On profinite Eisenstein periods in the monodromy of universal elliptic curves

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Contents

1. **Introduction**

Studies of monodromy representations in families of algebraic curves naturally arise in Grothendieck's anabelian geometry ([Gr1], [Gr2], [S97]), and many authors have enriched arithmetic and geometric phenomena surrounding the (outer) profinite universal monodromy representation

$$\varphi_{g,n} : \pi_1(M_{g,n}/\mathbb{Q}) \to \mathrm{Out}(\hat{\Pi}_{g,n})$$

of the Galois-Teichmüller modular group $\pi_1(M_{g,n}/\mathbb{Q})$ in the profinite surface group $\hat{\Pi}_{g,n}$ of type (g, n), i.e., of genus g and n punctures (cf. e.g. [AMO], [Lo12]).

Grothendieck has already pointed out in his *Esquisse* [Gr2] significance of the basic pieces $M_{0,4}$, $M_{0,5}$, $M_{1,1}$ and $M_{1,2}$ that should play fundamental roles in what he called the "Galois-Teichmüller tower". Recently, deep aspects of $\varphi_{1,1}$ (constructed from the projection $M_{1,2} \to M_{1,1}$) have been focused and revealed from several different viewpoints of topology, algebraic geometry, combinatorial group theory, and representation theory (e.g., [E14], [H], [HM], [BS], [MSS], [Br] and references therein). Through these works one could also view a number of bridges from the lego-structured skytree $\{\varphi_{g,n} : \pi_1(M_{g,n}/\mathbb{Q}) \to \mathrm{Out}(\hat{\Pi}_{g,n})\}_{2-2g-n<0}$ to marvelous landscapes of important reseach areas such as multiple zeta values, (motivic) elliptic polylogarithms ([BL94], [Go98], [HK99], [BK10], [Woj04] etc.), iterated modular forms (e.g., [GKZ06] [Ma06]), Johnson homomorphisms and related combinatorial objects (e.g., [CKV13], [ES14], [KK]).

The aim of this article is to introduce another arithmetic viewpoint in which $\varphi_{1,1}$ provides a profinite version of the Eisenstein cocycles $\Psi_k : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Sym}^{k-2}(\mathbb{Q}^2)$ ($k \in \mathbb{Z}_{\geq 2}$) that interpolate amplitudes of the Eichler-Shimura integrals of Eisenstein series. Traditionally in number theory, Eisenstein cocycles were extended to cocycles on $\mathrm{GL}_2(\mathbb{Q})$ to play certain roles in the theory of automorphic forms and L-functions (cf. [Scz], [St87]). However, taking account of the profinite topology of $\mathrm{SL}_2(\mathbb{Z})$ leads us to extend Ψ_k in another direction towards the congruence kernel $\mathrm{CSL}_2 := \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z})^\wedge \to \mathrm{SL}_2(\hat{\mathbb{Z}}))$. The resulting map $\widehat{\Psi}_k : \mathrm{CSL}_2 \to \mathrm{Sym}^{k-2}(\hat{\mathbb{Z}}^2)$, which we call the profinite Eisenstein periods, is the main subject to discuss below.

To outline our arithmetic strategy to approach the subject, let us introduce some more terminology. Given a smooth family of hyperbolic curves over a noetherian normal $\mathbb{Q}$-scheme S, there arises a natural exact sequences of etale fundamental groups

$$1 \to \pi_1(C_{\bar{\xi}}, \tilde{\xi}) \to \pi_1(C, \tilde{\xi}) \to \pi_1(S, \bar{\xi}) \to 1,$$

where $\bar{\xi}$ is a geometric point on S, $C_{\bar{\xi}}$ the geometric fiber of C/S over $\bar{\xi}$, and $\tilde{\xi}$ a lift of $\bar{\xi}$ on $C_{\bar{\xi}}$. This induces the associated outer monodromy representation

$$\varphi_{C/S} : \pi_1(S, \bar{\xi}) \to \mathrm{Out}(\pi_1(C_{\bar{\xi}}, \tilde{\xi})).$$

By anabelian philosophy, non-isotrivial information of the deformation of hyperbolic curves C/S should be reflected largely in the image of the arithmetic fundamental group $\pi_1(S)$ into the (outer) automorphisms of finitely generated profinite group $\pi_1(C_{\bar{\xi}}, \tilde{\xi})$. In particular, we may expect group-theoretical interpretation of arithmetic-geometric properties of C/S. Since the group homomorphism structure of $\varphi_{C/S}$ is independent of the choice of $\bar{\xi}, \tilde{\xi}$, we often abbreviate references to base points for π_1.

When S is the spectrum of a field k, $\pi_1(S)$ is the absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ and $\varphi_{C/k} : G_k \to \mathrm{Out}(\pi_1(C_{\bar{\xi}}))$ is called the outer Galois representation for the curve C/k. The most basic case $C/k = \mathbf{P}^1 - \{0, 1, \infty\}/\mathbb{Q}$ has been intensively studied by Y.Ihara ([I86], [I90], [I99]), where a highly arithmetic object called the universal Jacobi sum power series $\mathcal{J} : G_k \to \mathbb{Z}_l[[T_1, T_2]]^\times$ was discovered to represent the meta-abelian reduction of $\varphi_{C/k}$. Remarkably, the coefficient characters of his power series were explicitly connected to the Soule characters induced from the cyclotomic elements of K-theory.

In 1990s, the author tried to extend Ihara's theory to higher genus curves C/k to obtain hints to Grothendieck's anabelian conjecture (cf. [NTM]), and as a first step, looked at the case of C/k being an elliptic curve E minus one point partly in collaboration with H.Tsunogai. We investigated S. Bloch's construction [i)] of fundamental power series $\mathcal{E}^{(l)} : G_{k(E[l^\infty])} \to \mathbb{Z}_l[[T_1, T_2]]$ illustrated in [Bl84] (cf. [Tsu95a]), where $k(E[l^\infty])$ is the field obtained by adjoining the coordinates of all the torsion points of E of l-power orders. Like Ihara's power series, this captures the meta-abelian reduction of $\varphi_{E-\{O\}/k}$, and was found in [N95] that the coefficient characters are given explicitly by certain Galois characters defined by "theta invariants" of the elliptic curve. In this article, we consider the Bloch-Tsunogai construction for a family of elliptic curves $C := E - \{O\}$ over a base scheme S, and focus on the profinite monodromy representation

$$\mathcal{E} : \pi_1(S_\infty) \longrightarrow \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$$

with S_∞ the pro-etale cover of S trivializing the torsions of E. See (4.1) for more details. The representation $\mathcal{E}$ has the following two primary features:

- If the elliptic curve degenerates into a nodal cubic (Tate curve over $S = \mathbb{Q}[[q]]$), then $\mathcal{E}$ turns out to be reduced to a logarithmic derivative of $\mathcal{J}$ with one variable degenerate ([N99]).

[i)] Bloch constructed the power series modulo constant term and Tsunogai lifted it to be associated with constant term. For an explicit description of the constant term, see [N13] §6.10.

- If we vary elliptic curves in the universal family over the "fine" j-line as S, then we obtain $\mathcal{E}$ geometrically approximated by what we mean by the "profinite Eisenstein periods" in the title.

In §7-8 we will closely discuss the above second aspect which had been developed after a hint posed by T.Ibukiyama given to the author in 1993: Can one relate the universal power series $\mathcal{E}$ with "Dedekind sums" in some way analogous to the relation of Ihara's power series with Jacobi sums? The following table summarizes our answer with comparison of key words.

	fundamental group	special values	coefficients
$\mathcal{J}$	$\pi_1(\mathbf{P}^1 - \{0,1,\infty\}/\mathbb{Q})$	Jacobi sum	Soule characters
$\mathcal{E}$	$\pi_1(E - \{O\}/\overline{\mathbb{Q}}(j))$	Dedekind sum	Eisenstein cocycle

The entry of 'Dedekind sum' above should read, in more precise terms, 'period of division $\wp$-values' which is a certain linear sum of generalized Rademacher functions defined by Dedekind sums (see Theorem 8.5).

The main body of the following sections is based on my two Japanese articles [N02r], [N02j], except for §5 that consists of gadgets from [NT] (cited from [AN95]).

2. **Dedekind sums and Rademacher functions**

For a rational number $x \in \mathbb{Q}$, we denote by $\lfloor x \rfloor$ the maximal integer less than or equal to x and call $\lfloor * \rfloor : \mathbb{Q} \to \mathbb{Z}$ the flooring function. The flooring function which at first sight appears only destroying the precious algebraic structure of $\mathbb{Q}$ is, in fact, well known to play important roles in various subtle arithmetic, for example, in the third proof by Gauss of the reciprocity law of quadratic residue symbols.

R. Dedekind, in his note published in the collected work of Riemann [D], gave a transformation for the Dedekind η-function

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \qquad (\tau \in \mathbb{C},\, \Im(\tau) > 0)$$

by Riemann's method, in which introduced is the so-called the Dedekind sum $s(a,c)$ for a pair of mutually prime integers a, c $(c > 0)$:

$$s(a,c) = \sum_{i=0}^{c-1} P_1\left(\frac{i}{c}\right) P_1\left(\frac{ai}{c}\right).$$

Here, $P_1 : \mathbb{Q} \to \mathbb{Q}$ is the "saw-tooth" function defined by

$$P_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & (x \notin \mathbb{Z}); \\ 0, & (x \in \mathbb{Z}). \end{cases}$$

It is known that, when both a and c are positive, the Dedekind sum has the following beautiful reciprocity law

$$s(a,c) + s(c,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{c}{a} + \frac{a}{c} + \frac{1}{ac}\right).$$

The actual concern of the Dedekind sum to the transformation formula for the η-fucntion is given through a $\mathbb{Z}$-*valued* function φ on matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ $(c > 0)$ by

$$\varphi(A) = \begin{cases} \frac{b}{d}, & (c = 0); \\ \frac{a+d}{c} - 12s(a,c), & (c > 0). \end{cases}$$

It holds then the remarkable transformation formula

$$\eta(A\tau) = \begin{cases} e^{\frac{2\pi i}{24}\varphi(A)}\eta(\tau), & (c = 0); \\ e^{\frac{2\pi i}{24}\varphi(A)}\sqrt{\frac{c\tau+d}{i}}\eta(\tau), & (c > 0). \end{cases}$$

The algebraic properties of $\varphi : \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$ were investigated by Rademacher who derived for example the composition formula

$$\varphi(AB) = \varphi(A) + \varphi(B) - 3\mathrm{sgn}(c_A c_B c_{AB}) \qquad (A, B \in \mathrm{SL}_2(\mathbb{Z})).$$

Here, for a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$, c_A denotes the lower left entry of A. Following the nice article by Kerby-Melavin [KM], we shall call φ the Rademacher function.

Various kinds of generalization of Dedekind sums and their reciprocity laws are known. In many cases, the saw-tooth function P_1 is regarded as the first member of the periodic Bernoulli functions P_k $(k = 1, 2, ...)$, where P_k for $k \geq 2$ is a continuous function $P_k(x) := B_k(x - \lfloor x \rfloor)$ defined by the Bernoulli polynomial $B_k(T)$ satisfying the generating function $\frac{ze^{Tz}}{e^z-1} = \sum_{k=0}^{\infty} B_k(T)\frac{z^k}{k!}$.

One direction of generalization can be given by focusing on the fact that the logarithmic derivative of the η-function is the Eisenstein series E_2 of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ (see, e.g., [A] Chap.3, Ex. 5), in other words, the Rademacher function φ is a period integral of E_2 in a suitable sense. Let us consider a class of Eisenstein series of general weights and levels: For an even integer $k \geq 2$ and a pair $\mathbf{x} = (\frac{r_1}{N}, \frac{r_2}{N}) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$ ($\mathbf{x} \neq \mathbf{0}$ when $k = 2$), introduce the Eisenstein series on the upper half plane $\mathfrak{H} := \{\tau \in \mathbb{C}, \Im(\tau) > 0\}$ by

$$E_k^{(\mathbf{x})}(\tau) := \frac{(k-1)!}{(2\pi i)^k} \sum_{\substack{\mathbf{a} \in (\mathbb{Z}/N\mathbb{Z})^2 \\ \mathbf{a}=(a_1,a_2)}} e^{2\pi i(r_1 a_2 - r_2 a_1)/N} \sideset{}{'}\sum_{\mathbf{m} \equiv \mathbf{a}(N)} \frac{1}{(m_1\tau + m_2)^k} \tag{2.1}$$

Here, the summation $\sum'$ is taken over the pairs $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ with $m_1 \equiv a_1$, $m_2 \equiv a_2 \pmod N$.

Although Eisenstein series have nontrivial constant terms at cusps so that their naive Eichler-Shimura type period integrals do not converge, G.Stevens [St87] was able to obtain convergent integrals by putting "modular caps" on modular symbols. He then introduces generalizations of Rademacher function that represent period integrals of Eisenstein series. There are also detailed studies (e.g., by Sczech [Scz]) describing explicitly the Eichler-Shimura type 1-cocycle for Eisenstein series extended from $\Gamma(N)$ to $\mathrm{SL}_2(\mathbb{Z})$ or even to $\mathrm{GL}_2(\mathbb{Q})$. In the present article, however, we restrict ourselves to looking at a version on $\mathrm{PSL}_2(\mathbb{Z})$ associated to $E_k^{(\mathbf{x})}$ valued in $\mathrm{Sym}^{k-2}(\mathbb{Q}^2) = \mathbb{Q}[X,Y]_{\deg = k-2}$: we consider the generalized Rademacher function

$$\Phi_{\mathbf{x}}^{(k)} : \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathrm{Sym}^{k-2}(\mathbb{Q}^2) = \mathbb{Q}[X,Y]_{\deg=k-2} \tag{2.2}$$

$$A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto \sum_{r=0}^{k-2} \Phi_{\mathbf{x}}^{(r+1,k-1-r)}(A) X^r Y^{k-2-r}$$

defined for $A = \binom{ab}{cd} \in \mathrm{SL}_2(\mathbb{Z})$ and $\mathbf{x} = (x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2$ $(c \geq 0)$ by

$$
\Phi_{\mathbf{x}}^{(k)}\left(\binom{ab}{cd}\right) = \begin{cases} -\frac{P_k(x_1)}{k} \int_0^{\frac{b}{d}} (tX+Y)^{k-2} dt, & (c=0); \\ -\frac{P_k(x_1)}{k} \int_0^{\frac{a}{c}} (tX+Y)^{k-2} dt \\ -\frac{P_k(ax_1+cx_2)}{k} \int_{-\frac{d}{c}}^{0} (t(aX+cY)+bX+dY)^{k-2} dt \\ +\sum_{r=0}^{k-2} (-1)^r \binom{k-2}{r} X^r (aX+cY)^{k-2-r} s_{\mathbf{x}}^{(k-1-r,r+1)}(a,c), \\ & (c>0). \end{cases} \tag{2.3}
$$

Here, the last factor $s_{\mathbf{x}}^{(k-1-r,r+1)}(a,c)$ is (a genralization of) the classical Dedekind sum:

$$
s_{\mathbf{x}}^{(k-1-r,r+1)}(a,c) = \sum_{i=0}^{c-1} \frac{P_{k-1-r}(\frac{x_1+i}{c})}{k-1-r} \frac{P_{r+1}(x_2 + a\frac{x_1+i}{c})}{r+1} \tag{2.4}
$$

with $P_k(T)$ the above mentioned periodic Bernoulli polynomial.

Remark 2.1. Generalized Rademacher function in the above form was exhibited in [N03, §2]. This formula was also applied in a context of non-commutative geometry [CM06, p.105].

Remark 2.2. It is easy to see that $s_{\mathbf{x}}^{(k-1-r,r+1)}(a,c)$ is determined regardless of representatives (x_1, x_2) in $\mathbb{Q}^2$ for any given class $\mathbf{x} \in (\mathbb{Q}/\mathbb{Z})^2$. See also [HWZ] for reciprocity formulas involving this type of generalized Dedekind sums.

The following basic formations for $A, B \in \mathrm{SL}_2(\mathbb{Z})$ are known:

$$
\Phi_{\mathbf{0}}^{(2)} = -\frac{1}{12}\varphi(A); \tag{2.5}
$$

$$
\Phi_{\mathbf{x}}^{(k)}(A) = n^{k-2} \sum_{\mathbf{y} \in \frac{1}{n}\mathbf{x}} \Phi_{\mathbf{y}}^{(k)}(A) \qquad (n \geq 1); \tag{2.6}
$$

$$
\Phi_{\mathbf{x}}^{(k)}(AB) = \Phi_{\mathbf{x}}^{(k)}(A) + \rho(A).\Phi_{\mathbf{x}A}^{(k)}(B) + \frac{1}{4}\delta_{\mathbf{x}}^{k=2} \mathrm{sgn}(c_A c_B c_{AB}), \tag{2.7}
$$

where, $\delta_{\mathbf{x}}^{k=2}$ is the characteristic function that gives the value 1 only when $(k, \mathbf{x}) = (2, \mathbf{0})$, and $\rho(A)$ for a matrix $A = \binom{ab}{cd}$ denotes the left action on $\mathbb{Q}[X, Y]$:

$$
\rho(A) : \begin{cases} X \mapsto aX + cY, \\ Y \mapsto bX + dY; \end{cases} \qquad \delta_{\mathbf{x}}^{k=2} = \begin{cases} 1, & \text{if } (k, \mathbf{x}) = (2, \mathbf{0}); \\ 0, & \text{otherwise.} \end{cases}
$$

3. Measure function $\mathcal{E}$ on the congruence kernel

For a while, we leave S to be a general (noetherian normal) $\mathbb{Q}$-scheme as in Introduction, and suppose that we are given a family of elliptic curves $E - \{O\}/S$ with the Weierstrass equation $Y^2 = 4X^3 - g_2X - g_3$ $(g_2, g_3 \in B,\ \Delta := g_2^3 - 27g_3^2 \in B^\times)$. We shall write $\hat{\Pi} = \pi_1(E_{\bar{\xi}} \setminus \{O\})$ for the profinite fundamental group of a once punctured elliptic curve – a free profinite group of rank 2 presented by

$$
\hat{\Pi} = \langle x_1, x_2, z \mid [x_1, x_2]z = 1 \rangle, \tag{3.1}
$$

where $z = x_2 x_1 x_2^{-1} x_1^{-1}$ is taken as a topological loop on $E(\mathbb{C}) \setminus \{O\}$ around the puncture O. In the fundamental exact sequence

$$1 \to \hat{\Pi} \to \pi_1(E \setminus O, \tilde{\xi}) \to \pi_1(S, \bar{\xi}) \to 1, \tag{3.2}$$

a standard section homomorphism $\pi_1(S) \to \pi_1(E \setminus \{O\})$ is determined by the local coordinate $t = -2x/y$ at O that lifts the outer representation $\varphi : \pi_1(S) \to \mathrm{Out}(\hat{\Pi})$ to

$$\tilde{\varphi} : \pi_1(S) \to \mathrm{Aut}(\hat{\Pi}). \tag{3.3}$$

This action of $\pi_1(S)$ preserves the inertia subgroup $\langle z \rangle \subset \hat{\Pi}$ and acts on it by the cyclotomic character. Cf. [N99] for a quick account of these matters.

Denote the commutator (resp. double commuatator) subgroup of $\hat{\Pi}$ by $\hat{\Pi}'$ (resp. $\hat{\Pi}''$). Then, the abelianization $\hat{\Pi}^{ab} := \hat{\Pi}/\hat{\Pi}'$ is the free $\hat{\mathbb{Z}}$-module generated by $\bar{x}_1, \bar{x}_2$, the images of x_1, x_2, and the induced monodromy representation

$$\varphi^{ab} : \pi_1(S) \to \mathrm{GL}(\hat{\Pi}^{ab}) = \mathrm{GL}_2(\hat{\mathbb{Z}})$$

is nothing but the action on the (projective limit) of torsion points of E. We shall say the kernel $\pi_1(S_\infty)$ to be the *congruence kernel* of $E - \{O\}/S$.

The following proposition enables us to describe the action of $\pi_1(S_\infty)$ by φ on the meta-abelian quotient $\hat{\Pi}/\hat{\Pi}''$ in terms of a single measure function of two variables. Note that $\hat{\Pi}'/\hat{\Pi}''$ forms a free cyclic module over the complete group algebra $\hat{\mathbb{Z}}[[\hat{\Pi}^{ab}]]$ generated by the image $\bar{z} \in \hat{\Pi}'/\hat{\Pi}''$ of $z \in \hat{\Pi}'$ by linearly extending the conjugate action of $\hat{\Pi}^{ab}$ on $\hat{\Pi}'/\hat{\Pi}''$ ([I99]).

Proposition 3.1. *Suppose that an automorphism $\alpha \in \mathrm{Aut}(\hat{\Pi})$ of a free profinite group $\hat{\Pi} = \langle x_1, x_2, z \mid [x_1, x_2]z = 1 \rangle$ satisfies:*
(i) *$\alpha(\langle z \rangle) = \langle z \rangle$;*
(ii) *The action on $\hat{\Pi}/\hat{\Pi}'$ is trivial.*
Then, there exists a unique element $\mathcal{E}_\alpha \in \hat{\mathbb{Z}}[[\hat{\Pi}^{ab}]]$ such that

$$\alpha(x)x^{-1} \equiv \big((\bar{x} - 1)\mathcal{E}_\alpha\big) \cdot \bar{z} \qquad \mathit{mod}\ \hat{\Pi}''$$

holds for every $x \in \hat{\Pi}$ whose image in the abelianization is written as $\bar{x} \in \hat{\Pi}^{ab}$.

It is not difficult to check that $\alpha \mapsto \mathcal{E}_\alpha$ is an additive homomorphism, i.e.,

$$\mathcal{E}_{\alpha\beta} = \mathcal{E}_\alpha + \mathcal{E}_\beta.$$

The pro-ℓ version of $\mathcal{E}$ (cf. §5 below) is the same as the one studied by S.Bloch [Bl84] and H.Tsunogai [Tsu95a], and the proof idea of the above proposition dates back to their work. For a proof of general profinite case, we refer the reader to [N13] §3.6.

4. $\mathcal{E}$ in terms of geometric invariants

Now, let us focus on a family of elliptic curves $E - \{O\}/S$. For each element σ of the congruence kernel $\pi_1(S_\infty)$, the monodromy lift $\tilde{\varphi}(\sigma) \in \mathrm{Aut}(\hat{\Pi})$ satisfies assumptions (i), (ii) of Proposition 3.1. Therefore, we obtain the induced element $\mathcal{E}_\sigma \in \hat{\mathbb{Z}}[[\hat{\Pi}^{ab}]]$ which can be regarded as $\hat{\mathbb{Z}}$-valued measure on $\hat{\Pi}^{ab} \cong \hat{\mathbb{Z}}^2$. Naturally we are motivated to describe the correspondence

$$\begin{aligned} \mathcal{E} : \pi_1(S_\infty) &\longrightarrow \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] \\ \sigma &\longmapsto \mathcal{E}_\sigma \end{aligned} \tag{4.1}$$

in terms of geometric invariants of the family E/S of elliptic curves. To specify our invariants, for each $N \geq 1$, denote by $S_N \to S$ the cover corresponding to the kernel of monodromy action $\pi_1(S) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on the N-torsion points of elliptic curves. Our fixed coordinates of Tate modules $\bar{x}_1, \bar{x}_2$ determines a morphism from S_N to the modular curve $X(N)$ of principal level N. We shall look at theta functions

$$\theta_{\mathbf{x}}(\tau) = q_\tau^{6B_2(\frac{r_1}{N})} e^{12\pi i \frac{r_2}{N}(\frac{r_1}{N}-1)} \left[(1-q_z) \prod_{n\geq 1} (1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}) \right]^{12},$$

for the pairs $\mathbf{x} = (\frac{r_1}{N}, \frac{r_2}{N}) \in (\frac{\mathbb{Z}}{N})^2$. Set $z = (r_1\tau + r_2)/N$, $B_2(T) = T^2 - T + \frac{1}{6}$, and consider $\theta_{\mathbf{x}}$ as a holomorphic function on the upper half plane $\tau \in \mathfrak{H}$. Then, it gives a modular unit on $X(N^2)$ (whose divisor has support on cusps) known as the 12-th power of the so called Siegel units ([KL81]). Taking monodromy along power roots of $\theta_{\mathbf{x}}$ on $S_\infty \to S_{N^2} \to X(N^2)$, we obtain the Kummer character

$$\rho_{\mathbf{x}} : \pi_1(S_\infty) \to \hat{\mathbb{Z}} \quad \text{by} \quad \frac{\sigma(\theta_{\mathbf{x}}^{1/m})}{\theta_{\mathbf{x}}^{1/m}} = \zeta_m^{\rho_{\mathbf{x}}(\sigma)} \quad (\sigma \in \pi_1(S_\infty),\ m \geq 1). \tag{4.2}$$

Since shifting $\mathbf{x}$ by a pair in $\mathbb{Z}^2$ changes $\theta_{\mathbf{x}}$ by multiplcation by a root of unity, and since the structure ring of S_∞ contains the roots of unity, it follows that $\rho_{\mathbf{x}} : \pi_1(S_\infty) \to \hat{\mathbb{Z}}$ is a well-defined character for the congruence class $\mathbf{x}$ mod $\mathbb{Z}^2$.

Theorem 4.1. *For each $\sigma \in \pi_1(S_\infty)$, let*

$$\mathcal{E}_{\sigma,N} = \sum_{\mathbf{r}\in(\mathbb{Z}/N\mathbb{Z})^2} e_{\sigma,N}(\mathbf{r}) \cdot \mathbf{r}$$

be the N-th component of $\mathcal{E}_\sigma \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] = \varprojlim_N \hat{\mathbb{Z}}[(\mathbb{Z}/N\mathbb{Z})^2]$. Then, the coeffcients $e_{\sigma,N}(\mathbf{r})$ are given by

$$e_{\sigma,N}(\mathbf{r}) = \begin{cases} \frac{1}{12}\rho_{\mathbf{r}/N}(\sigma), & (\mathbf{r} \neq \mathbf{0}); \\ \frac{1}{12}\rho_\Delta(\sigma) - \rho_N(\sigma), & (\mathbf{r} = \mathbf{0}). \end{cases}$$

Here $\rho_{\mathbf{r}/N}$ denotes $\rho_{\mathbf{x}}$ for any $\mathbf{x} \in (\frac{\mathbb{Z}}{N})^2$ with $\mathbf{r} = N\mathbf{x}$ mod N, and ρ_Δ (resp. ρ_N) is the Kummer character along power roots of $\Delta := g_2^3 - 27g_3^2$ (resp. of N) $\in B^\times$ respectively.

The pro-l version of this theorem (where $\pi_1(S_\infty)$, $\hat{\mathbb{Z}}$ are replaced by $\pi_1(S_{l^\infty})$, $\mathbb{Z}_l$ respectively) was shown in [N95]. See [N13] for more general profinite version.

5. Pro-l version and Lie derivations

Fix a rational prime l and consider the above construction for the maximal pro-l quotient Π_l of $\hat{\Pi}$. Then, instead of $\mathcal{E}$, we obtain its image $\mathcal{E}^{(l)}$ projected in $\mathbb{Z}_l[[\mathbb{Z}_l^2]]$ which can be defined on $\pi_1(S_{l^\infty})$. Let us continue the notations x_1, x_2 and z of the presentation (3.1) to denote their images in Π_l. It is useful to denote by $\mathrm{Aut}_1^*(\Pi_l)$ (resp. of $\mathrm{Aut}_1^*(\Pi_l/\Pi_l'')$)

the group of automorphisms of Π_l (resp. Π_l/Π_l'') that preserves the image of $\langle z\rangle$ and trivially acts on $\Pi_l^{ab} = \Pi_l/\Pi_l'$. Then, $\mathcal{E}^{(l)}$ is the composite of the monodoromy representation $\varphi^{(l)} : \pi_1(S_{l^\infty}) \to \mathrm{Aut}_1^*(\Pi_l)$ with the natural meta-abelian reduction:

$$\mathcal{E}^{(l)} : \pi_1(S_{l^\infty}) \xrightarrow{\varphi^{(l)}} \mathrm{Aut}_1^*(\Pi_l) \longrightarrow \mathrm{Aut}_1^*(\Pi_l/\Pi_l'') \cong \mathbb{Z}_l[[\mathbb{Z}_l^2]], \tag{5.1}$$

where the last identification $\cong$ is due to the pro-l version of Proposition 3.1. Write $\mathcal{L}(\mathbb{Q}_l)$ for the Malcev Lie algebra of Π_l which is generated by $X_i = \log(x_i)$ $(i = 1, 2)$ and $Z := \log(z)$ subject to the single relation $[X_1, X_2] + Z = 0$. Introduce the lower central filtration $\mathcal{L}(\mathbb{Q}_l) = \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots$ with $\mathcal{L}_{i+1} = [\mathcal{L}_1, \mathcal{L}_i]$ $(i \geq 1)$, and let $\mathrm{Der}_+^*(\mathcal{L}(\mathbb{Q}_l))$ be the Lie algebra of derivations of $\mathcal{L}(\mathbb{Q}_l)$ conveying X_i $(i = 1, 2)$ into $\mathcal{L}_2 = \mathcal{L}(\mathbb{Q}_l)'$ and killing Z. Then, the logarithmic map $\mathrm{Log} : \mathrm{Aut}_1^*(\Pi_l) \to \mathrm{Der}_+^*(\mathcal{L}(\mathbb{Q}_l))$ induces the following commutative diagram:

$$\begin{array}{ccccccc} \pi_1(S_{l^\infty}) & \xrightarrow{\varphi^{(l)}} & \mathrm{Aut}_1^*(\Pi_l) & \longrightarrow & \mathrm{Aut}_1^*(\Pi_l/\Pi_l'') & \overset{\sim}{=\!=} & \mathbb{Z}_l[[\mathbb{Z}_l^2]] \\ & & \downarrow{\scriptstyle \mathrm{Log}} & & \downarrow{\scriptstyle \mathrm{Log}} & & \\ & & \mathrm{Der}_+^*(\mathcal{L}(\mathbb{Q}_l)) & \longrightarrow & \mathrm{Der}_+^*(\mathcal{L}(\mathbb{Q}_l)/\mathcal{L}'') & \overset{\sim}{=\!=} & \mathbb{Q}_l[[X_1, X_2]] \end{array} \tag{5.2}$$

The lower central filtration of Π_l also induces a filtration $\mathrm{Der}_+^*(\mathcal{L}(\mathbb{Q}_l)) = \mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots$, $\mathrm{Der}_+^*(\mathcal{L}(\mathbb{Q}_l)/\mathcal{L}'') = \overline{\mathcal{D}}_1 \supset \overline{\mathcal{D}}_2 \supset \cdots$ by the spaces $\mathcal{D}_i$, $\overline{\mathcal{D}}_i$ of derivations of degree $\geq i$ $(i \geq 1)$. Each of the graded quotients $\mathrm{gr}^i(\mathcal{D}) = \mathcal{D}_i/\mathcal{D}_{i+1}$ and $\mathrm{gr}^i(\overline{\mathcal{D}}) = \overline{\mathcal{D}}_i/\overline{\mathcal{D}}_{i+1}$ is canonically acted on by $\mathrm{GL}_2(\mathbb{Q}_l)$ and decomposed into a sum of irreducible representations $\chi^i S^j := \det^i \otimes \mathrm{Sym}^j(\mathbb{Q}_l^2)$ $(i, j \geq 0)$ of GL_2 as indicated in the following table:

i	$\mathrm{gr}^i\mathcal{D}$	$\mathrm{gr}^i\overline{\mathcal{D}}$
1	0	0
2	χS^0	χS^0
3	0	χS^1
4	χS^2	χS^2
5	0	χS^3
6	$\chi S^4 + \ 0\ + \chi^2 S^0$	χS^4
7	$0\ + \chi^2 S^3$	χS^5
8	$\chi S^6 + \ 0\ + 2\chi^3 S^2$	χS^6
9	$0\ + \chi^2 S^5 + \chi^3 S^3 + \chi^4 S^1$	χS^7
10	$\chi S^8 + \chi^2 S^6 + 3\chi^3 S^4 + \chi^4 S^2 + 3\chi^5 S^0$	χS^8
11	$0\ + \chi^2 S^7 + 2\chi^3 S^5 + 4\chi^4 S^3 + 2\chi^5 S^1$	χS^9
12	$\chi S^{10} + \chi^2 S^8 + 5\chi^3 S^6 + 4\chi^4 S^4 + 8\chi^5 S^2$	χS^{10}
13	$0\ + 2\chi^2 S^9 + 3\chi^3 S^7 + 8\chi^4 S^5 + 9\chi^5 S^3 + 6\chi^6 S^1$	χS^{11}
14	$\chi S^{12} + \chi^2 S^{10} + 7\chi^3 S^8 + 9\chi^4 S^6 + 18\chi^5 S^4 + 11\chi^6 S^2 + 11\chi^7 S^0$	χS^{12}
15	$0\ + 2\chi^2 S^{11} + 5\chi^3 S^9 + 14\chi^4 S^7 + 21\chi^5 S^5 + 26\chi^6 S^3 + 17\chi^7 S^1$	χS^{13}

Note that, for $i \geq 2$, $\mathrm{gr}^i\overline{\mathcal{D}}$ is given by the space of homogeneous polynomials in $\mathbb{Q}_l[[X_1, X_2]]$ of degree $i - 2$, where $\mathrm{GL}_2(\mathbb{Q}_l)$ acts by the natural action on $\mathbb{Q}_l X_1 \oplus \mathbb{Q}_l X_2$ twisted once by the det-character. Tsunogai, already in [Tsu92], showed that $\mathrm{gr}^i\mathcal{D} = \sum_{j\geq 1} b_{j-1}^{(i)} \chi^j S^{i-2j}$

with

$$(5.3)\qquad b_0^{(i)} = \begin{cases} 0 & (i : \text{odd}), \\ 1 & (i : \text{even}); \end{cases} \qquad b_1^{(i)} = \begin{cases} \lfloor \frac{i-1}{6} \rfloor - 1 & (i \equiv 2 \bmod 6,\ i > 2), \\ \lfloor \frac{i-1}{6} \rfloor & (i \not\equiv 2 \bmod 6), \end{cases}$$

and noted that $b_1^{(i)}$ is the dimension of the cusp forms of weight $2(i-1)$ for $\mathrm{SL}_2(\mathbb{Z})$ [ii]. It is also noteworthy that the highest weight part of $\mathrm{gr}^{2i}\mathcal{D}$ $(i \geq 1)$ under the action of $\mathfrak{sl}_2$ appears multiplicity freely (according to $b_0^{(2i)} = 1$ in the above notation) and is generated by the graded derivation

$$(5.4)\qquad \epsilon_{2i} : \begin{cases} X_1 & \mapsto (\mathrm{ad}X_1)^{2i}(X_2), \\ X_2 & \mapsto \displaystyle\sum_{r=0}^{i-2} [(\mathrm{ad}X_1)^r(X_2), (\mathrm{ad}X_1)^{2i-1-r}(X_2)] \end{cases}$$

on $\mathrm{Gr}(\mathcal{L}(\mathbb{Q}_l))$, as figured out first by Tsunogai in [Tsu95b, §3].

In particular, it follows that the image of $\mathcal{E}^{(l)} : \pi_1(S_{l^\infty}) \to \mathbb{Z}_l[[\mathbb{Z}_l^2]]$ lies in the even degree part of $\mathbb{Z}_l[[\mathbb{Z}_l^2]]$, i.e., the invariant part of $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_l)$. This latter fact is also a geometric consequence of the fact that the elliptic curve E/S has involution by the multiplication by (-1) of E, as noted in [N95].

On the other side, one also observes in the above table that $\mathfrak{sl}_2$-invariant components S^0 start to appear in $\mathrm{gr}^6\mathcal{D}$, $\mathrm{gr}^{10}\mathcal{D}$, $\mathrm{gr}^{14}\mathcal{D}$,... In fact, we know that the image of $\varphi^{(l)}$ for arithmetic base S contains an $\mathfrak{sl}_2$-invariant free Lie algebra which has one non-trivial generator in each degree $2k$ ($k \geq 3$, odd) surviving via the Soule character in $H^1(\mathbb{Q}, \mathbb{Z}_l(k))$. This is a consequence of the Deligne-Oda conjecture (specialized to genus one case) which was settled by several Japanese authors in 1990s, combined with F.Brown's more recent result on the mixed Tate motives. For this topic, we refer the readers to a comprehensive survey article by M.Matsumoto [Mat13] and references therein.

6. $\mathcal{E}$ on CSL_2

Let us apply the construction of §3-4 to the case where $S = M_{1,1} \otimes \overline{\mathbb{Q}}$ (the moduli stack of elliptic curves) and $E \setminus \{O\} \cong M_{1,2} \otimes \overline{\mathbb{Q}}$ (the universal elliptic curve over S). There is a theoretical treatment of etale fundamental groups for moduli stacks (cf. [Od97]) which provides (3.1), (3.2) and (4.1) in a well-behaved manner [while the exact sequence (3.2) does not split so that a lifted monodromy $\tilde{\varphi}$ (3.3) is unavailable and needs a careful alternative]. In this case, $\pi_1(S)$ is naturally identified with the profinite completion of $\mathrm{SL}_2(\mathbb{Z})$. We shall consider two different kinds of topology : the congruence topology and the non-congruence topology on $\mathrm{SL}_2(\mathbb{Z})$. In the congruence topology, two matrices A and B are closer to each other as $A \equiv B$ modulo a (multiplicatively) bigger integer. In the non-congruence topology, they are regarded as closer when the ratio AB^{-1} lies in smaller subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. There is a projection of the completion in the latter topology

$$\mathrm{SL}_2(\mathbb{Z})^\wedge := \varprojlim_{(\mathrm{SL}_2(\mathbb{Z}):U)<\infty} (\mathrm{SL}_2(\mathbb{Z})/U)$$

[ii] See [MSS] Proposition 8.2 for a proof. The latter was also observed and enlightened by D.Zagier on the occasion of my talk at Bristol conference in May, 2011.

onto the completion in the former topology $\mathrm{SL}_2(\hat{\mathbb{Z}})$. The *congruence kernel* is by definition $\pi_1(S_\infty) = \mathrm{CSL}_2 := \ker(\mathrm{SL}_2(\mathbb{Z})^\wedge \to \mathrm{SL}_2(\hat{\mathbb{Z}}))$:

$$1 \longrightarrow \mathrm{CSL}_2 \longrightarrow \mathrm{SL}_2(\mathbb{Z})^\wedge \longrightarrow \mathrm{SL}_2(\hat{\mathbb{Z}}) \longrightarrow 1.$$

To see CSL_2 in a more down-to-earth way, let $\Gamma(n) \subset \mathrm{SL}_2(\mathbb{Z})$ be the principal congruence subgroup of level $n \geq 3$, and let $P_n \subset \mathrm{SL}_2(\mathbb{Z})$ be the normal subgroup generated by the inertia subgroups in $\Gamma(n)$ at cusps on the modular curve $X(n)$. In other words, the P_n is the smallest normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing the matrix $\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$. According to K.Wohlfahrt [Woh64], we know that $\{\Gamma(n)/P_n\}_n$ forms a *surjective* projective system with respect to the levels n multiplicatively. Note that the quotient group $\Gamma(n)/P_n$ is isomorphic to the fundaemental group of the compactified modular curve $X(n)$, in particular, is of residually finite.

Lemma 6.1. (i) CSL_2 *is isomorphic to the projective limit* $\varprojlim_n (\Gamma(n)/P_n)^\wedge$.

(ii) *For a prime l, let* $\mathrm{CSL}_2^{(l)}$ *be the kernel of natural map* $\mathrm{SL}_2(\mathbb{Z})^\wedge \to \mathrm{SL}_2(\mathbb{Z}_l)$. *Then,* $\mathrm{CSL}_2^{(l)}$ *is isomorphic to* $\varprojlim_n (\Gamma(l^n)/P_{l^n})^\wedge$.

(iii) *Both* CSL_2 *and* $\mathrm{CSL}_2^{(l)}$ *are free profinite groups on a countable number of generators.*

Proof. To prove (i), we shall trace an arugment in [Se70, §2.5] carefully in our context. First, we remark that every finite index subgroup H of $\mathrm{SL}_2(\mathbb{Z})$ contains P_m for some large enough m. [Indeed, without loss of generality we may assume H is normal in $\mathrm{SL}_2(\mathbb{Z})$, and then may pick $m > 0$ so that $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) \in H$.] Now, for every positive integer n, we have an exact sequence of profinite groups:

$$1 \to (\Gamma(n)/P_n)^\wedge \to (\mathrm{SL}_2(\mathbb{Z})/P_n)^\wedge \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \to 1.$$

The assertion follows from taking the projective limit of the above sequence with respect to n multiplicatively:

$$1 \to \varprojlim_n (\Gamma(n)/P_n)^\wedge \to \mathrm{SL}_2(\mathbb{Z})^\wedge \to \mathrm{SL}_2(\hat{\mathbb{Z}}) \to 1.$$

Note here that the middle term identification with $\mathrm{SL}_2(\mathbb{Z})^\wedge$ follows from the above remark, and that the surjectivity onto $\mathrm{SL}_2(\hat{\mathbb{Z}})$ is a consequence of (kind of Mittag-Leffler condition deduced from) the above mentioned surjectivity result by Wohlfahrt [Woh64]. (ii) follows from an exactly similar argument to (i). (iii): It is well known that CSL_2 is a free profinite group with countably many generators (Melnikov, Lubotzky [Lu82]). As for $\mathrm{CSL}_2^{(l)}$, we first note that $\mathrm{CSL}_2^{(l)}$ is a closed normal subgroup of a free profinite group $\Gamma(l^2)$ with some e $(2 \leq e < \infty)$ free generators. To show the assertion, by [Lu82, Theorem 2.1(b)], it suffices to see that the quotient group $Q := \Gamma(l^2)/\mathrm{CSL}_2^{(l)}$ is not 'e-freely indexed' in the sense of loc.cit. This follows from that fact Q is an analytic pro-l group $(\subset \mathrm{SL}_2(\mathbb{Z}_l))$ whose open subgroups have a stable rank [i.e., the minimal number of generators of open subgroups $H \subset Q$ does not increase with $(Q : H)$]. □

To approximate each $\sigma \in \mathrm{CSL}_2$ by a sequence of 2×2 integral matrices in the non-congruence topology, we introduce for $n \geq 1$,

$$V_n := \bigcap \mathscr{H}_n, \tag{6.1}$$

where $\mathscr{H}_n$ is the collection of all subgroups H of $\Gamma(n)$ with $(\Gamma(n) : H) | n$ and $P_n \subset H$.

Lemma 6.2. *Notations being as above, we have*

(i) $V_m \supset V_n$ *if* $m|n$.
(ii) *Every finite index subgroup* H *of* $\mathrm{SL}_2(\mathbb{Z})$ *contains* V_n *for some* n.

Proof. (i) Set $m = N$ and $n = MN$. Remark that, by the above mentioned theorem of Wohlfahrt, the natural map $\pi : \Gamma(MN)/P_{MN} \to \Gamma(N)/P_N$ is a surjective homomorphism. Pick any $\alpha \in V_{MN}$ and $H \in \mathscr{H}_N$. It suffices to show $\alpha \in V_N$. Now, from the above remark it follows that the index of $H' := H \cap \Gamma(MN)$ in $\Gamma(MN)$ is the same as that of H in $\Gamma(N)$, hence $(\Gamma(MN) : H')$ divides N, in particular, divides MN. Thus $H' \in \mathscr{H}_{MN}$; hence $\alpha \in V_{MN} \subset H' \subset H$. Letting H vary in the members of $\mathscr{H}_N$, we obtain $\alpha \in V_N$.

(ii) Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Without loss of generality, we may assume H is a normal subgroup. Pick a positive integer N with $\left(\begin{smallmatrix}1 & N\\ 0 & 1\end{smallmatrix}\right) \in H$. Then $P_N \subset H$. Let $H' := H \cap \Gamma(N)$, $M := (\Gamma(N) : H')$ and set $H'' := \Gamma(MN) \cap H'$. Then, again by the above theorem of Wohlfahrt tells the natural map $\pi : H''/P_{MN} \to H/P_N$ is a surjective homomorphism, hence $H' = H'' \cdot P_N$. From this it follows that $(\Gamma(MN) : H'') = (\Gamma(N) : H') = M$ divides MN and that $H'' \in \mathscr{H}_{MN}$. Therefore we conclude $V_{MN}(\subset H'') \subset H$, which settles the proof. □

Definition 6.3. We shall say that a matrix sequence $\{A_n \in \Gamma(n)\}$ *converges to* $\sigma \in \mathrm{CSL}_2$ *in the non-congruence topology*, if there is an integer N such that, for every positive multiple n of N, the images of A_n and σ in the finite quotient $\widehat{\Gamma(n)}/\widehat{V_n}(= \Gamma(n)/V_n)$ coincide to each other.

We would like to look closely at the measure function $\mathcal{E} : \mathrm{CSL}_2 \to \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$ in the case of $S = M_{1,1} \otimes \overline{\mathbb{Q}}$ in terms of a sequence of matrices converging to $\sigma \in \mathrm{CSL}_2$. Let $\varepsilon : \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] \to \hat{\mathbb{Z}}$ be the augmentation homomorphism. We shall take the unique lift of $\mathrm{CSL}_2 = \pi_1(S_\infty)$ in $\mathrm{Aut}_1^*(\hat{\Pi})$ such that $\varepsilon(\mathcal{E}_\sigma) = 0$ for all $\sigma \in \mathrm{CSL}_2$. (The existence and uniqueness of such a lift can be seen group-theoretically from the fact that $\mathrm{gr}^2\mathcal{D} \cong \mathbb{Q}_l(2)$ for all primes l [iii]. See also Lemma 9.1 below.) This is equivalent to vanishing of $\frac{1}{12}\rho_\Delta$ on CSL_2. If we denote the image of $\mathcal{E}_\sigma$ in $\hat{\mathbb{Z}}[(\mathbb{Z}/N\mathbb{Z})^2]$ by

$$\mathcal{E}_{\sigma,N} = \sum_{\mathbf{r} \in (\mathbb{Z}/N\mathbb{Z})^2} e_{\sigma,N}(\mathbf{r}) \cdot \mathbf{r}$$

as in the setting of Theorem 4.1, then we have $e_{\sigma,N}(\mathbf{0}) = 0$. (Note that $\theta_{\mathbf{x}}$ changes to its certain constant multiples when $\mathbf{x} \in \mathbb{Q}^2$ varies modulo $\mathbb{Z}^2$, whereas $E_2^{(\mathbf{x})}$ relies only on the class $\mathbf{x} \in (\mathbb{Q}/\mathbb{Z})^2$.) As for the other $\mathbf{r} = (r_1, r_2) \in (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{\mathbf{0}\}$, $e_{\sigma,N}(\mathbf{r})$ are approximated by monodromy actions on the power roots of $\theta_{\mathbf{r}/N}$ through the identity

$$\frac{d}{d\tau}\log\theta_{\mathbf{x}}(\tau) = -(24\pi i)E_2^{(\mathbf{x})}(\tau) \qquad \left(\mathbf{x} = (\frac{r_1}{N}, \frac{r_2}{N}) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \setminus \{\mathbf{0}\}\right). \tag{6.2}$$

More precisely, if a sequence of matrices $\{A_n \in \Gamma(n)\}_n$ converges to $\sigma \in \mathrm{CSL}_2$ in the non-congruence topology, then, for every $m \geq 1$, there exists a large enough n such that $A = A_n$ satisfies

$$\frac{\sigma(\theta_{\mathbf{x}}(\tau))^{1/m}}{\theta_{\mathbf{x}}(\tau)^{1/m}} = \frac{\theta_{\mathbf{x}}(\tau)^{1/m}|_{{}^tA}}{\theta_{\mathbf{x}}(\tau)^{1/m}} = \exp\left(-\frac{24\pi i}{m}\int_\tau^{{}^tA\tau} E_2^{(\mathbf{x})}(\tau)d\tau\right) = \zeta_m^{-12\Phi_{\mathbf{x}}^{(2)}({}^tA)}.$$

[iii] This remark could be traced back to [N95] (4.4).

Thus, we have

$$\rho_{\mathbf{x}}(\sigma) = -12 \lim_{n \to \times\infty} \Phi^{(2)}_{\mathbf{x}}({}^t A_n),$$

where $\lim\limits_{n\to\times\infty}$ means the limit in n (multiplicatively). In the setting of Theorem 4.1, the above discussion can be summarized as

$$e_{\sigma,N}(r_1, r_2) = \begin{cases} -\lim\limits_{n\to\times\infty} \Phi^{(2)}_{(\frac{r_1}{N},\frac{r_2}{N})}({}^t A_n), & (r_1, r_2) \not\equiv \mathbf{0} \pmod N; \\ 0, & (r_1, r_2) \equiv \mathbf{0} \pmod N). \end{cases} \tag{6.3}$$

By the multiplicative independence of Siegel units (cf. [KL81, Chap. 5, §7]), the image of $\mathcal{E}$ covers most of the even part $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]_+$, which is the invariant part under the involution '$\mathbf{x} \leftrightarrow -\mathbf{x}$'. This means that $\mathcal{E}$ maps CSL_2 into $(\ker(\varepsilon)^4)_+$ so as to have an open image in every quotient $(\ker(\varepsilon)^4/\ker(\varepsilon)^N)_+$ for $N \geq 4$. More effective estimation of that image size should amount to calculation of the integrals of the logarithmic derivative of $\theta_{\mathbf{x}}(\tau)$ along any path on $\mathfrak{H}$ from a point τ_0 to $A\tau_0$ for various matrices $A \in \mathrm{SL}_2(\mathbb{Z})$, i.e., periods of Eisenstein series. This will be one of the ultimate goals of our discussions below bringing into play the classical formulas for such Eisenstein periods via (generalized) Dedekind sums.

7. **Eisenstein periods in l-adic expansion of $\mathcal{E}$**

Let us first review classical Eisenstein cocycles. For an integer $k > 2$, the Eisenstein series

$$E_k(\tau) := \frac{(k-1)!}{(2\pi i)^k} \sideset{}{'}\sum_{(m_1,m_2)} \frac{1}{(m_1\tau + m_2)^k} \qquad (\tau \in \mathfrak{H}),$$

where the summation is over the pairs $(m_1, m_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and its q-expansion has the constant term $-B_k/k$ ($q = \exp(2\pi i\tau)$). Note that the above E_k is a special case $E_k^{(\mathbf{0})}$ of (2.1) and that $E_k \equiv 0$ for odd $k > 2$. Choose a $(k-1)$-th indefinite integral called the "Eichler integral of the first type" by

$$F_k(\tau) = -\frac{1}{(k-2)!} \int_{\tau}^{i\infty} \left(E_k(u) + \frac{B_k}{k} \right) (\tau - u)^{k-2} du - \frac{B_k}{k} \frac{\tau^{k-1}}{(k-1)!}.$$

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ act on $\mathfrak{H}$ by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. Consider an amplitude of the above $F_k(\tau)$ under transformation by $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:

$$\phi_k(A) := F_k(A\tau)(c\tau + d)^{k-2} - F_k(\tau).$$

It is easy to see that $\phi_k(A)$ turns out to be a polynomial in τ of degree $(k-2)$. Moreover, the (coefficientwise) real part $\mathrm{Re}(\phi_k)$ is known to be in $\mathbb{Q}[\tau]$. In fact, from a series of computations similar to [G80, Remark 4.7], [H83, §1.4], we obtain

$$X^{k-2}\mathrm{Re}(\phi_k)(A)\left(-\frac{Y}{X}\right) = \frac{(-1)^{k-1}}{(k-2)!} \Phi^{(k)}_{\mathbf{0}}(A^{-1})(X, Y) \tag{7.1}$$

in $\mathbb{Q}[X, Y]_{\deg = k-2}$ for $A \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 7.1 (Eisenstein cocycle)**.** For $i, j \geq 0$ with $i + j = k - 2 \geq 0$, define the map $\psi_{ij} : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}$ by the following identity of polynomials in $\mathbb{Q}[\tau]$:

$$\sum_{i+j=k-2} \psi_{ij}(A)\tau^i := -\mathrm{Re}(\phi_k)({}^t A) \qquad (A \in \mathrm{SL}_2(\mathbb{Z})).$$

We then define a polynomial $\Psi_k(A)$ $(A \in \mathrm{SL}_2(\mathbb{Z}))$ in two variables X, Y by

$$\Psi_k(A)(X,Y) := \sum_{\substack{i+j=k-2\\ i,j\geq 0}} \psi_{ij}(A) X^j Y^i \left(= -X^{k-2}\mathrm{Re}(\phi_k)({}^tA)\left(\frac{Y}{X}\right).\right)$$

The above definition 7.1 combined with (2.2), (7.1) implies

$$-\sum_{i+j=k-2} \psi_{ij}({}^tA) X^j (-Y)^i = \frac{(-1)^{k-1}}{(k-2)!} \sum_{i+j=k-2} \Phi_{\mathbf{0}}^{(j+1.i+1)}(A^{-1}) X^j Y^i$$

for every matrix $A \in \mathrm{SL}_2(\mathbb{Z})$. In particular, it holds that

$$\psi_{ij}({}^tA) = \frac{(-1)^j}{(k-2)!}\Phi_{\mathbf{0}}^{(j+1,i+1)}(A^{-1}) \tag{7.2}$$

for $i,j \geq 0$, $i+j=k-2$.

l-adic (and adelic) Eisenstein periods.

Let l be a fixed rational prime. For the universal elliptic curve $S = M_{1,1}\otimes\overline{\mathbb{Q}}$, the power series representation $\mathcal{E}^{(l)}$ of (5.1) is an additive homomorphism from the l-congruence kernel $\mathrm{CSL}_2^{(l)} = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z})^\wedge \to \mathrm{SL}_2(\mathbb{Z}_l))$ to $\mathbb{Z}_l[[\mathbb{Z}_l^2]]$. Write $\bar{x}_1, \bar{x}_2$ for the images of generators x_1, x_2 of $\hat{\Pi}$ in the pro-l abelianization. Then the target ring $\mathbb{Z}_l[[\mathbb{Z}_l^2]]$ is identified with the commutative power series ring $\mathbb{Z}_l[[T_1,T_2]]$ in $T_i = \bar{x}_i - 1$ $(i=1,2)$, which may be embedded into the rational power series ring $\mathbb{Q}_l[[U_1,U_2]]$ generated by $U_i := \log(\bar{x}_i)$ $(i=1,2)$. Expand $\mathcal{E}_\sigma^{(l)}$ for $\sigma \in \mathrm{CSL}_2^{(l)}$ in the form:

$$\mathcal{E}_\sigma^{(l)} = \sum_{i,j=0}^{\infty} e_{ij}^{(l)}(\sigma)\frac{U_1^i U_2^j}{i!j!},$$

and consider the coefficient character $e_{ij}^{(l)} : \mathrm{CSL}_2 \to \mathbb{Z}_l$. By the symmetry condition, we immediately see that $e_{00}^{(l)} = 0$ and that if $i+j = \mathit{odd}$ then $e_{ij}^{(l)} = 0$. In the following, we shall interpret the other coefficients as "l-adic limits" of Eisenstein cocycles.

For our fixed prime l and given $i,j\geq 0$ with $i+j=k-2\geq 0$, we shall introduce the l-adic limit

$$\psi_{ij}^{(l)} : \mathrm{CSL}_2^{(l)} \to \mathbb{Q}_l \tag{7.3}$$

of the coefficient character $\psi_{ij} : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}$ as follows. First, since $\mathrm{SL}_2(\mathbb{Z})$ is finitely generated, from the composition rule (2.7), we know that the denominator of the image of $\mathrm{SL}_2(\mathbb{Z})$ by ψ_{ij} is bounded. Moreover, recalling from (2.3) that

$$\Phi_{\mathbf{0}}^{(k)}\left(\left(\begin{smallmatrix}1 & l^n\\ 0 & 1\end{smallmatrix}\right)\right) = -\frac{B_k}{k}\int_0^{l^n}(tX+Y)^{k-2}dt,$$

we see from (7.1) that there is a constant integer c so that for every $n\geq 1$ the image of P_{l^n} by ψ_{ij} is divisible by l^{n-c}. Thus, there are sufficiently large integers α, β such that

- For every $\sigma \in \mathrm{SL}_2(\mathbb{Z})$, it holds that $\alpha\cdot\psi_{ij}(\sigma)\in\mathbb{Z}$;
- For every $n\geq 1$ and $\sigma\in P_{l^{n+\beta}}$, it holds that $\alpha\cdot\psi_{ij}(\sigma)\in l^n\mathbb{Z}$.

Consider the projective limit

$$\alpha\cdot\psi_{ij}:\mathrm{CSL}_2^{(l)}=\varprojlim_n(\Gamma(l^{n+\beta})/P_{l^{n+\beta}})^\wedge\longrightarrow\varprojlim_n(\mathbb{Z}_l/l^n\mathbb{Z}_l)=\mathbb{Z}_l$$

and define $\psi_{ij}^{(l)}:\mathrm{CSL}_2^{(l)}\to\mathbb{Q}_l$ to be the α^{-1}-multiple of it. This is well defined and determined independently of the choice of α,β. Accordingly we define

$$\Psi_k^{(l)}(\sigma)(X,Y):=\sum_{\substack{i+j=k-2\\ i,j\geq 0}}\psi_{ij}^{(l)}(\sigma)X^jY^i\qquad(\sigma\in\mathrm{CSL}_2^{(l)}).\tag{7.4}$$

Theorem 7.2. *For each $\sigma\in\mathrm{CSL}_2^{(l)}$, the coefficient characters in the expansion*

$$\mathcal{E}_\sigma^{(l)}=\sum_{i,j}e_{ij}^{(l)}(\sigma)\frac{U_1^iU_2^j}{i!j!}$$

are given by:

$$\frac{e_{ij}^{(l)}(\sigma)}{i!j!}=\begin{cases}\psi_{ij}^{(l)}(\sigma) & (i+j\geq 2,\ \textit{even}),\\ 0 & (\textit{otherwise}).\end{cases}$$

We will illustrate a proof of this theorem in the next section.

Before closing this section, let us introduce the profinite version of Ψ_k and ψ_{ij}. Note first that $\mathrm{CSL}_2=\bigcap_{l:\mathrm{prime}}\mathrm{CSL}_2^{(l)}$, since $\mathrm{SL}_2(\hat{\mathbb{Z}})=\prod_l\mathrm{SL}_2(\mathbb{Z}_l)$.

Definition 7.3. We define $\widehat{\psi_{ij}}:\mathrm{CSL}_2\to\mathbb{Q}\otimes\hat{\mathbb{Z}}$ to be the mapping

$$\sigma\longmapsto(\psi_{ij}^{(l)}(\sigma))_l\in\prod_l\left(\frac{1}{i!j!}\mathbb{Z}_l\right)\subset\mathbb{Q}\otimes\hat{\mathbb{Z}},$$

and accordingly define $\widehat{\Psi}_k:\mathrm{CSL}_2\to(\mathbb{Q}\otimes\hat{\mathbb{Z}})[X,Y]$ by

$$\widehat{\Psi}_k(\sigma)(X,Y)=\sum_{\substack{i+j=k-2\\ i,j\geq 0}}\widehat{\psi_{ij}}(\sigma)X^jY^i\qquad(\sigma\in\mathrm{CSL}_2).$$

8. **Trading levels for weights: Proof of Theorem 7.2**

Derivation of Theorem 7.2 from Theorem 4.1 amounts to deriving information on Eichler-Shimura integrals for Eisenstein series of any higher weights $k\geq 4$ for the fixed level $\Gamma(1)=\mathrm{SL}_2(\mathbb{Z})$ from those for the Eisenstein series of weight $k=2$ fixed but for all levels $\Gamma(N)$ $(N\geq 1)$. In the terminology of generalized Rademacher functions introduced in §2, we may reduce Theorem 7.2 to combination of Theorem 4.1 with the following

Lemma 8.1. *Let N, r, k be integers with $N\geq 1$, $k\geq 2$, $0\leq r\leq k-2$, and let $\mathbb{Z}'_N\subset\mathbb{Q}$ be the ring of rational numbers whose denominators are prime to N. Then, there exits a positive integer $D_{k,r}$ depending only on k,r such that for each $A\in\Gamma(N)$ the following congruence holds*

$$\binom{k-2}{r}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}x^{k-2-r}(-y)^r\Phi^{(2)}_{(\frac{x}{N},\frac{y}{N})}(A)\equiv\Phi_{\mathbf{0}}^{(r+1,k-1-r)}(A)$$

mod $\binom{k-2}{r}\frac{N}{12D_{k,r}}\mathbb{Z}'_N$.

See [N03] for a proof of this lemma and for an explicit description of $D_{k,r}$ as the common denominator of finitely many Bernoulli numbers. The lemma was first shown in the special case $r = 0$ by the author, and the general case was conjectured by Mr.Morimoto in his Master thesis ([Mo02]) who detected the binomial coefficient $\binom{k-2}{r}$ by numerical experiments. Then it was not difficult to conjecture a polynomial congruence in the form

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}(xY-yX)^{k-2}\Phi^{(2)}_{(\frac{x}{N},\frac{y}{N})}(A) \equiv \Phi^{(k)}_{\mathbf{0}}(A)(X,Y) \tag{8.1}$$

modulo $\frac{N}{D_k}\mathbb{Z}'_N$ with a bounded denominator D_k in $\mathbb{Q}[X,Y]$. Soon afterwards an elementary proof of the above lemma was obtained as in [N03].

Example 8.2. Let us discuss a simple numerical example for congruences of Lemma 8.1. Take a matrix $A = \begin{pmatrix} 12 & -55 \\ 55 & -252 \end{pmatrix} \in \Gamma(11)$, and consider the case $k = 6$, $N = 11$. In this case, for all $0 \le r \le 4$, we have $\binom{4}{r}\frac{11}{12D_{6,r}}\mathbb{Z}'_{11} = 11\mathbb{Z}'_{11}$. The RHS (the real part of Eichler-Shimura integral) is then

$$\Phi^{(6)}_{\mathbf{0}}(A)(X,Y) = \frac{1398479}{42}X^4 + \frac{1537687159}{2520}X^3Y + \frac{58706693}{14}X^2Y^2 + \frac{6455052203}{504}XY^3 + \frac{308055833}{21}Y^4,$$

while LHS (the moment sum) is

$$\sum_{x,y=0}^{10}(xY-yX)^4\Phi^{(2)}_{(\frac{x}{11},\frac{y}{11})}(A) = \frac{52009}{4}X^4 - 19121\,X^3Y + \frac{68901}{2}X^2Y^2 - 19121\,XY^3 + \frac{52009}{4}Y^4.$$

The above coefficients are all prime to 11. The difference RHS−LHS is computed as

$$\begin{aligned}&\Phi^{(6)}_{\mathbf{0}}(A)(X,Y) - \sum_{x,y=0}^{10}(xY-yX)^4\Phi^{(2)}_{(\frac{x}{11},\frac{y}{11})}(A)\\ &= \frac{1704769}{84}X^4 + \frac{1585872079}{2520}X^3Y + \frac{29112193}{7}X^2Y^2 + \frac{6464689187}{504}XY^3 + \frac{1231131143}{84}Y^4 \equiv 0\end{aligned}$$

mod $11\mathbb{Z}'_{11}$.

Now, let $\sigma \in \mathrm{CSL}_2$ and pick a matrix sequence $\{A_n \in \Gamma(n)\}_n$ converging to σ in the non-congruence topology. Putting (6.3) and (7.2) with $A = A_n$ into Lemma 8.1, and taking the limit $n \to \times\infty$ for n multiples of N, we obtain

$$-\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}x^iy^je_{\sigma,N}(x,y) \equiv \lim_{N|n\to\times\infty}\frac{(-1)^j i!\,j!}{(k-2)!}\Phi^{(j+1,i+1)}_{\mathbf{0}}({}^tA_n) = i!\,j!\,\widehat{\psi_{ij}}(\sigma^{-1}) \tag{8.2}$$

mod $\frac{N}{12D_{k,r}}\hat{\mathbb{Z}}$ for N subject to being multiples of $12D_{k,r}$. Now, the element

$$\mathcal{E}_\sigma = \varprojlim_N \mathcal{E}_{\sigma,N} = \varprojlim_N \sum_{\mathbf{r}\in(\mathbb{Z}/N\mathbb{Z})^2} e_{\sigma,N}(\mathbf{r})\cdot\mathbf{r}$$

of the complete group algebra

$$\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] = \varprojlim_N \hat{\mathbb{Z}}[(\mathbb{Z}/N\mathbb{Z})^2] = \varprojlim_{m,n}(\mathbb{Z}/m\mathbb{Z})\left[(\mathbb{Z}/n\mathbb{Z})^2\right]$$

may be regarded as a $\hat{\mathbb{Z}}$-valued measure $d\mathcal{E}_\sigma$ on the profinite space $\hat{\mathbb{Z}}^2$. Write x, y to denote the projection map $\hat{\mathbb{Z}}^2 \to \hat{\mathbb{Z}}$ to the first, second component respectively. Then, we can compute the moment integral $\int_{\hat{\mathbb{Z}}^2} x^i y^j d\mathcal{E}_\sigma(x,y)$ by letting $N \to \times\infty$ in the above congruence (8.2):

Theorem 8.3. *For $\sigma \in \mathrm{CSL}_2$, we have*

$$\int_{\hat{\mathbb{Z}}^2} x^i y^j d\mathcal{E}_\sigma(x,y) = i!\, j!\, \widehat{\psi_{ij}}(\sigma) \qquad (i,j \geq 0).$$

If we restrict the above argument to prime power levels $N = l^m$, then moment sums of the LHS of Lemma 8.1 give the Taylor coefficients of $\mathcal{E}_\sigma^{(l)}$, while RHS gives the period polynomial of E_k. Then, in the exactly similar discussion to Theorem 8.3, we obtain the proof of Theorem 7.2.

We can also collect the congruence formulas (8.1) into the following form of identity

Corollary 8.4. *For $\sigma \in \mathrm{CSL}_2$, we have*

$$\int_{\hat{\mathbb{Z}}^2} (xY + yX)^{k-2} d\mathcal{E}_\sigma(x,y) = (k-2)!\, \widehat{\Psi_k}(\sigma)(X,Y)$$

in $\hat{\mathbb{Z}}[X,Y]_{\deg = k-2}$.

Proof. The left hand side equals

$$\sum_{\substack{i+j=k-2 \\ i,j\geq 0}} \binom{k-2}{i} \left[\int_{\hat{\mathbb{Z}}^2} x^i y^j d\mathcal{E}_\sigma(x,y)\right] X^j Y^i.$$

The corollary follows from the the above theorem and Definition 7.3. □

Special values at roots of unity.

Next, we consider special values of $\mathcal{E}_\sigma \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$ ($\sigma \in \mathrm{CSL}_2$) at pairs of roots of unity. Regard $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$ as the projective limit of $\hat{\mathbb{Z}}[(\mathbb{Z}/n\mathbb{Z})^2] = \hat{\mathbb{Z}}[x,y]/(x^n-1, y^n-1)$. Then, according to Theorem 4.1 and (6.3), the image $\mathcal{E}_{\sigma,N}$ of $\mathcal{E}_\sigma$ at the N-th layer can be written as

$$\mathcal{E}_{\sigma,N}(x,y) \equiv -\sum_{\substack{i,j=0 \\ (i,j)\neq(0,0)}}^{n-1} \left(\lim_{n\to\times\infty} \Phi^{(2)}_{(\frac{i}{N},\frac{j}{N})}({}^tA_n)\right) x^i y^j \mod (x^N-1,\ y^N-1), \tag{8.3}$$

where $\{A_n\}$ is any matrix sequence converging to $\sigma \in \mathrm{CSL}_2$ in the non-congruence topology. From this description, for each pair of $\zeta, \zeta' \in \mu_N$, the value $\mathcal{E}_\sigma(\zeta,\zeta') \in \hat{\mathbb{Z}} \times \mathbb{Q}(\zeta_N)$ is well defined to be $\mathcal{E}_{\sigma,N}(\zeta,\zeta')$.

Now, the division values of the Weierstrass $\wp$-function $\wp\left(\frac{a\tau+b}{N};\tau,1\right)$ for a fixed pair $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{\mathbf{0}\}$ is a holomorphic modular form in $\tau \in \mathfrak{H}$ of weight 2 for $\Gamma(N)$. If $A \in \Gamma(N)$, then the period integral

$$\frac{1}{(2\pi i)^2}\int_z^{Az} \wp\left(\frac{a\tau+b}{N};\tau,1\right) d\tau$$

is independent of $z \in \mathfrak{H}$ or of an integral path $z \rightsquigarrow Az$. This integral is known to be valued in $\mathbb{Q}(\zeta_N)$. (Moreover if $A \in \Gamma(12N^2)$ then the value is in $\mathbb{Z}[\zeta_N]$.)

Proposition 8.5. [iv)] *Let $\{A_n \in \Gamma(n)\}_n$ be a matrix sequence converging to $\sigma \in \mathrm{CSL}_2$ in the non-congruence topology. Then, for $(s,t) \in (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{\mathbf{0}\}$,*

$$\mathcal{E}_\sigma(\zeta_N^s, \zeta_N^t) = \lim_{N|n\to\times\infty} \frac{1}{(2\pi i)^2} \int_z^{({}^tA_n^{-1})z} \wp\left(\frac{t\tau - s}{N}; \tau, 1\right) d\tau \qquad \big(\in \hat{\mathbb{Z}} \otimes \mathbb{Q}(\zeta_N)\big).$$

Here, $\lim_{N|n\to\times\infty}$ means the limit when n runs multiplicatively over the integers that are multiples of N.

Proof. By (8.3), we have

$$\mathcal{E}_\sigma(\zeta_N^s, \zeta_N^t) = \sum_{\substack{(i,j)\in(\mathbb{Z}/N\mathbb{Z})^2 \\ (i,j)\neq \mathbf{0}}} e_{\sigma,N}(i,j)\zeta_N^{si+tj} = -\lim_{n\to\times\infty} \sum_{\substack{i,j=0 \\ (i,j)\neq(0,0)}}^{N-1} \Phi^{(2)}_{(\frac{i}{N},\frac{j}{N})}({}^tA_n)\zeta_N^{si+tj}.$$

Recall also that $\Phi^{(2)}_{\mathbf{x}}({}^tA^{-1}) = -\Phi^{(2)}_{\mathbf{x}}({}^tA)$ for $A \in \Gamma(N)$ and $\mathbf{x} \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$. Then, comparing the periods of both sides of the classical identity (cf. [Sch, VII §3.2]):

$$\sum_{\substack{(i,j)\in(\mathbb{Z}/N\mathbb{Z})^2 \\ (i,j)\neq(0,0)}} \zeta_N^{si+tj} E_2^{(\frac{i}{N},\frac{j}{N})}(\tau) = \frac{1}{(2\pi i)^2} \wp\left(\frac{t\tau - s}{N}; \tau, 1\right),$$

we settle the proof of the assertion. □

9. **Braid groups and Weierstrass equation**

Our Eisenstein representation $\mathcal{E} : \mathrm{CSL}_2 \to \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$ can be constructed from braid groups in a purely group theoretical way as follows. Consider the Artin braid group with 4 strands

$$B_4 := \langle \tau_1, \tau_2, \tau_3 \mid \tau_1\tau_3 = \tau_3\tau_1,\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}\ (i=1,2)\rangle,$$

and regard the subgroup generated by τ_1, τ_2 as the braid group B_3 with three strands. There is also a rank 2 free subgroup F_2 in B_4 freely generated by $x_1 := \tau_1^{-1}\tau_3\tau_2\tau_1\tau_3^{-1}\tau_2^{-1}$ and $x_2 := \tau_1\tau_3^{-1}$. If we put $z := (\tau_1\tau_2)^6(\tau_1\tau_2\tau_3)^{-4} \in F_2$, then $x_1x_2x_1^{-1}x_2^{-1}z = 1$ holds, so that we may identify

$$F_2 \cong \Pi = \langle x_1, x_2, z \mid [x_1, x_2]z = 1\rangle$$

as the fundamental group of a once-punctured elliptic curve. These subgroups B_3, Π of B_4 give the semi-direct decomposition $B_4 = \Pi \rtimes B_3$, in which the defining conjugate action $\varphi : B_3 \to \mathrm{Aut}(\Pi)$ is given by

$$\varphi(\tau_1) : \begin{cases} x_1 \mapsto x_1x_2^{-1}, \\ x_2 \mapsto x_2; \end{cases} \qquad \varphi(\tau_2) : \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto x_2x_1. \end{cases} \tag{9.1}$$

Taking profinite completion $\hat{B}_4 = \hat{F}_2 \rtimes \hat{B}_3$, we obtain

$$\tilde{\varphi} : \hat{B}_3 \to \mathrm{Aut}(\hat{F}_2)$$

whose image turns out to fix $z \in \hat{\Pi}$. Let $\mathcal{C}B_3$ be the congruence kernel for $\hat{B}_3$, i.e., the kernel of the abelian reduction $\varphi^{\mathrm{ab}} : \hat{B}_3 \to \mathrm{SL}_2(\hat{\mathbb{Z}}) \subset \mathrm{Aut}(\Pi^{\mathrm{ab}})$. Then, by Proposition 3.1, we obtain the Eisenstein representation

$$\mathcal{E} : \mathcal{C}B_3 \longrightarrow \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]. \tag{9.2}$$

[iv)] The appearance of $\wp$ was hinted from a comment by M.Kaneko on the occasion of my talk [N02j] in Muroran, 2002.

Now, let us construct a canonical embedding $\mathrm{CSL}_2 \hookrightarrow \mathcal{C}B_3$ so that the restriction $\mathcal{E}|_{\mathrm{CSL}_2}$ of (9.2) gives the one considered in §6-8. In fact, we can canonically lift the commutator subgroup $\mathrm{SL}_2' := [\mathrm{SL}_2(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z})]$ into B_3 in the following way. Let $\rho_\Delta : B_3 \to \mathbb{Z}$ be the abelianization homomorphism sending both τ_1 and τ_2 to $-1 \in \mathbb{Z}$, and let $\rho : B_3 \to \mathrm{SL}_2(\mathbb{Z})$ be the matrix representation induced by (9.1); these two homomorphisms are characterized by

$$\rho_\Delta : B_3 \to \mathbb{Z} \begin{cases} \rho_\Delta(\tau_1) & = -1, \\ \rho_\Delta(\tau_2) & = -1; \end{cases} \qquad \rho : B_3 \to \mathrm{SL}_2(\mathbb{Z}) \begin{cases} \rho(\tau_1) & = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right), \\ \rho(\tau_2) & = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right). \end{cases}$$

Lemma 9.1. *Notations being as above, we have a canonical lift* $\mathrm{SL}_2' \hookrightarrow B_3$ *as well as* $\widehat{\mathrm{SL}_2'} \hookrightarrow \hat{B}_3$. *It then induces a canonical embedding* $\mathrm{CSL}_2 \hookrightarrow \mathcal{C}B_3$ *whose image is identified with the kernel of* $\hat{\rho}_\Delta : \mathcal{C}B_3 \twoheadrightarrow 12\hat{\mathbb{Z}}$.

Proof. Since $\mathrm{SL}_2(\mathbb{Z}) = B_3/\langle(\tau_1\tau_2)^6\rangle$, the abelianization of $\mathrm{SL}_2(\mathbb{Z})$ is induced from $\bar{\rho}_\Delta : B_3 \to \mathbb{Z}/12\mathbb{Z}$, i.e., SL_2' is isomorphic to the subquotient $\mathrm{Ker}(\bar{\rho}_\Delta)/\langle(\tau_1\tau_2)^6\rangle$. Consequently it follows that $\mathrm{Ker}(\rho_\Delta) \cong \mathrm{SL}_2'$, for ρ_Δ injectively maps $\langle(\tau_1\tau_2)^6\rangle$ onto $12\mathbb{Z}$. It is easy to see that the profinite completion functor preserves the above procedure faithfully. The restriction from $\widehat{\mathrm{SL}_2'}$ to CSL_2 gives the last assertion. □

The geometric origin of $\mathcal{E} : \mathrm{CSL}_2 \to \hat{\mathbb{Z}}[\hat{\mathbb{Z}}^2]]$ is the universal family of affine Weierstrass elliptic curves $E \setminus \{O\} := \{Y^2 = 4X^3 - g_2X - g_3\}$ over the space of coefficients $S := \{(g_2, g_3) | \Delta = g_2^3 - 27g_3^2 \neq 0\}$. It turns out that $\pi_1(E \setminus \{O\})$ (resp. $\pi_1(S)$) is an extension of $G_\mathbb{Q}$ by $\hat{B}_4$ (resp. $\hat{B}_3$) and that suitably chosen paths τ_1, τ_2, τ_3 with $\mathbb{Q}$-rational tangential basepoints give the monodromy action (9.1) and the Galois actions in the form

$$\begin{cases} x_1 & \mapsto z^{\frac{1-\chi(\sigma)}{2}} f_\sigma(x_1x_2x_1^{-1}, z)x_1 f_\sigma(x_2^{-1}, z)^{-1}, \\ x_2 & \mapsto f_\sigma(x_2^{-1}, z)x_2^{\chi(\sigma)} f_\sigma(x_2^{-1}, z)^{-1}, \\ z & \mapsto z^{\chi(\sigma)} \end{cases} \tag{9.3}$$

for $\sigma \in G_\mathbb{Q}$, where $(\chi(\sigma), f_\sigma) \in \hat{\mathbb{Z}}^\times \times \hat{F}_2$ denotes the standard image in the Grothendieck-Teichmüller group $\widehat{GT}$ ([N99]; see also [N13] §5). At this stage, we would become inclined to reconstruct a whole view on Eisenstein invariants in spirit of anabelian geometry around fundamental pieces $M_{0,n}$ $(n = 3, 4, 5)$, $M_{1,n}$ $(n = 1, 2)$.

As a sequel motivated by the above viewpoint, we later posed in [N13] a certain series of monodromy invariants in the form

$$\mathbb{E}_m : \pi_1(S) \times \hat{\mathbb{Z}}^2 \longrightarrow \hat{\mathbb{Z}} \quad (m \in \mathbb{N})$$

associated to any family of elliptic curves $E \setminus O$ over S in the general setting of §3. This invariant series $\{\mathbb{E}_m\}_{m\in\mathbb{N}}$ turned out to recover $\mathcal{E} : \pi_1(S_\infty) \to \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$ as well as to enjoy some elementary congruence properties investigated in [N12]. Based on those results in loc. cit., we can obtain a good series of extensions of $\mathcal{E}$ to finite levels: $\mathbb{E}_{m\rceil M} : \pi_1(S_{mN}) \to (\mathbb{Z}/M\mathbb{Z})[(\mathbb{Z}/m\mathbb{Z})^2]$ for $m, M \in \mathbb{N}$ (where $N := M$, $2M$ according as $2 \nmid M$, $2|M$ respectively) that should encode highly arithmetic information on Eisenstein quotients of modular curves. We will discuss some more progress in our subsequent work [N16].

References

[A] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Math. **41**, 1976.

[AMO] M. Asada, M. Matsumoto, T. Oda *Local monodromy on the fundamental groups of algebraic curves along a degenerate stable curve*, J. Pure and Appl. Alg. **103** (1995), 235–283.

[AN95] M. Asada, H. Nakamura, *On the graded quotient modules of mapping class groups of surfaces*, Israel. J. Math. **90** (1995) 93–113.

[BK10] K.Bannai, S.Kobayashi, Algebraic theta functions and the p-adic interpolation of Eisenstein-Kronecker numbers, Duke Math. J. **153** (2010), 229–295.

[BS] S. Baumard, L. Schneps, *On the derivation representation of the fundamental Lie algebra of mixed elliptic motives*, Preprint arXiv:1510.05549.

[BL94] A.Beilinson, A.Levin, *The elliptic polylogarithm*, Proc. Symp. Pure Math. **55** (1994), Part 2, 123–190.

[Bl84] S. Bloch, *letter to P.Deligne*, 1984.

[Br] F. Brown, Zeta elements in depth 3 and the fundamental Lie algebra of a punctured elliptic curve, Preprint arXiv:1504.04737.

[CKV13] J.Conant, M.Kassabov, K.Vogtmann, *Hairy graphs and the unstable homology of $Mod(g,s)$, $Out(F_n)$ and $Aut(F_n)$*, Journal of Topology, **6** (2013), 119–153.

[CM06] A.Connes, H.Moscovici, *Transgression of the Godbillon-Vey class and Rademacher functions*, in "Noncommutative Geometry and Number Theory" (C.Consani, M.Marcolli eds.) Aspects of Mathematics (Vieweg) **E37** (2006), 79–107.

[D] R. Dedekind, *Erläuterungen zu zwei Fragmenten von Riemann*, in "Riemann's Gesammelte Math. Werke, 2nd edition", 1892, 466–472.

[ES14] N.Enomoto, T.Satoh, *New series in the Johnson cokernels of the mapping class groups of surfaces*, Algebraic & Geometric Topology, **14** (2014), 627–669.

[E14] B. Enriquez, *Elliptic associators*, Selecta Math. New Series **20** (2014), 491-584.

[GKZ06] H. Gangl, M. Kaneko, D. Zagier, *Double zeta values and modular forms*, in "Automorphic forms and zeta functions" (S. Böcherer et. al. eds.) , 71–106, 2006, World Scientific.

[Go98] A.Goncharov, *Mixed elliptic motives*, London Math. Soc. Lect. Note Ser. **254** (1998), 147–221.

[G80] L. J. Goldstein, *Zeta functions and Eichler integrals*, Acta Arith. **36** (1980), 229–256.

[Gr1] A. Grothendieck, *La Longue Marche à travers la théorie de Galois*, partially available at http://www.grothendieckcircle.org

[Gr2] A. Grothendieck, *Esquisse d'un Programme 1984*, in "Geometric Galois Actions, I" (P.Lochak, L.Schneps eds.), London Math. Soc. Lect. Note Ser. **242** (1997), 5–48.

[H83] K. Haberland, *Perioden von Modulformen einer Variabler und Gruppencohomologie, I*, Math. Nachr. **112** (1983), 245–282.

[H] R. Hain, *The Hodge-de Rham Theory of Modular Groups*, arXiv:1403.6443 (to appear in the Vancouver Hodge theory volume).

[HM] R.Hain, M.Matsumoto, Universal mixed elliptic motives, Preprint arXiv:1512.03975

[HWZ] R. R. Hall, J. C. Wilson, D. Zagier, *Reciprocity formulae for general Dedekind-Rademacher sums*, Acta Arith. **73** (1995), 389–396.

[HK99] A.Huber, G.Kings, *Degeneration of l-adic Eisenstein classes and of the elliptic polylog*, Invent. Math. **135** (1999), 545–594.

[I86] Y. Ihara, *Profinite braid groups, Galois representations, and complex multiplications*, Ann. of Math. **123** (1986), 43–106.

[I90] Y. Ihara, *Braids, Galois groups and some arithmetic functions*, Proc. ICM, Kyoto (1990), 99–120.

[I99] Y. Ihara, *On beta and gamma functions associated with the Grothendieck-Teichmüller group*, in "Aspects of Galois Theory" (H.Völklein, et. al eds.) London Math. Soc. Lect. Note Ser. **256** (1999), 144–179.

[KK] N.Kawazumi, Y.Kuno, *The Goldman-Turaev Lie bialgebras and the Johnson homomorphisms*, Preprint arXiv:1304.1885, to appear in "Handbook of Teichmuller theory V" (A. Papadopoulos ed.).

[KM] R. Kirby, P. Melvin, *Dedekind sums, μ-invariants and the signature cocycle*, Math. Ann. **299** (1994), 231–267.

[KL81] D. Kubert, S. Lang, *Modular Units*, Springer 1981.

[Lo12] P. Lochak, *Results and conjectures in profinite Teichmüller theory*, in "Galois-Teichmüller theory and arithmetic geometry" (H.Nakamura, F.Pop, L.Schneps, A.Tamagawa eds.), Adv. Stud. Pure Math., **63** (2012), 263–335.

[Lu82] A. Lubotzky, *Free quotients and the congruence kernel of SL_2*, J. of Algebra **77** (1982), 411–418.

[Ma06] Y.Manin, *Iterated integrals of modular forms and noncommutative modular symbols*, in "Algebraic Geometry and Number Theory" (V. Ginzburg ed.), Progr. Math. **253** (2006), 565–597, Birkhauser.

[Mat13] M. Matsumoto, *Introduction to arithmetic mapping class groups*, in "Moduli Spaces of Riemann Surfaces", IAS/Park City Mathematical Series, Vol. 20, 321356, Amer. Math. Soc., 2013.

[Mo02] Y. Morimoto, *Arithmetic behavior of functions related with generalized Dedekind sums* (in Japanese), Maser thesis at Tokyo Metropolitan Univ. January 2002.

[MSS] S. Morita, T. Sakasai, M. Suzuki, *Structures of symplectic invariant Lie subalgebras of symplectic derivation Lie algebras*, Preprint arXiv:1404.3351.

[N95] H. Nakamura, *On exterior Galois representations associated with open elliptic curves*, J. Math. Sci. Univ. Tokyo **2** (1995), 197–231.

[N99] H. Nakamura, *Tangential base points and Eisenstein power series*, in "Aspects of Galois Theory" (H. Völkein et. al. eds.) London Math. Soc. Lect. Note Ser. **256** (1999), 202–217.

[N02r] H. Nakamura, *On exterior monodromy representation associated with elliptic curves and a certain Eisenstein measure function (in Japanese)*, in "Study on automorphic forms and related Dirichlet series" (H.Katsurada ed.), RIMS Kokyuroku **1281** (2002), 176–183.

[N02j] H. Nakamura, *On Magnus representation arising from elliptic curves and Eisenstein power series (in Japanese)*, in "Report Collection of 47th Algebra Symposium in 2002" (A.Tamagawa, S.Yoshihara eds.), 143-149.

[N03] H. Nakamura, *Generalized Rademacher functions and some congruence properties* in "Galois theory and Modular forms" (K.Hashimoto, et. al eds.) Developments in Math. **11** (2003), 375–394.

[N12] H. Nakamura, *Some congruence properties of Eisenstein invariants associated to elliptic curves*, in "Galois-Teichmüller theory and arithmetic geometry", (H.Nakamura, F.Pop, L.Schneps, A.Tamagawa eds.), Advanced Studies in Pure Math. **63** (2012), 813–832.

[N13] H. Nakamura, *On arithmetic monodromy representations of Eisenstein type in fundamental groups of once punctured elliptic curves*, Publ. RIMS, Kyoto University, **49** (2013), 413–496.

[N16] H. Nakamura, *Variations of Eisenstein invariants for elliptic actions on a free profinite group*, preprint (to appear).

[NTM] H. Nakamura, A. Tamagawa, S. Mochizuki, *On Grothendieck conjecture on the fundamental groups of algebraic curves*, Sugaku Expositions **14** (2001), 31–53.

[NT] H. Nakamura, H. Tsunogai, *Atlas of pro-l mapping class groups and related topics*, file collection since 1992.

[Od97] T. Oda, *Etale homotopy type of the moduli spaces of algebraic curves*, in "Geometric Galois Actions, I" (P.Lochak, L.Schneps eds.), London Math. Soc. Lect. Note Ser. **242** (1997), 85–96.

[S97] L. Schneps, *Grothendieck's "Long March through Galois theory"*, in "Geometric Galois Actions, I" (P.Lochak, L.Schneps eds.), London Math. Soc. Lect. Note Ser. **242** (1997), 59–66.

[Sch] B. Schoeneberg, *Elliptic Modular Functions*, Springer 1974.

[Scz] R. Sczech, *Eisenstein cocycles for $GL_2\mathbb{Q}$ and values of L-functions in real quadratic fields*, Comment. Math. Helvetici **67** (1992), 363–382.

[Se70] J.-P.Serre, *Le problèm des groupes de congruence pour SL_2*, Ann. of Math. **92** (1970), 489–527.

[St87] G. Stevens, *The Eisenstein measure and real quadratic fields*, in "Proceedings of the International Conference on Number Theory (Quebec, PQ, 1987)", (1987) 887–927, de Gruyter, Berlin.

[Tsu92] H. Tsunogai, *Tokyo GT-seminar note*, June 19, 1992.

[Tsu95a] H. Tsunogai, *On the automorphism group of a free pro-l meta-abelian group and an application to Galois representations*, Math. Nachr. **171** (1995), 315–324.

[Tsu95b] H. Tsunogai, *On some derivations of Lie algebras related to Galois representations*, Publ. RIMS, Kyoto Univ. **31** (1995), 113–134.

[Tsu03] H. Tsunogai, *The stable derivation algebras for higher genera*, Israel J. Math. **136** (2003), 221–250.

[Woh64] K. Wohlfahrt, *An extension of F.Klein's level concept*, Illinois J. Math., **8** (1964), 529–535.

[Woj04] Z.Wojtkowiak, On a torsor of paths of an elliptic curve minus a point, J. Math. Sci. Univ. Tokyo, **11** (2004), 353–399.

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