Unknown Book Proceedings Series
Volume **00**, XXXX

Harmonic and equianharmonic equations in the Grothendieck-Teichmüller group, II

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Abstract. In [LS], Lochak and Schneps introduced the "harmonic parameter g" of the Grothendieck-Teichmüller group $\widehat{GT}$. We closely study the behavior of g on the absolute Galois group $G_{\mathbb{Q}}$ using a family of lemniscate elliptic curves. We obtain a relationship of the adelic beta function and the harmonic parameter specialized in the matrix group $\mathrm{SL}_2(\hat{\mathbb{Z}})$.

§1. Introduction.

Let $\widehat{GT}$ be the profinite Grothendieck-Teichmüller group introduced and studied by Drinfeld [Dr], Ihara [I1]. It contains the Galois group $G_{\mathbb{Q}}=\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and has the standard parametrization $(\lambda,f):\widehat{GT}\hookrightarrow\hat{\mathbb{Z}}^{\times}\times\hat{F}_2$ ($\hat{\mathbb{Z}}$, $\hat{F}_2$ are respectively the profinite completions of the integer ring $\mathbb{Z}$ and of F_2, the free group generated by non-commutative symbols x and y.) In [NT], we closely studied Galois behaviors of the auxiliary parameters g and $h:\widehat{GT}\to\hat{F}_2$ which had been introduced in Lochak-Schneps [LS] so as to decompose (uniquely) the main parameter $f:\widehat{GT}\to\hat{F}_2$ (actually f terminates into the commutator subgroup $[\hat{F}_2,\hat{F}_2]$ by definition) as follows:

$$f(x,y)=g(y,x)^{-1}g(x,y)=\begin{cases} y^{-\frac{\lambda-1}{2}}h(y,z)^{-1}h(x,y) & (\lambda\equiv 1 \mod 6),\\ y^{-\frac{\lambda-1}{2}}h(y,z)^{-1}y^{-1}h(x,y) & (\lambda\equiv -1 \mod 6),\end{cases}$$

where $z=(xy)^{-1}\in\hat{F}_2$. We showed in [NT] that the parameters g and h can be directly written by (λ,f) on the image of $G_{\mathbb{Q}}$ in $\widehat{GT}$, and presented several new-type equations satisfied by the Galois image.

In this paper, we continue our study with mainly concentrating on the parameter g. Our motivating problem to the present paper concerns with the matrix specialization of this parameter. In fact, in [N-I] Corollary 4.13, one of the authors explicitly computed the matrix $f_\sigma\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right)\right)\in\mathrm{SL}_2(\hat{\mathbb{Z}})$, which turned out later

1991 *Mathematics Subject Classification.* Primary 14H30; Secondary 11G05, 20E18, 20F36.
Key words and phrases. adelic beta functions, lemniscate elliptic curve.

in [NS] Remark 2.7 to be decomposed as the following "intriguing" form:

$$(*)\qquad f_\sigma\left(\begin{pmatrix}1&2\\0&1\end{pmatrix},\begin{pmatrix}1&0\\-2&1\end{pmatrix}\right)=(-1)^{\frac{\lambda_\sigma-1}{2}}\begin{pmatrix}1&0\\-8\rho_2(\sigma)&1\end{pmatrix}\begin{pmatrix}\lambda_\sigma^{-1}&0\\0&\lambda_\sigma\end{pmatrix}\begin{pmatrix}1&-8\rho_2(\sigma)\\0&1\end{pmatrix}.$$

Here, $\rho_2: G_{\mathbb{Q}}\to\hat{\mathbb{Z}}$ designates the Kummer 1-cocycle along the positive roots of 2. This symmetric matrix should be decomposed into the form ${}^tG\cdot G$ for some G corresponding to the image $G=g_\sigma\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right)\right)$. However, unlike the decomposition $f(x,y)=g(y,x)^{-1}g(x,y)$ in $\hat{F}_2$, the determination of G in $\mathrm{SL}_2(\hat{\mathbb{Z}})$ does not follow from the decomposition property $f_\sigma(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right))={}^tG\cdot G$; namely, there remains apriori ambiguity modulo left multiplication by "rotation matrices" on G. In this paper, we shall determine the true choice of the matrix G representing $g_\sigma\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right)\right)$. Our answer is that the main factor is given by the special value of Anderson-Ihara's adelic beta function $B_\sigma(\frac14,\frac14)$. In terms of the measure version $\mathbb{B}_\sigma\in\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$ of B_σ (see §2 below), we prove:

Theorem D. (§5 (5.5)) *For each $\sigma\in G_{\mathbb{Q}}$, we have*

$$g_\sigma\left(\begin{pmatrix}1&2\\0&1\end{pmatrix},\begin{pmatrix}1&0\\-2&1\end{pmatrix}\right)=(-1)^{\frac{\lambda_\sigma-1}{2}}\mathbb{B}_\sigma\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\cdot\begin{pmatrix}\lambda_\sigma^{-1}&-8\rho_2(\sigma)\lambda_\sigma^{-1}\\0&(-1)^{\frac{\lambda_\sigma-1}{2}}\end{pmatrix}$$

in $\mathrm{GL}_2(\hat{\mathbb{Z}})$.

Once Theorem D is obtained, it is easy to recover $(*)$ by a basic property of $\mathbb{B}_\sigma$ (see (2.5.1)). The connection of the above two sides is, naively speaking, via the classical link between the harmonic ratio and the lemniscate elliptic curve $Y^2=X^3-X$. The right hand side reflects the adelic Galois version of the classical period integral via the beta function:

$$\int_0^1\frac{dt}{\sqrt{1-t^4}}=\frac{1}{4\sqrt2}B(\frac14,\frac14).$$

Since the lemniscate elliptic curve is dominated by the Fermat quartic, it is not surprising that the Galois action on the Tate module is given by the adelic beta values at the pair of 4-th roots of unity. However, we wish to take an explicit choice of a basis of this Tate module which allows well controls to be linked with the Grothendieck-Teichmüller parameters. Thereby we construct in §3 an explicit $G_{\mathbb{Q}}$-compatible immersion of the fundamental group of the (affine) lemniscate elliptic curve into the braid group with 4 strands. A close comparison of tangential base-points on the braid configuration space makes clear the appearance of the harmonic ratio $(0,\frac12,1,\infty)$ at an addressed object (§4). We settle the proof of Theorem D in §5.

In [NT] Theorem A, Proposition 6.1 (2), we proved the following equations in the braid group $\hat{B}_3=\langle\tau_1,\tau_2\mid\tau_1\tau_2\tau_1=\tau_2\tau_1\tau_2=:\eta\rangle$ for the image of $G_{\mathbb{Q}}\hookrightarrow\widehat{GT}$:

$$(\mathrm{GF}_0)\qquad g(\tau_1^2,\tau_2^2)=\eta^{2\rho_2-\rho_3}f(\tau_1,\eta)\tau_1^{-2\rho_2+3\rho_3},$$

$$(\mathrm{GF}_1)\qquad g(\tau_1^2,\tau_2^2)=f(\tau_1^2,\eta)\tau_1^{4\rho_2},$$

$$(\mathrm{GG})\qquad g(\tau_1^2,\tau_2^2)=\eta^{2\rho_2-\rho_3}g(\tau_1,\tau_2)\tau_1^{-4\rho_2+3\rho_3}.$$

Here and henceforth, we sometimes omit σ from formulas for simplicity, when no confusion could occur from contexts. Combining Theorem D with the above (GF0), (GF1) and (GG) specialized with $\tau_1\mapsto\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\tau_2\mapsto\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$, we immediately obtain

COROLLARY E. *On the image of $G_{\mathbb{Q}}$ in $\widehat{GT}$, we have*

$$f\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right)=(-1)^{\frac{\lambda-1}{2}}\mathbb{B}\left(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right)\cdot\left(\begin{smallmatrix}\lambda^{-1}&-12\rho_2\lambda^{-1}\\0&(-1)^{\frac{\lambda-1}{2}}\end{smallmatrix}\right).$$

$$f\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right)=(-1)^{\frac{\lambda-1}{2}-\rho_2}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)^{\rho_3}\mathbb{B}(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right))\begin{pmatrix}\lambda^{-1}&(-6\rho_2-3\rho_3)\lambda^{-1}\\0&(-1)^{\frac{\lambda-1}{2}}\end{pmatrix},$$

$$g\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right)=(-1)^{\frac{\lambda-1}{2}-\rho_2}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)^{\rho_3}\mathbb{B}(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right))\begin{pmatrix}\lambda^{-1}&(-4\rho_2-3\rho_3)\lambda^{-1}\\0&(-1)^{\frac{\lambda-1}{2}}\end{pmatrix}.$$

One obvious remaining subject is to pursue the analogy of these results for the equianharmonic ratio $(0,e^{2\pi i/6},1,\infty)$. The lemniscate elliptic curve will then be replaced by the mordell elliptic curve $Y^2=X^4-X$ (*cf.* (3.4)). We hope to continue the investigation in a subsequent work. In the present paper, all algebraic varieties are treated as defined over $\mathbb{Q}$. We fix once and for all the embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$. The notation ζ_n always represents the specified primitive n-th root of unity $\exp(2\pi i/n)\in\mathbb{C}$.

§2. Short review of the adelic beta function.

(2.1) We shall start with the standard action of $\widehat{GT}$ on the free profinite group $\hat{F}_2$ generated by the symbols x and y. For each $\sigma\in\widehat{GT}$ associated are two parameters $\lambda=\lambda_\sigma\in\hat{\mathbb{Z}}^\times$ and $f=f_\sigma(x,y)\in[\hat{F}_2,\hat{F}_2]$ which are unique to satisfy $\sigma(x)=x^\lambda$, $\sigma(y)=f^{-1}y^\lambda f$. The abelianization $\hat{\mathbb{Z}}^2(1)$ of $\hat{F}_2$ is generated by the images $\bar{x},\bar{y}$ of x,y respectively. The complete group algebra $\hat{\mathbb{Z}}[\![\hat{F}_2]\!]$ and its abelianization $\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$ play fundamental roles in Ihara's definition of the adelic beta function $\mathbb{B}_\sigma$. Recall that in the former ring $\hat{\mathbb{Z}}[\![\hat{F}_2]\!]$, one has free differential operators $\partial_x:=\frac{\partial}{\partial x}$, $\partial_y:=\frac{\partial}{\partial y}$ as the profinite analog of Fox's operators in the discrete case (cf. [I1] Appendix). Using them, one may associates the Jacobian matrix

$$J_\sigma:=\begin{pmatrix}\partial_x\sigma(x)&\partial_y\sigma(x)\\\partial_x\sigma(y)&\partial_y\sigma(y)\end{pmatrix}\ \left(=\begin{pmatrix}\frac{x^\lambda-1}{x-1}&0\\(\partial_x f)(y^\lambda-1)&\frac{y^\lambda-1}{y-1}+(\partial_y f)(y^\lambda-1)\end{pmatrix}\right)$$

for each $\sigma\in\widehat{GT}$ whose abelianization image J_σ^{ab} belongs to $\mathrm{GL}_2(\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!])$. Then,

DEFINITION. (Ihara [I2])

$$\mathbb{B}_\sigma(\bar{x},\bar{y}):=\frac{\bar{x}-1}{\bar{x}^\lambda-1}\frac{\bar{y}-1}{\bar{y}^\lambda-1}\det(J_\sigma^{\mathrm{ab}})\ \Big(=1+(\partial_y f)^{\mathrm{ab}}(\bar{y}-1)\Big).$$

Since $\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$ is identified with $\varprojlim_n\big(\hat{\mathbb{Z}}[\bar{x},\bar{y}]/(\bar{x}^n-1,\bar{y}^n-1)\big)$, one may speak about the specialization $\mathbb{B}_\sigma(\zeta,\zeta')$ where the variables $\bar{x}$, $\bar{y}$ are specialized to any roots of unity ζ,ζ'. The specialization is valued in $\hat{\mathbb{Z}}\otimes\mathbb{Z}(\zeta,\zeta')$. As an element of $\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$, $\mathbb{B}_\sigma$ can be determined by the collection of all specialization values at pairs of roots of unity. G.Anderson [A] introduces the adelic beta function $B_\sigma:(\mathbb{Q}/\mathbb{Z})^2\to\hat{\mathbb{Z}}\otimes\mathbb{Q}^{\mathrm{ab}}$ by setting $B_\sigma(\frac{a}{n},\frac{b}{n}):=\mathbb{B}_\sigma(\zeta_n^a,\zeta_n^b)$ which is a closer analogue of the classical beta function $B(\frac{a}{n},\frac{b}{n}):=\int_0^1 t^{\frac{a}{n}-1}(1-t)^{\frac{b}{n}-1}dt$. These special values control the complex multiplication theory of the quotients of Fermat Jacobians, especially the Galois actions on the torsion points of them. In this paper, however, we will establish the control directly for the lemniscate elliptic curve in an elementary way, so that we do not enter details of Anderson's deep device here. Also, we mainly use $\mathbb{B}_\sigma$ (the measure version of B_σ) for the usefulness of its Galois theoretic formations.

Some of the first basic properties of $\mathbb{B}_\sigma$ are:

$$\text{(B0)} \qquad \mathbb{B}_\sigma(\bar{x},\bar{y}) \in \hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]^\times,$$

$$\text{(B1)} \qquad \mathbb{B}_\sigma(\bar{x},\bar{y}) = \mathbb{B}_\sigma(\bar{y},\bar{x}),$$

$$\text{(B2)} \qquad \mathbb{B}_\sigma(1,\bar{x}) = 1,$$

$$\text{(B3)} \qquad \mathbb{B}_\sigma(\bar{x},\bar{x}^{-1}) = \frac{1-\bar{x}}{1-\bar{x}^\lambda}\cdot \bar{x}^{\frac{\lambda-1}{2}}\cdot\lambda_\sigma,$$

$$\textit{in particular},\ \mathbb{B}_\sigma(-1,-1) = (-1)^{\frac{\lambda-1}{2}}\lambda_\sigma.$$

(2.2) An alternative useful characterization of $\mathbb{B}_\sigma$ may be given as follows. Let the commutator subgroup $[\hat{F}_2,\hat{F}_2]$ be denoted $\hat{F}_2'$, and let the double commutator subgroup $[\hat{F}_2',\hat{F}_2']$ be denoted $\hat{F}_2''$. Then, the quotient $\hat{F}_2'/\hat{F}_2''$ is acted on by $\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$ via (linear extension of) conjugation, and Ihara showed that $\hat{F}_2'/\hat{F}_2''$ is a free cyclic $\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$-module generated by $[x,y]=xyx^{-1}y^{-1}$. We shall write this conjugate action by $*$. Then, there exists a unique $\mathbb{A}_\sigma(\bar{x},\bar{y})\in\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$ such that

$$\text{(FA)} \qquad f_\sigma(x,y) \equiv \mathbb{A}_\sigma(\bar{x},\bar{y}) * [x,y] \bmod \hat{F}_2''.$$

Then, the theory of Blanchfield-Lyndon sequence relates this quantity with the adelic beta function as follows:

$$\text{(BA)} \qquad \mathbb{B}_\sigma(\bar{x},\bar{y}) = 1 + \mathbb{A}_\sigma(\bar{x},\bar{y})\cdot(\bar{x}-1)(\bar{y}-1).$$

One remarkable use of $\mathbb{A}_\sigma$ is that if we define $\rho_N : \widehat{GT}\to\hat{\mathbb{Z}}$ by

$$\rho_N(\sigma) := \frac{1}{N}\sum_{c=0}^{N-1}(\zeta_N^c - 1)\mathbb{A}_\sigma(\zeta_N^c,1),$$

then, restricted on $G_{\mathbb{Q}}\subset\widehat{GT}$, ρ_N gives the Kummer 1-cocycle on positive roots of N, i.e., $\sigma(\sqrt[k]{N}) = \zeta_N^{\rho_N(\sigma)}\sqrt[k]{N}$ for all $k=1,2,\ldots$ and $\sigma\in G_{\mathbb{Q}}$. From this, one can easily see that $\mathbb{A}_\sigma(-1,1) = -\rho_2(\sigma)$ (see Appendix (A4); *cf.* also [NS] Proposition 5.3).

(2.3) Let $\mathbb{W}_p$ be the ring of restricted Witt vectors of $\bar{\mathbb{F}}_p$, and let $\mathbb{W}=\prod_p\mathbb{W}_p$ be the direct product for all primes p which is topologized by the collection $\{n\mathbb{W}\}_{n\in\mathbb{N}}$ as the fundamental system of neighborhoods of 0. One can consider the enlarged ring $\mathbb{W}[\![\hat{\mathbb{Z}}^2(1)]\!]$ naturally containing $\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!]$. Ihara [I2] extends Anderson's hyperadelic Gamma function $\mathit{\Gamma}_\sigma$ for all $\sigma\in\widehat{GT}$. In particular, he deduces that the 5-cyclic relation of $\widehat{GT}$ allows the "Γ-decomposition" of $\mathbb{B}_\sigma$:

$$\text{(B}\Gamma\text{)} \qquad \forall\sigma\in\widehat{GT},\ \exists\,\mathit{\Gamma}_\sigma(\bar{x})\in\mathbb{W}[\![\hat{\mathbb{Z}}]\!]\ \text{s.t.}\ \mathbb{B}_\sigma = \frac{\mathit{\Gamma}_\sigma(\bar{x})\,\mathit{\Gamma}_\sigma(\bar{y})}{\mathit{\Gamma}_\sigma(\bar{x}\bar{y})}.$$

G.Anderson [A] proved that, for $\sigma\in G_{\mathbb{Q}}$, the hyperadelic Gamma function $\mathit{\Gamma}_\sigma$ satisfies an analog of the Gauss n-multiplication formula:

$$(\Gamma_n) \qquad \prod_{nc=0}\frac{\mathit{\Gamma}_\sigma(\zeta_n^c\bar{x})}{\mathit{\Gamma}_\sigma(\zeta_n^c)}\cdot\frac{1}{\mathit{\Gamma}_\sigma(\bar{x}^n)} = \bar{x}^{\,n\rho_n(\sigma)}.$$

It is unknown whether (Γ_n) holds on the total $\widehat{GT}$. Recently, Furusho [F] showed that the property (Γ_2) holds on a subgroup of $\widehat{GT}$ satisfying certain geometric relations (IV) and (K_2). The following lemma indicates that the two main parameters λ and f of $\widehat{GT}$ are "not totally independent."

(2.4) Lemma. *If $\sigma \in \widehat{GT}$ satisifies (Γ_2) (e.g., $\sigma \in G_{\mathbb{Q}}$), then,*

$$(-1)^{\rho_2(\sigma)} = \frac{\sqrt{-1}^{\frac{\lambda_\sigma - 1}{2}}(1 - \sqrt{-1}^{\lambda_\sigma})}{1 - \sqrt{-1}}.$$

Proof. For $\sigma \in G_{\mathbb{Q}}$, one can separately check that each side coincides with 1 when $\lambda_\sigma \equiv \pm 1 \mod 8$, and with -1 when $\lambda_\sigma \equiv \pm 5 \mod 8$ (Notice that $(-1)^{\rho_2(\sigma)}$ depends only on $\lambda \mod 8$ since $\sqrt{2} \in \mathbb{Q}(\zeta_8)$.) For general $\sigma \in \widehat{GT}$, the property (Γ_2) reads: $\frac{\varGamma_\sigma(\bar{x})\varGamma_\sigma(-\bar{x})}{\varGamma_\sigma(1)\varGamma_\sigma(-1)} \cdot \frac{1}{\varGamma_\sigma(\bar{x}^2)} = \bar{x}^{2\rho_2(\sigma)}$. Putting $\bar{x} = \sqrt{-1}$, we then obtain

$$(-1)^{\rho_2(\sigma)} = (-1)^{-\rho_2(\sigma)} = \frac{\mathbb{B}_\sigma(-1,-1)}{\mathbb{B}_\sigma(\sqrt{-1}, -\sqrt{-1})}.$$

The lemma follows from this and (B3). □

(2.5) We finally complement a few observations on the specialization $\mathbb{B}_\sigma((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))$ in the 2 by 2 matrix algebra $M_2(\hat{\mathbb{Z}})$. Let R be the commutative subring consisting of matrices of the form $(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix})$ $(a, b \in \hat{\mathbb{Z}})$. The issued $\mathbb{B}_\sigma((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))$ is defined as the specialization image of $\mathbb{B}_\sigma$ under the map $\hat{\mathbb{Z}}[\![\hat{\mathbb{Z}}^2(1)]\!] \to R$ $(x, y \mapsto (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))$. Then, for all $\sigma \in \widehat{GT}$,

$$ {}^t\mathbb{B}_\sigma((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})) \cdot \mathbb{B}_\sigma((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})) = (-1)^{\frac{\lambda-1}{2}} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{2.5.1}$$

Proof. Since the matrix transposition gives an algebra automorphism on R, the result amounts to the formula $\mathbb{B}_\sigma(\sqrt{-1}^{-1}, \sqrt{-1}^{-1})\mathbb{B}_\sigma(\sqrt{-1}, \sqrt{-1}) = (-1)^{\frac{\lambda-1}{2}}\lambda$. This is a simple result of (B2) (and (BΓ)). □

Since the transposed inverse of $B = (\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix})$ $(a, b \in \hat{\mathbb{Z}})$ is $\det(B)^{-1}B$, from (2.5.1) it is immediate to see

$$\det \mathbb{B}_\sigma((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})) = (-1)^{\frac{\lambda-1}{2}}\lambda. \tag{2.5.2}$$

§3. Lemniscate curve.

(3.1) In this section, we shall consider the lemniscate elliptic curve

$$E^{\mathrm{lem}} : Y^2 = X^3 - X$$

as a cyclic branched cover over the projective t-line $\mathbf{P}^1_t$ ramified only over $t = 0, 1, \infty$. The fundamental group of $E^{\mathrm{lem}} - \{O\}$ is then recognized as a suitable subquotient of $\pi_1(\mathbf{P}^1_t - \{0, 1, \infty\}, \overrightarrow{01})$. We recall that the standard action of $\sigma = (\lambda, f) \in \widehat{GT}$ (where $\lambda \in \hat{\mathbb{Z}}^\times$, $f \in [\hat{F}_2, \hat{F}_2]$) on the latter group is given as follows:

$$\begin{cases} x & \mapsto x^\lambda, \\ y & \mapsto f(y,x)y^\lambda f(x,y), \\ z & \mapsto x^{\frac{\lambda-1}{2}} f(z,x) z^\lambda f(x,z) x^{\frac{1-\lambda}{2}}, \end{cases} \tag{3.1.1}$$

where x, y, z are loops of $\pi_1(\mathbf{P}^1_t - \{0,1,\infty\}, \overrightarrow{01})$ with $xyz = 1$ chosen as in [NT] §1. The purpose of this section is to give explicit descriptions of the Galois representations in the Tate module of E^{lem} in terms of the adelic beta functions.

(3.2) The lemniscate elliptic curve $E^{\mathrm{lem}} : Y^2 = X^3 - X$ can be realized as a cyclic ramified cover of $\mathbf{P}^1_t$ of degree 4 by

$$X = \frac{1}{\sqrt{t}} \quad \text{and} \quad Y = \sqrt{\frac{1-t}{\sqrt{t}^3}}.$$

The geometric fundamental group of $E^{\mathrm{lem}} - \{O\}$ is naturally regarded as a subgroup of $\pi_1(\mathbf{P}^1_t - \{0,1,\infty\}; e_1|2, e_\infty|4) = \widehat{\Delta}(\infty, 2, 4)$ with free generators $x_1 := x^{-1}z$, $x_2 := xz^{-1}$. The commutator $[x_1, x_2] = x^{-4}$ generates an inertia subgroup over the point $O \in E^{\mathrm{lem}}$. Therefore, if the extra condition '$e_0|4$' is imposed on the above fundamental group of $\mathbf{P}^1_t - \{0,1,\infty\}$, then the finite-adelic Tate module $T_f(E^{\mathrm{lem}})$ appears as the corresponding subgroup fitting in the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(E^{\mathrm{lem}}_{\overline{\mathbb{Q}}} - \{O\}) & \longrightarrow & \widehat{\Delta}(\infty,2,4) & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & T_f(E^{\mathrm{lem}}) & \longrightarrow & \widehat{\Delta}(4,2,4) & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 1.
\end{array}$$

Note here that $\Delta(4,2,4)$ denotes the triangle group of Euclidean type corresponding to a plane tiling by isosceles right triangles which refines the regular square tessellation. Since the quotient $\pi_1(\mathbf{P}^1_t - \{0,1,\infty\}; e_0|4, e_1|2, e_\infty|4)$ is preserved by the operation of $\widehat{GT}$, the $G_{\mathbb{Q}}$-action on the Tate module $T_f(E^{\mathrm{lem}})$ is naturally extended to the $\widehat{GT}$-action on it.

(3.3) Proposition. *Let $\bar{x}_1, \bar{x}_2$ be the basis of the Tate module $T_f(E^{\mathrm{lem}})$ which are the images of x_1, x_2 respectively, and define the action of the ring $\hat{\mathbb{Z}}[\sqrt{-1}]$ on $T_f(E^{\mathrm{lem}})$ by $\sqrt{-1} : \bar{x}_1 \mapsto -\bar{x}_2$, $\bar{x}_2 \mapsto \bar{x}_1$. Then, each element $\sigma = (\lambda, f) \in \widehat{GT}$ acts on $T_f(E^{\mathrm{lem}})$ by*

$$\begin{cases} \bar{x}_1 & \mapsto (-1)^{\frac{\lambda-1}{2} + \rho_2(\sigma)} \mathbb{B}_\sigma(\sqrt{-1}, \sqrt{-1}) \cdot \bar{x}_1, \\ \bar{x}_2 & \mapsto (-1)^{\rho_2(\sigma)} \mathbb{B}_\sigma(\sqrt{-1}, \sqrt{-1}) \cdot \bar{x}_2. \end{cases}$$

Proof. First, we compute the action of $\sigma = (\lambda, f)$ on $x_2 = xz^{-1}$. The formula in (3.1) implies

$$\begin{aligned} \sigma(x_2) &= x^\lambda x^{\frac{\lambda-1}{2}} f(z,x) z^{-\lambda} f(x,z) x^{\frac{1-\lambda}{2}} \\ &= \mathrm{Int}(x)^{\frac{\lambda-1}{2}} \left(\mathrm{Int}(x)^\lambda (f(x,z)^{-1}) \cdot x^\lambda z^{-\lambda} f(x,z) \right). \end{aligned}$$

We shall use the notation $\equiv$ to designate the congruence modulo the kernel of the projection of $\widehat{\Delta}(\infty,2,4)$ onto $\widehat{\Delta}(4,2,4)$. Since $[x,z] \equiv x_1^{-1} x_2$ in $T_f(E^{\mathrm{lem}})$, the formula (FA) of §2 implies that

$$f(x,z) \equiv (1 - \sqrt{-1}) \mathbb{A}_\sigma(\sqrt{-1}, \sqrt{-1}) \cdot \bar{x}_2.$$

Here, note that the inner (conjugate) actions $\mathrm{Int}(x)$ and $\mathrm{Int}(z)$ on $T_f(E^{\mathrm{lem}})$ are given by multiplication by $\sqrt{-1}$. We also have the congruence formula:

$$x^\lambda z^{-\lambda} \equiv \frac{1-\sqrt{-1}^\lambda}{1-\sqrt{-1}} \cdot \bar{x}_2 \qquad (\lambda \in \hat{\mathbb{Z}})$$

which can be easily proved by induction on $\lambda \in \mathbb{Z}_{>0}$ and by the standard continuity argument. Then, in total, it follows that

$$\sigma(x_2) \equiv \frac{\sqrt{-1}^{\frac{\lambda-1}{2}}(1-\sqrt{-1}^\lambda)}{1-\sqrt{-1}}\big(1+\mathbb{A}_\sigma(\sqrt{-1},\sqrt{-1})(1-\sqrt{-1})^2\big)\cdot \bar{x}_2.$$

Since the value $\sqrt{-1}^{\frac{\lambda-1}{2}}(1-\sqrt{-1}^\lambda)/(1-\sqrt{-1})$ is equal to $(-1)^{\rho_2(\sigma)}$ (*cf*. Lemma (2.4)), we finally obtain

$$\sigma(x_2) \equiv (-1)^{\rho_2(\sigma)}\mathbb{B}_\sigma(\sqrt{-1},\sqrt{-1})\cdot \bar{x}_2.$$

To compute $\sigma(\bar{x}_1)$, we apply the above formula to $\bar{x}_1 \equiv \mathrm{Int}(x)(x_2)$. Then,

$$\sigma(x_1) \equiv \mathrm{Int}(x^\lambda)\big((-1)^{\rho_2(\sigma)}\mathbb{B}_\sigma(\sqrt{-1},\sqrt{-1})\cdot\bar{x}_2\big) = \sqrt{-1}^{\lambda-1}(-1)^{\rho_2(\sigma)}\mathbb{B}_\sigma(\sqrt{-1},\sqrt{-1})\cdot\bar{x}_1.$$

This completes the proof. □

(3.4) Before closing this section, we shall discuss briefly the case of the *mordell elliptic curve* $E^{\mathrm{mor}} : Y^2 = X^4 - X$. This curve can be realized as a cyclic ramified cover of $\mathbf{P}^1_t$ of degree 6 by $t = X^3$: The function field of E^{mor} is generated by $X = \sqrt[3]{t}$ and $Y = \sqrt[6]{t}\sqrt{t-1}$. The geometric fundamental group of $E^{\mathrm{mor}} - \{O\}$ is naturally regarded as a subgroup of $\pi_1(\mathbf{P}^1_t - \{0,1,\infty\}; e_1|2, e_\infty|3) = \widehat{\Delta}(\infty,2,3)$ freely generated by $x_1' := x^{-2}z$, $x_2' := x^2z^{-1}$. An inertia group over the origin of E^{mor} is generated by the commutator $[x_1', x_2'] = x^{-6}$. Therefore, if the extra condition '$e_0|6$' is imposed on the fundamental group of $\mathbf{P}^1_t - \{0,1,\infty\}$, then the finite-adelic Tate module $T_f(E^{\mathrm{mor}})$ appears as the corresponding subgroup. This fits in the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(E^{\mathrm{mor}}_{\overline{\mathbb{Q}}} - \{O\}) & \longrightarrow & \widehat{\Delta}(\infty,2,3) & \longrightarrow & \mathbb{Z}/6\mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & T_f(E^{\mathrm{mor}}) & \longrightarrow & \widehat{\Delta}(6,2,3) & \longrightarrow & \mathbb{Z}/6\mathbb{Z} & \longrightarrow & 1.
\end{array}$$

The triangle group $\Delta(6,2,3)$ is of Euclidean type corresponding to a plane tiling by 30-60-90 triangles which refines the regular triangle tessellation. Since the quotient $\pi_1(\mathbf{P}^1_t - \{0,1,\infty\}; e_0|6, e_1|2, e_\infty|3)$ is preserved by the operation of $\widehat{GT}$, the $G_{\mathbb{Q}}$-action on the Tate module $T_f(E^{\mathrm{mor}})$ is naturally extended to the $\widehat{GT}$-action on it. By the similar argument to Proposition (3.3), we can show

(3.5) PROPOSITION. *Let $\bar{x}_1', \bar{x}_2'$ be the basis of the Tate module $T_f(E^{\mathrm{mor}})$ which are the images of x_1', x_2' respectively, and define the action of the ring $\hat{\mathbb{Z}}[\zeta_6]$ on $T_f(E^{\mathrm{mor}})$ by $\zeta_6 : \bar{x}_1' \mapsto (\bar{x}_1' - \bar{x}_2')$, $\bar{x}_2' \mapsto \bar{x}_1'$. Then, each element $\sigma = (\lambda, f) \in \widehat{GT}$ acts on $T_f(E^{\mathrm{mor}})$ by*

$$\begin{cases} \bar{x}_1' & \mapsto \zeta_6^{\lambda-1}\mathbb{B}_\sigma(\zeta_6,\zeta_6^2)\cdot\bar{x}_1', \\ \bar{x}_2' & \mapsto \mathbb{B}_\sigma(\zeta_6,\zeta_6^2)\cdot\bar{x}_2'. \end{cases}$$

□

We leave the proof as reader's exercise. This result may be regarded as the Galois correspondent of the classical integral:

$$\int_0^\infty \frac{dt}{\sqrt{t^3+1}} = \frac{1}{3}B(\frac{1}{6}, \frac{1}{3}).$$

§4. Family of quartics and tangential base points.

(4.1) The purpose of this section is to give suitable embedding of $\mathbf{P}^1_t - \{0, 1, \infty\}$ to the space of quartics which enables us to consider the fundamental group of $E^{\mathrm{lem}} \setminus \{O\}$ in the framework of braid groups. Let $\mathbf{A}^4_u$ be the affine 4-space (over $\mathbb{Q}$) where each point $u = (u_1, u_2, u_3, u_4)$ is regarded as a quartic $f_u(T) := T^4 + u_1T^3 + u_2T^2 + u_3T + u_4$ of one formal variable T. Let D be the discriminant locus on $\mathbf{A}^4_u$ corresponding to the quartics with duplicate roots. Over the affine variety $\mathbf{A}^4_u - D$, one has a Galois etale cover $\mathbf{A}^4_v - \Delta$ called *the space of solutions* whose points are the ordered tuples $v = (v_1, v_2, v_3, v_4)$ with distinct entries. (The divisor Δ denotes the hyper-diagonals of $\mathbf{A}^4_v$.) The morphism of $\mathbf{A}^4_v - \Delta$ to $\mathbf{A}^4_u - D$ is given by $v \mapsto f_u(T) := (T - v_1)(T - v_2)(T - v_3)(T - v_4)$. Over $\mathbf{A}^4_v - \Delta$, one has a standard tangential basepoint $\vec{b}$ given by $v = (0, t_1t_2t_3, t_2t_3, t_3)$ with values in the formal Puiseux power series ring $\overline{\mathbb{Q}}\{\{t_1, t_2, t_3\}\}$ (the maximal etale extension of $\overline{\mathbb{Q}}[\![t_1, t_2, t_3]\!][(t_1t_2t_3)^{-1}]$). This is so called the Ihara-Matsumoto tangential basepoint ([IM]). Writing the image of $\vec{b}$ on $\mathbf{A}^4_u - D$ by the same symbol $\vec{b}$, one can naturally identify $\pi_1(\mathbf{A}^4_u - D, \vec{b}) = G_{\mathbb{Q}} \ltimes \hat{B}_4$. The Galois action here can be extended to that of $\widehat{GT}$ in the form of the famous formula of Drinfeld ([Dr], see [IM]), i.e., each $\sigma = (\lambda, f) \in GT$ maps starndard generators as follows:

$$\begin{cases} \tau_1 & \mapsto \tau_1^\lambda, \\ \tau_2 & \mapsto f(\tau_1^2, \tau_2^2)^{-1}\tau_2^\lambda f(\tau_1^2, \tau_2^2), \\ \tau_3 & \mapsto f(\eta^2, \tau_3^2)^{-1}\tau_3^\lambda f(\eta^2, \tau_3^2). \end{cases} \tag{4.1.1}$$

Here τ_1, τ_2, τ_3 are standard generators of the braid group B_4 ($\tau_1\tau_3 = \tau_3\tau_1$, $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ $(i = 1, 2)$), and $\eta = \tau_1\tau_2\tau_1$.

(4.2) The space $\mathbf{A}^4_v - \Delta$ is an $\mathbf{A}^1$-torsor by parallel transformations $v \mapsto (v_1+a, v_2+a, v_3+a, v_4+a)$ $(a \in \mathbf{A}^1)$. This action does not affect the fundamental groupoid of $\mathbf{A}^4_v - \Delta$ as $\pi_1(\mathbf{A}^1_{\overline{\mathbb{Q}}}) = 1$. Two (tangential) basepoints b_1, b_2 on $\mathbf{A}^4_v - \Delta$ will be called equivalent (written $b_1 \sim b_2$) if they are transformed to each other by this $\mathbf{A}^1$-action. The multiplicative group $\mathbf{G}_m$ acts on $\mathbf{A}^4_v - \Delta$ by simultaneous multiplication on entries. The induced $\mathbf{G}_m$-action on $\mathbf{A}^4_u - D$ is given by $f_u(T) \mapsto a^4 f_u(a^{-1}T)$ $(a \in \mathbf{G}_m)$. We shall write those quotient spaces as

$$\mathcal{A}^4_v := (\mathbf{A}^4_v - \Delta)/\mathbf{G}_m, \quad \mathcal{A}^4_u := (\mathbf{A}^4_u - \Delta)/\mathbf{G}_m.$$

Their geometric fundamental groups are given by moding out by the center $\langle\omega_4\rangle$ $(\omega_4 = (\tau_1\tau_2\tau_3)^4)$, i.e., $\pi_1(\mathcal{A}^4_u \otimes \overline{\mathbb{Q}})$ is isomorphic to $\hat{B}^*_4 := \hat{B}_4/\langle\omega_4\rangle$, and $\pi_1(\mathcal{A}^4_v \otimes \overline{\mathbb{Q}})$ is its pure part $\hat{P}^*_4$ which is by definition the kernel of the "string-permutation" homomorphism $\hat{B}^*_4 \to S_4$. Remarkable observation here is that we can find braids in $\hat{B}^*_4$ which behaves like the generators of the triangle group $\Delta(\infty, 2, 4)$: One can take

$$\eta := \tau_1\tau_2\tau_1, \quad \eta_2 := \tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\tau_1^{-1}\tau_2^{-1}\tau_1^{-1}, \quad \eta_4 := \tau_1\tau_2\tau_3 \tag{4.2.1}$$

so that $\eta\eta_2\eta_4 = 1$, ${\eta_2}^2 (= \omega_4^{-1}) = 1$, ${\eta_4}^4 (= \omega_4) = 1$ in $\hat{B}_4^*$. Therefore, one has a homomorphism of $\widehat{\Delta}(\infty, 2, 4)$ into $\hat{B}_4^*$ with $x \mapsto \eta$, $y \mapsto \eta_2$, $z \mapsto \eta_4$. Under this mapping, the generators x_1, x_2 of the $\pi_1(E^* \setminus \{O\}) \subset \widehat{\Delta}(\infty, 2, 4)$ will be mapped into the braids ξ_1, ξ_2 of $\hat{B}_4^*$:

$$\text{(4.2.2)} \qquad \begin{cases} x_1 = x^{-1}z & \mapsto \xi_1 := \tau_3\tau_1^{-1}, \\ x_2 = xz^{-1} & \mapsto \xi_2 := \tau_2\tau_1\tau_3^{-1}\tau_2^{-1}\tau_3\tau_1^{-1} \end{cases}$$

so that $[x_1, x_2] \mapsto [\xi_1, \xi_2] = \omega_4\eta^{-4} = \eta^{-4}$. It is known that these two braids generate a free profinite subgroup in $\hat{B}_4^*$ (*cf.* §5 (5.2) below). From this one easily sees the injectivity of $\widehat{\Delta}(\infty, 2, 4) \hookrightarrow \hat{B}_4^*$.

(4.3) Now, let us introduce the embedding $f^{\mathrm{lem}} : \mathbf{P}_t^1 - \{0, 1, \infty\} \to \mathbf{A}^4 \setminus D$ by giving the corresponding quartics as follows:

$$f_t^{\mathrm{lem}}(T)(= f^{\mathrm{lem}}(t)(T)) := T^4 - 2T^3 + \frac{3}{2} \cdot \frac{t}{t-1} T^2 - \frac{1}{2} \cdot \frac{t}{t-1} T + \frac{1}{16} \cdot \frac{t^2}{(t-1)^2}$$

The discriminant of $f_t^{\mathrm{lem}}(T)$ is $\frac{4t^3}{(t-1)^6}$ so that the morphism f^{lem} is well defined. The injectivity is obvious from the coefficients in T. The tasks we are now going to work on are:

(a) to find and fix a suitable path from $\vec{b}$ to the image of $\overrightarrow{01}_t$ so that the image of loops in $\pi_1(\mathbf{P}_t^1 - \{0, 1, \infty\}, \overrightarrow{01}_t)$ can be uniquely identified with loops in $\mathcal{A}_u^4$;

(b) to check the coincidence of the image of x, y with η, η_2 respectively under the above identification of (a).

To approach (a), we shall solve the quartic equation $f_t^{\mathrm{lem}}(T) = 0$ by the Cardano-Ferrari formula. The result is that the 4 zeros are given by

$$\text{(4.3.1)} \qquad T = \frac{1}{2} + \sqrt{\frac{1}{4(1-t)}} + \sqrt{\frac{1+\sqrt{t}}{4(1-t)}} + \sqrt{\frac{1-\sqrt{t}}{4(1-t)}}$$

where the (outer) three $\sqrt{*}$'s are taken so that the product comes to be $1/8(1-t)$. Near t being small > 0, after declaring that all $\sqrt{*}$ mean positive roots, one can specify 4 solutions as :

$$\text{(4.3.2)} \qquad T_1 \approx -\frac{1}{2}\sqrt{t}, \quad T_2 \approx 0, \quad T_3 \approx \frac{1}{2}\sqrt{t}, \quad T_4 \approx 2.$$

The corresponding lift of the tangential basepoint $f^{\mathrm{lem}}(\overrightarrow{01}_t)$ on $\mathbf{A}_v^4 - \Delta$ is then estimated as :

$$(0, t_1t_2t_3, t_2t_3, t_3) = (-\frac{1}{2}\sqrt{t}, 0, \frac{1}{2}\sqrt{t}, 2) \sim (0, \frac{1}{2}\sqrt{t}, \sqrt{t}, 2 + \frac{1}{2}\sqrt{t}),$$

where $\sim$ means equivalence by $\mathbf{A}^1$-action ($cf.$(4.2)); namely $t_1 = 1/2$, $t_2 = \frac{\sqrt{t}}{2} = \sqrt{\frac{t}{4}}$ and $t_3 \risingdotseq 2$. Moding out $t_3 \in \mathbf{G}_m$, we can define a path r on $\mathcal{A}_v^4$ from (the image of) $\vec{b}$ to (that of) $f^{\mathrm{lem}}(\frac{1}{4}\overrightarrow{01}_t)$ by letting t_1 move from $\overrightarrow{01}_{t_1}$ to $\frac{1}{2}$. Let ε denote the straight path from $\frac{1}{4}\overrightarrow{01}_t$ to $\overrightarrow{01}_t$ on $\mathbf{P}^1_t-\{0,1,\infty\}$.

$$\begin{tikzcd}
 & \frac{1}{4}\overrightarrow{01}_t \arrow[r, "\varepsilon"] \arrow[d, "f^{\mathrm{lem}}"] & \overrightarrow{01}_t(\rightleftharpoons x,y) \\
\vec{b} \arrow[r, "r"] & f^{\mathrm{lem}}(\frac{1}{4}\overrightarrow{01}_t) &
\end{tikzcd}$$

By using (the images on $\mathcal{A}^4_u$ of) these r and ε, the loops x, y of $\pi_1(\mathbf{P}^1_t-\{0,1,\infty\},\overrightarrow{01}_t)$ are mapped to the loops $rf^{\mathrm{lem}}(\varepsilon x\varepsilon^{-1})r^{-1}$, $rf^{\mathrm{lem}}(\varepsilon y\varepsilon^{-1})r^{-1}$ of $\pi_1(\mathcal{A}^4_u,\vec{b})$. For (b), it suffices to show that these correspond to the braids ξ_1 and ξ_2 of (4.2.2) respectively. The first one is not difficult: the image $rf^{\mathrm{lem}}(\varepsilon x\varepsilon^{-1})r^{-1}$ of x can be seen from (4.3.2) where t moves on a small circle around 0 in the anticlockwise way. The result is that T_1 and T_3 go around T_2 to be interchanged to each other with leaving T_2 (inside) and T_4 (outside) invariant. This is nothing but the braid $\eta=\tau_1\tau_2\tau_1$. When t is close to 1, the four roots T_1,...,T_4 behave approximately as follows:

$$T_1\approx 1-\sqrt{\frac{1}{1-t}},\quad T_2\approx\frac{1-\sqrt{2}}{2}\sqrt{\frac{1}{1-t}},\quad T_3\approx\frac{\sqrt{2}-1}{2}\sqrt{\frac{1}{1-t}},\quad T_4\approx 1+\sqrt{\frac{1}{1-t}}.$$

From this one sees that, when t goes on a small circle around 1 in the anticlockwise way, the points T_1 and T_4 (resp. T_2 and T_3) move on a large circle clockwise with angle π to be switched to one another. This identifies $rf^{\mathrm{lem}}(\varepsilon y\varepsilon^{-1})r^{-1}$ with the braid $\eta_2=\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\tau_1^{-1}\tau_2^{-1}\tau_1^{-1}$.

We summarize the above discussion as

(4.4) Proposition. *Let r and ε be paths introduced as above. Then, each $\sigma\in G_{\mathbb{Q}}$ acts on them by*

$$\begin{cases}\sigma(r) &= g_\sigma(\tau_1^2,\tau_2^2)^{-1}\cdot r,\\ \sigma(\varepsilon) &= \varepsilon\cdot x^{-2\rho_2(\sigma)}.\end{cases}$$

The induced homomorphism $(*)\mapsto rf^{\mathrm{lem}}(\varepsilon(*)\varepsilon^{-1})r^{-1}$ of $\pi_1(\mathbf{P}^1_t-\{0,1,\infty\},\overrightarrow{01}_t)$ into $\pi_1(\mathcal{A}^4_u)$ is injective, and maps $x\mapsto\eta$, $y\mapsto\eta_2$ and $z\mapsto\eta_4$. $\square$

§5. Galois actions and proof of Theorem D.

(5.1) Proposition. *Under the standard action of $G_{\mathbb{Q}}(\subset\widehat{GT})$ on $\hat{B}_4$ (4.1.1), the elements $\eta:=\tau_1\tau_2\tau_1$, $\eta_2:=\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\tau_1^{-1}\tau_2^{-1}\tau_1^{-1}$, $\eta_4:=\tau_1\tau_2\tau_3$, $\xi_1:=\tau_3\tau_1^{-1}$ and $\xi_2:=\tau_2\tau_1\tau_3^{-1}\tau_2^{-1}\tau_3\tau_1^{-1}$ are explicitly transformed as follows:*

(1) $\quad\eta\mapsto g(\tau_1^2,\tau_2^2)^{-1}\eta^\lambda g(\tau_1^2,\tau_2^2),=f(\tau_2^2,\tau_1^2)\eta^\lambda$

(2) $\quad\eta_2\mapsto g(\tau_1^2,\tau_2^2)^{-1}\eta^{-2\rho_2}f(\eta,\eta_2)\eta_2^\lambda f(\eta_2,\eta)\eta^{2\rho_2}g(\tau_1^2,\tau_2^2),$

(3) $\quad\eta_4\mapsto g(\tau_1^2,\tau_2^2)^{-1}\eta^{\frac{\lambda-1}{2}-2\rho_2}f(\eta_4,\eta)\eta_4^\lambda f(\eta,\eta_4)\eta^{-\frac{\lambda-1}{2}+2\rho_2}g(\tau_1^2,\tau_2^2);$

(4) $\quad\xi_1\mapsto\eta^{-8\rho_2}f(\xi_1,z_0)\xi_1^\lambda f(z_0,\xi_1)\eta^{8\rho_2},$

(5) $\quad\xi_2\mapsto f(\tau_2^2,\tau_1^2)\eta^{\lambda-1-8\rho_2}f(\xi_2^{-1},z_0)\xi_2^\lambda f(z_0,\xi_2^{-1})\eta^{1-\lambda+8\rho_2}f(\tau_1^2,\tau_2^2).$

Here $z_0=\eta^4\omega_4^{-1}$ is the braid satisfying $[\xi_1,\xi_2]z_0=1$.

PROOF. The result (1) dates back to [N-I] Proposition 4.12. The validity of the first three formulae (1)∼(3) in $\hat{B}_4^*$ are consequences of Proposition (4.4) and the $G_{\mathbb{Q}}$-action on x, y, z described in (3.1). To lift them to $\hat{B}_4$, note that the kernel of $\hat{B}_4 \to \hat{B}_4^*$ is the center $\langle\omega_4\rangle$ of $\hat{B}_4$ and that the $G_{\mathbb{Q}}$-actions on η^4, $\eta_2^2 = \omega_4^{-1}$ and $\eta_4^4 = \omega_4$ are given by their λ powers. The abelianization map $\hat{B}_4 \to \hat{\mathbb{Z}}$ assures the coincidence of both sides of the issued lifts. (4) follows from the standard $G_{\mathbb{Q}}$-action on $\xi_1 = \tau_3\tau_1^{-1} \in \hat{B}_3$ according to Drinfeld's formula (4.1.1) together with the relation (IV) first proved in [N-I] Theorem 4.16:

$$\text{(IV)}\qquad f(\tau_3^2, \eta^2) = \eta^{-8\rho_2} f(\xi_1, \eta^4\omega_4^{-1})\tau_1^{4\rho_2}\tau_3^{-4\rho_2} = \eta^{-8\rho_2} f(\xi_1, \eta^4\omega_4^{-1})\tau_1^{4\rho_2}\xi_1^{-4\rho_2}$$

Note that $[\xi_1, \xi_2](\eta^4\omega_4^{-1}) = 1$. Applying (1),(4) to $\xi_2 = \eta\xi_1^{-1}\eta^{-1}$, we obtain (5). □

(5.2) Let $M_{1,2}$ be the moduli stack of smooth genus 1 curves with 2 ordered marked points and let $f_2 : M_{1,2} \to M_{1,1}$ be the morphism to the moduli stack of elliptic curves which forgets the second marked points. Each quartic $f_u(T)$ of $\mathbf{A}^4 \setminus D$ gives the elliptic curve $Y^2 = f_u(T)$ with two ordered points at infinity. This gives rise to an isomorphism of fundamental groups $\pi_1(\mathcal{A}_u^4) \xrightarrow{\sim} \pi_1(M_{1,2})$ (*cf.* [N-I] p.353, [N-II] §7.8). The homotopy exact sequence of this fibration is of the form:

$$\begin{array}{ccccccccc} & & & & \pi_1(\mathcal{A}_u^4) & & & & \\ & & & & \big\downarrow {\scriptstyle isom.} & & & & \\ 1 & \longrightarrow & \hat{F}_2 & \longrightarrow & \pi_1(M_{1,2}) & \xrightarrow{f_2} & \pi_1(M_{1,1}) & \longrightarrow & 1. \end{array}$$

The kernel part $\hat{F}_2$ corresponds to the geometric fundamental group of a fibre elliptic curve minus origin. In [N-I] §4.9, we saw that ξ_1, ξ_2 give a generator system of this $\hat{F}_2$. Thus we see: ξ_1, ξ_2 *generates a free profinite subgroup in* $\pi_1(\mathcal{A}_u^4)$, *hence also in* $\hat{B}_4$.

(5.3) REMARK. The above forgetful projection f_2 can be lifted to $\pi_1(\mathbf{A}^4 \setminus D_4) \to \pi_1(\mathbf{A}^3 \setminus D_3)$ coming from a certain morphism $\mathcal{F} : \mathbf{A}^4 \setminus D_4 \to \mathbf{A}^3 \setminus D_3$. This morphism $\mathcal{F}$, called the *Ferrari morphism* (cf. [N01]) associates to quartic polynomials their resolvent cubics, and will play basic roles in the theory of Weierstrass tangential base points. (See the forthcoming paper partly based on [N01]).

(5.4) In [N-I], we studied various actions on the free subgroup $\hat{F}_2 = \langle\xi_1, \xi_2\rangle$ of $\pi_1(M_{1,2}) \cong \hat{B}_4^*$ which embodies the fundamental group of punctured elliptic curves. Geometric monodromy on them can be summarized as the inner actions by τ_1, τ_2. This action was determined in the following simple way:

$$\text{(5.4.1)}\qquad \mathrm{Int}(\tau_1) : \begin{cases} \xi_1 \mapsto \xi_1, \\ \xi_2 \mapsto \xi_2\xi_1, \end{cases} \quad \mathrm{Int}(\tau_2) : \begin{cases} \xi_1 \mapsto \xi_2\xi_1^{-1}, \\ \xi_2 \mapsto \xi_2. \end{cases}$$

Once given an elliptic curve with Weierstrass equation $Y^2 = 4X^3 - g_2X - g_3$, then we may associate in a standard way the Weierstrass tangential basepoint on it and the Galois action on $\hat{F}_2$ (*cf.* also [N98]). One important example is the case of the Tate elliptic curve

$$\mathrm{Tate}(q) : Y^2 = 4X^3 - g_2(q)X - g_3(q)$$

over the formal Laurant power series field $\mathbb{Q}((q))$, where $g_2(q)$, $g_3(q)$ are given as Eisenstein q-series of weight 4, 6 (constant terms are $1/12$, $-1/216$) respectively. From the corresponding Weierstrass tangential basepoint, the Galois group $G_{\mathbb{Q}}$ acts by

$$\text{(5.4.2)} \qquad \begin{cases} \xi_1 \mapsto f(\xi_1, z_0)\xi_1^{\lambda} f(z_0, \xi_1), \\ \xi_2 \mapsto f(\xi_1, z_0)x_1^{\frac{1-\lambda}{2}} f(\xi_2\xi_1^{-1}\xi_2^{-1}, \xi_1)\xi_2\xi_1^{\frac{\lambda-1}{2}} f(z_0, \xi_1), \\ z_0 \mapsto z_0^{\lambda}, \end{cases}$$

where $[\xi_1, \xi_2]z_0 = 1$. The difference between this action and the Drinfeld-Ihara-Matsumoto action has been determined in [N-I] §4. (In fact, the conjugation by $\eta^{-8\rho_2}\tau_1^{8\rho_2}$ makes (5.4.2) coincide with Proposition (5.1) (4)-(5).) Using this circle of ideas, we showed in [N-I] Corollary 4.13:

$$\text{(5.4.3)} \qquad f_\sigma\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\right) = (-1)^{\frac{\lambda_\sigma - 1}{2}} \begin{pmatrix} 1 & 0 \\ -8\rho_2(\sigma) & 1 \end{pmatrix} \begin{pmatrix} \lambda_\sigma^{-1} & 0 \\ 0 & \lambda_\sigma \end{pmatrix} \begin{pmatrix} 1 & -8\rho_2(\sigma) \\ 0 & 1 \end{pmatrix}.$$

Now, we are ready to prove the following

(5.5) THEOREM. *For $\sigma \in G_{\mathbb{Q}}$, we have the following equation:*

$$g_\sigma\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\right) = (-1)^{\frac{\lambda_\sigma - 1}{2}} \mathbb{B}_\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot \begin{pmatrix} \lambda_\sigma^{-1} & -8\rho_2(\sigma)\lambda_\sigma^{-1} \\ 0 & (-1)^{\frac{\lambda_\sigma - 1}{2}} \end{pmatrix}.$$

PROOF. Identify $T_f(E^{\mathrm{lem}})$ with $\hat{\mathbb{Z}}^2$ by $\bar{x}_1 \mapsto \binom{1}{0}$, $\bar{x}_2 \mapsto \binom{0}{1}$. By Proposition (3.3), the $G_{\mathbb{Q}}$-action is given as the right multiplication by the matrix

$$(\#) \qquad (-1)^{\rho_2(\sigma)} \mathbb{B}_\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot \begin{pmatrix} (-1)^{\frac{\lambda-1}{2}} & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, by Proposition (5.1) (4)-(5), the standard action from $\vec{b}$ on the abelianization of $\hat{F}_2 = \langle \xi_1, \xi_2 \rangle$ is given by

$$\left((-1)^{-4\rho_2}\begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \; f_\sigma\left(\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right)(-1)^{\frac{\lambda-1}{2} - 4\rho_2}\begin{pmatrix} 0 \\ \lambda \end{pmatrix}\right) = \begin{pmatrix} \lambda & 8\rho_2(\sigma) \\ 0 & 1 \end{pmatrix},$$

Therefore, taking into consideration the factor coming from our path $rf^{\mathrm{lem}}(\varepsilon)$ from $\vec{b}$ to $f^{\mathrm{lem}}(\overrightarrow{01}_t)$, we see that the matrix (#) is equal to

$$(-1)^{\rho_2(\sigma)} g_\sigma\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\right) \begin{pmatrix} \lambda & 8\rho_2(\sigma) \\ 0 & 1 \end{pmatrix}.$$

This completes the proof. □

Appendix. A note on the cocycle $\Psi_n^{(0)}$.

In [NT] Remark (6.3), we mentioned about Ihara's 1-cocycle $\Psi_n^{(0)}$ ($n \in \mathbb{N}$) whose minus sign $-\Psi_n^{(0)}$ extends the Kummer 1-cocycle ρ_n on the positive roots of n. In fact, originated from a difference of path conventions between Ihara's papers [I1,I2,I3] and ours, the association manner $G_{\mathbb{Q}} \to \widehat{GT}$ delicately differs from each other so that the minus sign of "$-\Psi_n^{(0)}$" in that claim should be dropped if $\Psi_n^{(0)}$ is replaced by its italicized version "$\mathit{\Psi}_n^{(0)}$" introduced in our convention exactly in the same way as $\Psi_n^{(0)}$ in Ihara's paper [I1]. To normalize ambiguities buried in contexts of literatures, in this appendix we shall give an explicit exposition starting from the definition of our italicized $\Psi_n^{(0)}$ and then shall give a proof of the regularized statement:

(A1) PROPOSITION. *The 1-cocycle* $\Psi_n^{(0)} : \widehat{GT} \to \hat{\mathbb{Z}}(1)$ *extends the Kummer 1-cocycle* $\rho_n : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}(1)$ *attached to the real positive roots of* n.

Let $\hat{F}_2$ be the free profinite group freely generated by the symbols x, y. For a positive integer n, let H_n be the kernel of the homomorphism $\hat{F}_2 \to \mu_n$ defined by $x \mapsto \zeta_n, y \mapsto 1$, which is a free profinite subgroup $\hat{F}_{n+1}$ freely generated by the $x^{-a}yx^a$ ($a \in \mathbb{Z}/n\mathbb{Z}$) and x^n. Since, for any $\sigma = (\lambda_\sigma, f_\sigma) \in \widehat{GT}$, f_σ belongs to $\hat{F}_2' \subset H_n$, there exists a unique element ($n+1$-term proword) $f_\sigma^{(n)} = f_\sigma^{(n)}(\xi, \eta_0, \eta_1, \dots, \eta_{n-1}) \in \hat{F}_{n+1}$ satisfying $f_\sigma^{(n)}(x^n, y, xyx^{-1}, \dots, x^{n-1}yx^{-(n-1)}) = f_\sigma(x, y)$. For $a \in \mathbb{Z}/n\mathbb{Z}$, consider the homomorphism $\psi_n^{(a)} : F_{n+1} \to \hat{\mathbb{Z}}$ given by $\eta_a \mapsto 1$ and $\xi, \eta_i \mapsto 0$ ($i \neq a$), and then:

(A2) DEFINITION. Define $\Psi_n^{(a)}(\sigma) := \psi_n^{(a)}(f_\sigma^{(n)})$ ($a \in \mathbb{Z}/n\mathbb{Z}$).

By the same argument as [I2] p.171, these $\Psi_n^{(a)}$ can also be characterized by the formula on $\widehat{GT}$:

$$\text{(A3)} \quad \left(\frac{\mathbb{B}_\sigma(\bar{x}, \bar{y}) - 1}{\bar{y} - 1}\right)_{\bar{y}=1} = (\bar{x} - 1)\mathbb{A}_\sigma(\bar{x}, 1) \equiv \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \Psi_n^{(a)}(\sigma)\bar{x}^{-a} \mod (\bar{x}^n - 1).$$

In particular,

$$\text{(A4)} \qquad \Psi_n^{(0)}(\sigma) := \frac{1}{n}\sum_{c=0}^{n-1}(\zeta_n^c - 1)\mathbb{A}_\sigma(\zeta_n^c, 1).$$

To prove Proposition (A1), we shall compute $\Psi_n^{(0)}(\sigma)$ for $\sigma \in G_{\mathbb{Q}}$. Let $X_w = \mathbf{P}^1 \smallsetminus \{0, \mu_n, \infty\} \to X_z = \mathbf{P}^1 \smallsetminus \{0, 1, \infty\}$ be the cyclic cover given by $z = w^n$, and consider the tangential base points $\overrightarrow{01}_z$, $\overrightarrow{10}_z$ on the z-line and $\overrightarrow{01}_w$, $\overrightarrow{10}_w$, $n\overrightarrow{10}_w$ on the w-line, whose value fields are $\overline{\mathbb{Q}}\{\{z\}\}$, $\overline{\mathbb{Q}}\{\{z_1\}\}$ ($z_1 = 1 - z$), $\overline{\mathbb{Q}}\{\{w\}\}$, $\overline{\mathbb{Q}}\{\{w_1\}\}$ ($w_1 = 1 - w$), $\overline{\mathbb{Q}}\{\{nw_1\}\}$ respectively. Let $L_{\overrightarrow{01}_z}$ (resp. $L_{\overrightarrow{10}_z}$) be the maximal algebraic extention of $\mathbb{Q}(z)$ in $\overline{\mathbb{Q}}\{\{z\}\}$ (resp. $\overline{\mathbb{Q}}\{\{z_1\}\}$) unramified outside $0, 1, \infty$ (viz. over X_z), and let $L_{\overrightarrow{01}_w}$ (resp. $L_{\overrightarrow{10}_w}$, $L_{n\overrightarrow{10}_w}$) be the maximal algebraic extention of $\mathbb{Q}(w)$ in $\overline{\mathbb{Q}}\{\{w\}\}$ (resp. $\overline{\mathbb{Q}}\{\{w_1\}\}$, $\overline{\mathbb{Q}}\{\{nw_1\}\}$) unramified outside $0, \mu_n, \infty$ (viz. over X_w). We shall connect the above tangential base points by the standard paths along the real axis:

$$\text{(A5)} \qquad \begin{cases} p_z \in \pi_1(X_z; \overrightarrow{01}_z, \overrightarrow{10}_z) \simeq \mathrm{Isom}_{\mathbb{Q}(z)}(L_{\overrightarrow{10}_z}, L_{\overrightarrow{01}_z}), \\ p_w \in \pi_1(X_w; \overrightarrow{01}_w, \overrightarrow{10}_w) \simeq \mathrm{Isom}_{\mathbb{Q}(w)}(L_{\overrightarrow{10}_w}, L_{\overrightarrow{01}_w}), \\ \delta_n \in \pi_1(X_w; n\overrightarrow{10}_w, \overrightarrow{10}_w) \simeq \mathrm{Isom}_{\mathbb{Q}(w)}(L_{\overrightarrow{10}_w}, L_{n\overrightarrow{10}_w}). \end{cases}$$

The covering map $\beta_n : X_w \to X_z$ and the path δ_n give an embedding $\pi_1(X_w; \overrightarrow{01}_w) \hookrightarrow \pi_1(X_z; \overrightarrow{01}_z)$ such that $\xi \mapsto x^n, \eta_i \mapsto x^i y x^{-i}$ ($i = 0, 1, \dots, n-1$). Meanwhile, one has a natural open immersion $\iota_n : X_w \to X_z$ ($w \mapsto z$) which induces the surjection of $\pi_1(X_w; \overrightarrow{01}_w)$ onto $\pi_1(X_z; \overrightarrow{01}_z)$ such that $\xi \mapsto x$, $\eta_0 \mapsto y$, $\eta_i \mapsto 1$ ($i = 1, \dots, n-1$).

Writing the action of $\sigma \in G_{\mathbb{Q}}$ at a tangential base point b as σ_b, one may describe Galois effects on the above paths as follows:

$$\text{(A6)}\quad \begin{cases} \sigma(p_z) & = \sigma_{\overrightarrow{01}_z} p_z \sigma^{-1}_{\overrightarrow{10}_z} = f_\sigma(x,y)^{-1} p_z, \\ \sigma(p_w) & = \sigma_{\overrightarrow{01}_w} p_w \sigma^{-1}_{\overrightarrow{10}_w} =: F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})^{-1} p_w, \\ \sigma(p_w\delta_n^{-1}) & = \sigma_{\overrightarrow{01}_w} p_w \delta_n^{-1} \sigma^{-1}_{n\overrightarrow{10}_w} = f_\sigma^{(n)}(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})^{-1} p_w \delta_n^{-1}. \end{cases}$$

(A7) Lemma. *(1)* $\sigma(\delta_n) = \sigma_{n\overrightarrow{10}_w} \delta_n \sigma^{-1}_{\overrightarrow{10}_w} = \delta_n \eta_0^{\rho_n(\sigma)}$.
(2) $\beta_n(n\overrightarrow{10}_w) = \overrightarrow{10}_z$ *(hence* $\beta_n(p_w\delta_n^{-1}) = p_z$*).*
(3) $f_\sigma^{(n)}(\xi,\eta_0,\eta_1,\dots,\eta_{n-1}) = \eta_0^{\rho_n(\sigma)} F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})$.
(4) $\psi_n^{(0)}(F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})) = 0$.

Proof. Calculations:
(1) It suffices to check the effect of both sides on $w_1^{1/N} \in \mathbb{Q}\{\{w_1\}\}$:

$$\begin{aligned} \sigma(\delta_n)(w_1^{1/N}) &= \sigma_{n\overrightarrow{10}_w} \delta_n \sigma^{-1}_{\overrightarrow{10}_w}(w_1^{1/N}) = \sigma_{n\overrightarrow{10}_w} \delta_n (w_1^{1/N}) = \sigma_{n\overrightarrow{10}_w}(\sqrt[N]{n}^{-1}(nw_1)^{1/N}) \\ &= \zeta_N^{-\rho_n(\sigma)} \sqrt[N]{n}^{-1}(nw_1)^{1/N} = \delta_n(\zeta_N^{-\rho_n(\sigma)} w_1^{1/N}) = \delta_n \eta_0^{\rho_n(\sigma)}(w_1^{1/N}). \end{aligned}$$

Notice that $\eta_0(w_1^{1/N}) = \zeta_N^{-1} w_1^{1/N}$ in our convention.
(2) $\beta_n^*(z_1) = 1-(1-w_1)^n = nw_1(1-\dfrac{n-1}{2}w_1+\cdots) \in \mathbb{Q}\{\{w_1\}\}$.
(3) Recalling the above definition of F_σ in (A6), one has

$$\begin{aligned} \sigma(p_w\delta_n^{-1}) &= F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})^{-1} p_w (\delta_n \eta_0^{\rho_n(\sigma)})^{-1} \\ &= F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})^{-1} \eta_0^{-\rho_n(\sigma)} p_w \delta_n^{-1} \\ &= (\eta_0^{\rho_n(\sigma)} F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1}))^{-1} p_w \delta_n^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} f_\sigma(x,y) &= \beta_n(\eta_0^{\rho_n(\sigma)} F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})) \\ &= y^{\rho_n(\sigma)} F_\sigma(x^n, y, xyx^{-1}, \dots, x^{n-1}yx^{-(n-1)}). \end{aligned}$$

(4) Since $\iota_n(p_w) = p_z$, it follows that

$$f_\sigma(x,y) = \iota_n(F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})) = F_\sigma(x,y,1,\dots,1).$$

Thus, if we put $F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1}) \equiv \xi^a \eta_0^{b_0} \eta_1^{b_1} \cdots \eta_{n-1}^{b_{n-1}} \bmod \hat{F}'_{n+1}$, we have $a = b_0 = 0$ because $f_\sigma(x,y) \in \hat{F}'_2$. □

Proof of Proposition (A1): By Lemma (A2) and Lemma (A7) (3),(4), one sees

$$\begin{aligned} \Psi_n^{(0)}(\sigma) &= \psi_n^{(0)}(f_\sigma^{(n)}(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})) \\ &= \psi_n^{(0)}(\eta_0^{\rho_n(\sigma)} F_\sigma(\xi,\eta_0,\eta_1,\dots,\eta_{n-1})) = \rho_n(\sigma). \quad \square \end{aligned}$$

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