $\binom{4}{r}\frac{GCD\{6,N\}11}{12D_{6,r}}\mathbb{Z}'_{11} = 11\mathbb{Z}'_{11}$, hence $\frac{N}{12D(6)}\mathbb{Z}'_{11}$ may be taken as $11\mathbb{Z}'_{11}$ Computation on RHS shows

$$\begin{aligned}
&\Phi^{(6)}_{\mathbf{0}}(A)(X,Y)\\
&= \frac{6157810527168637}{315}X^4 + \frac{117260782677249595}{504}YX^3\\
&\quad + \frac{37381997569467617}{36}Y^2X^2 + \frac{5190578682530622937}{2520}Y^3X\\
&\quad + \frac{1930511018334372017}{1260}Y^4,
\end{aligned}$$

while that on LHS shows

$$\begin{aligned}
&\sum_{x,y=0}^{10}(xY - yX)^4\Phi^{(2)}_{(\frac{x}{11},\frac{y}{11})}(A)\\
&\quad = \frac{157339}{6}X^4 + \frac{557102}{3}YX^3 - 102157Y^2X^2\\
&\quad\quad - \frac{230146}{3}Y^3X + \frac{717205}{6}Y^4.
\end{aligned}$$

In both sides, each coefficient except for that of X^2Y^2 is prime to 11. Now, their difference may be computed as:

$$\begin{aligned}
&\Phi^{(6)}_{\mathbf{0}}(A)(X,Y) - \sum_{x,y=0}^{10}(xY - yX)^4\Phi^{(2)}_{(\frac{x}{11},\frac{y}{11})}(A)\\
&= \frac{12315621037816679}{630}X^4 + \frac{117260782583656459}{504}YX^3\\
&\quad + \frac{37381997573145269}{36}Y^2X^2 + \frac{5190578682723945577}{2520}Y^3X\\
&\quad + \frac{1930511018183758967}{1260}Y^4\\
&\equiv 0 \pmod{11\mathbb{Z}'_{11}}.
\end{aligned}$$

Example (4.7). This example was suggested by Y. Morimoto. Let $A = \begin{pmatrix} 6643 & 56295 \\ 1539 & 13042 \end{pmatrix} \in \Gamma(81)$, and consider the case of weight 10. Then, for $0 \le r \le 8$, $\binom{8}{r} \frac{GCD\{6,81\}81}{12D_{10,r}} \mathbb{Z}_3' = 3\mathbb{Z}_3'$. The coefficient $\Phi_{\mathbf{0}}^{(5,5)}(A)$ of X^4Y^4 in $\Phi_{\mathbf{0}}^{(10)}(A)$ is computed as

$$\Phi_{\mathbf{0}}^{(5,5)}(A) = -\frac{116073541903991309357774956489289951 9}{8}$$

which is prime to 3, while

$$\sum_{x,y=0}^{80} \binom{8}{4} x^4(-y)^4 \Phi^{(2)}_{(\frac{x}{81},\frac{y}{81})}(A) = -8836456074579123550$$

which is also prime to 3. Now the difference of these two values is

$$\Phi_{\mathbf{0}}^{(5,5)}(A) - \sum_{x,y} \binom{8}{4} x^4(-y)^4 \Phi^{(2)}_{(\frac{x}{81},\frac{y}{81})}(A)$$
$$= -\frac{1160735419039913022886100968259911119}{8}$$

which is divisible by 3 (but not divisible by 9).

Example (4.8). Let us consider the case of weight 4 for $A = \begin{pmatrix} 385 & 15616 \\ 15744 & 638593 \end{pmatrix} \in \Gamma(128)$. Then, $\binom{2}{r} \frac{GCD\{6,128\}128}{12D_{6,r}} \mathbb{Z}_2'$ is $4\mathbb{Z}_2'$ for $r = 0, 2$, and $8\mathbb{Z}_2'$ for $r = 1$. The computation shows

$$\Phi_{\mathbf{0}}^{(4)}(A) = \frac{1236394022}{45} X^2 + \frac{33706991732}{15} XY + \frac{229732849464}{5} Y^2,$$

and

$$\sum_{x,y=0}^{127} (xY - yX)^2 \Phi^{(2)}_{(\frac{x}{128},\frac{y}{128})}(A)$$
$$= \frac{80854186}{3} X^2 - \frac{10808380}{3} XY - \frac{80165720}{3} Y^2.$$

Both of them do not belong to $4\mathbb{Z}_2'[X, Y]$ (because of the coefficients of X^2). Now the difference of these two polynomials is computed as

$$\Phi_{\mathbf{0}}^{(4)}(A) - \sum_{x,y=0}^{127} (xY - yX)^2 \Phi^{(2)}_{(\frac{x}{128},\frac{y}{128})}(A)$$
$$= \frac{23581232}{45} X^2 + \frac{33761033632}{15} XY + \frac{689599376992}{15} Y^2$$

which belongs to $4\mathbb{Z}_2'[X,Y]$ (actually to $16\mathbb{Z}_2'[X,Y]$).

At the time of writing this paper, the author does not have an example assuring whether our estimate of the modulus is best possible with respect to 2-powers.

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