# On distribution formulas for complex and $\ell$-adic polylogarithms

*Dedicated to the memory of Professor Jean-Claude Douai*

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Abstract. We study an $\ell$-adic Galois analogue of the distribution formulas for polylogarithms with special emphasis on path dependency and arithmetic behaviors. As a goal, we obtain a notion of certain universal Kummer-Heisenberg measures that enable interpolating the $\ell$-adic polylogarithmic distribution relations for all degrees.

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## 1. Introduction

One of the most important and useful functional equations of classical complex polylogarithms is a series of distribution relations

$$Li_k(z^n) = n^{k-1}\left(\sum_{i=0}^{n-1} Li_k(\zeta_n^i z)\right) \qquad (\zeta_n = e^{2\pi i/n}). \tag{1}$$

J. Milnor [10, (7), (32)] says that a function $\mathscr{L}_s(z)$ has (multiplicative) Kubert identities of degree $s \in \mathbb{C}$, if it satisfies

$$\mathscr{L}_s(z) = n^{s-1}\sum_{w^n=z} \mathscr{L}_s(w) \tag{2}$$

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for every positive integer $n$. The aforementioned classical identity (1) for $Li_k(z)$ is, of course, a typical example of Kubert identity of degree $k$, assuming, however, correct choice of branches of the multivalued function $Li_k$ in all terms of the identity. To avoid the ambiguity of branch choice, we would rather consider $\mathscr{L}_s(z)$ as a function $\mathscr{L}_s(z;\gamma)$ of paths $\gamma$ on $\mathbf{P}^1 - \{0,1,\infty\}$ from the unit vector $\overrightarrow{01}$ to $z$. The main aim of this paper is to study generalizations of the above distribution relation for multiple polylogarithms and their $\ell$-adic Galois analogs ($\ell$-adic iterated integrals) with special emphasis on path dependency.

Let $K$ be a subfield of $\mathbb{C}$ with the algebraic closure $\overline{K} \subset \mathbb{C}$. The *$\ell$-adic polylogarithmic characters*

$$\tilde{\chi}_k^z : G_K \to \mathbb{Z}_\ell \qquad (k = 1, 2, \dots)$$

are introduced in [12] as $\mathbb{Z}_\ell$-valued 1-cochains on the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ for any given path $\gamma$ from $\overrightarrow{01}$ to a $K$-rational point $z$ on $\mathbf{P}^1 - \{0,1,\infty\}$. Our study in [NW2] showed that $\tilde{\chi}_k^z : G_K \to \mathbb{Z}_\ell$ behave nicely as $\ell$-adic analogues of the classical polylogarithms $Li_k(z)$. The $\ell$-adic polylogarithms and $\ell$-adic iterated integrals are $\mathbb{Q}_\ell$-valued variants (and generalizations) of the above 1-cochains $\tilde{\chi}_k^z : G_K \to \mathbb{Z}_\ell$. (See §2 and §3 for their precise definitions.) We will give a geometrical proof of distribution relations for classical multiple polylogarithms and their $\ell$-adic analogues in considerable generality. In particular, we will obtain several versions of Kubert identities with explicit path systems for:

- classical multiple polylogarithms (Theorem 4),
- $\ell$-adic iterated integrals (Proposition 6, Theorem 17),
- $\ell$-adic polylogarithms and polylogarithmic characters (Theorem 19, Corollary 21).

The polylogarithm is interpreted as a certain coefficient of an extension of the Tate module by the logarithm sheaf arising from the fundamental group of $V_1 := \mathbf{P}^1 - \{0,1,\infty\}$. The motivic construction dates back to the fundamental work of Beilinson-Deligne [1], Huber-Wildeshaus [7] (see also [5] §6 and references therein for more recent generalizations). In this article, we mainly work on the $\ell$-adic realization which forms a $\mathbb{Z}_\ell$- or $\mathbb{Q}_\ell$-valued 1-cochain on the Galois group $G_K$. In the collaboration [3] of the last author with J.-C.Douai, it was shown that certain linear combinations of $\ell$-adic polylogarithms at various points give rise to 1-cocycles on $G_K$, which lead to an $\ell$-adic version of Zagier's conjecture. See also Remark 9 and [11] §3.2

We will intensively make use of a system of simple cyclic covers $V_n := \mathbf{P}^1 - \{0, \mu_n, \infty\}$ over $V_1 = \mathbf{P}^1 - \{0,1,\infty\}$, where $\mu_n$ is the group of $n$-th roots of unity $\{1, \zeta_n, \dots, \zeta_n^{n-1}\}$ ($\zeta_n := e^{2\pi i/n}$), and $\{0, \mu_n, \infty\}$ denotes $\{0,\infty\} \cup \mu_n$ by abuse of notation. We consider the family of cyclic coverings $V_n \to V_1$ and open immersions $V_n \hookrightarrow V_1$ together with induced relations between their fundamental group(oid)s. Our basic idea is to understand the distribution relations of polylogarithms as the "trace property" of relevant coefficients ("iterated integrals") arising in those fundamental groups.

As observed in [13] and will be seen in §3 below, unlike in the classical complex case, there generally occur lower degree terms in $\ell$-adic case when a distribution relation is naively derived. This problem prevents artless approaches to $\ell$-adic Kubert identities i.e., distribution formulas of homogeneous form (with no lower degree terms). Our line of studies in §2-6 will lead us to understand why and how to make use of $\mathbb{Q}_\ell$-paths ($\ell$-adic paths with 'denominators') to eliminate such lower degree terms dramatically. Consequently in §7, as a primary goal of this paper, we arrive at introducing a generalization of the Kummer-Heisenberg measure of [12] so as to interpolate those $\ell$-adic distribution relations of polylogarithms for all degrees.

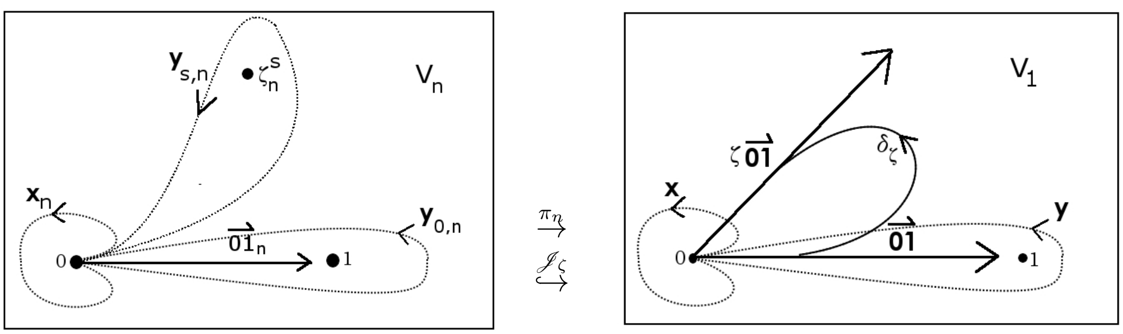
**Remark 1.** We have already studied in [20] and [19] the distribution relations for those $\ell$-adic polylogarithms under certain restricted assumptions (see [20, Prop. 11.1.4] for $\ell$-adic dilogarithms, [20, Cor. 11.2.2, 11.2.4] for $\ell$-adic polylogarithms on restricted Galois groups, and [19, Th. 2.1] for $\ell$-adic polylogarithmic characters with $\ell \nmid n$).

**Basic setup, notations and convention**:

Below, we understand that all algebraic varieties are geometrically connected over a fixed field $K \subset \mathbb{C}$ and that all morphisms between them are $K$-morphisms. A path on a $K$-variety $V$ is a topological path on $V(\mathbb{C})$ or an étale path on $V \otimes \overline{K}$ whose distinction will be obvious in contexts. The notation $\gamma : x \rightsquigarrow y$ means a path from $x$ to $y$, and write $\gamma_1\gamma_2$ for the composed path tracing $\gamma_1$ first and then $\gamma_2$ afterwards. We write $\chi : G_K \to \mathbb{Z}_\ell^\times$ for the $\ell$-adic cyclotomic character ($\ell$: a fixed prime). The Bernoulli polynomials $B_k(T)$ $(k = 0, 1, \dots)$ are defined by the generating function $\frac{ze^{Tz}}{e^z-1} = \sum_{k=0}^{\infty} B_k(T)\frac{z^k}{k!}$, and the Bernoulli numbers are set as $B_k := B_k(0)$. For a vector space $H$, we write $H^*$ for its dual vector space.

Assume $K \supset \mu_n$. We shall be concerned with two kinds of standard morphisms defined by

$$\begin{cases} \mathscr{I}_\zeta : V_n \hookrightarrow V_1 & \mathscr{I}_\zeta(z) = \zeta z \qquad (\zeta \in \mu_n); \\ \pi_n : V_n \to V_1 & \pi_n(z) = z^n. \end{cases}$$




As easily seen, each $\mathscr{I}_\zeta$ is an open immersion, while $\pi_n$ is an $n$-cyclic covering. Write $\overrightarrow{01}_n$ for the tangential base point represented by the unit tangent vector on $V_n$. Since $\mathscr{I}_1 : V_n \hookrightarrow V_1$ maps $\overrightarrow{01}_n$ to $\overrightarrow{01}_1$(often written just $\overrightarrow{01}$), it induces the surjection homomorphism

$$\pi_1(V_n, \overrightarrow{01}_n) \twoheadrightarrow \pi_1(V_1, \overrightarrow{01}). \tag{3}$$

On the other hand, although the image $\pi_n(\overrightarrow{01}_n)$ is not exactly the same as $\overrightarrow{01}_1$ as a tangent vector, they give the same tangential base point on $V_1$ in the sense that they give equivalent fiber functors on the Galois category of finite étale covers of $V_1$. Henceforth, for simplicity, we shall regard $\pi_n(\overrightarrow{01}_n) = \overrightarrow{01}_1 = \overrightarrow{01}$, and regard $\pi_1(V_n, \overrightarrow{01}_n)$ as a subgroup of $\pi_1(V_1, \overrightarrow{01})$ by the homomorphism

$$\pi_1(V_n, \overrightarrow{01}_n) \hookrightarrow \pi_1(V_1, \overrightarrow{01}) \tag{4}$$

induced from $\pi_n$.

For each $\zeta \in \mu_n$, introduce a path $\delta_\zeta : \overrightarrow{01} \rightsquigarrow \zeta\overrightarrow{01} = \mathscr{J}_\zeta(\overrightarrow{01}_n)$ on $V_1$ to be the arc from $\overrightarrow{01}$ to $\zeta\overrightarrow{01}$ anti-clockwise oriented. Using the path $\delta_\zeta$, we obtain the identification $\pi_1(V_1, \overrightarrow{01}) \xrightarrow{\sim} \pi_1(V_1, \zeta\overrightarrow{01})$.

Let $x$, $y$ be standard loops based at $\overrightarrow{01}_1$ on $V_1 = \mathbf{P}^1 - \{0, 1, \infty\}$ turning around the punctures 0, 1 once anticlockwise respectively. We introduce loops $x_n$, $y_{0,n} \dots, y_{n-1,n}$ based at $\overrightarrow{01}_n$ on $V_n$ characterized by:

$$\begin{cases} x_n & := \pi_n^{-1}(x^n) = \mathscr{J}_1^{-1}(x), \\ y_{s,n} & := \mathscr{J}_1^{-1}(\delta_\zeta) \cdot \mathscr{J}_{\zeta^{-1}}^{-1}(y) \cdot \mathscr{J}_1^{-1}(\delta_\zeta)^{-1} \qquad (\zeta = e^{\frac{2\pi i s}{n}},\ s = 0, \dots, n-1) \end{cases}$$

so that $x_n$, $y_{0,n}, \dots, y_{n-1,n}$ freely generate $\pi_1(V(\mathbb{C}), \overrightarrow{01}_n)$.

Note that, in view of the above inclusion (4), we have the identifications:

$$x_n = x^n, \quad y_{s,n} = x^s y x^{-s}. \tag{5}$$

## 2. Complex distribution relations

For $n = 1, 2, \dots$, let

$$\omega(V_n) := \frac{dz}{z} \otimes \left(\frac{dz}{z}\right)^* + \sum_{i=0}^{n-1} \frac{dz}{z - \zeta_n^i} \otimes \left(\frac{dz}{z - \zeta_n^i}\right)^* \in \Omega^1_{\log}(V_n) \otimes \Omega^1_{\log}(V_n)^*$$

be the canonical one-form on $V_n$. Traditionally, we set

$$X_n := \left(\frac{dz}{z}\right)^* \quad \text{and} \quad Y_{i,n} := \left(\frac{dz}{z - \zeta_n^i}\right)^*.$$

Let $\mathscr{R}_n := \mathbb{C}\langle\langle X_n, Y_{i,n} \mid 0 \le i < n \rangle\rangle$ be the non-commutative algebra of formal power series over $\mathbb{C}$ generated by non-commuting variables $X_n$ and $Y_{i,n}$ $(0 \le i < n)$. Consider the trivial bundle

$$\mathscr{R}_n \times V_n \to V_n$$

equipped with the (flat) connection $\nabla$: $\Phi \mapsto d\Phi - \Phi\,\omega(V_n)$ for smooth functions $\Phi : V_n \to \mathscr{R}_n$. For a piecewise smooth path $\gamma : [0, 1] \to V_n$ from $\gamma(0) = a$ to $\gamma(1) = z$, let $\Phi : [0, 1] \to \mathscr{R}_n$ be the solution to the differential equation $d\Phi = \Phi\,\omega(V_n)$ pulled back on $\gamma$ with $\Phi(0) = 1$ and define $\Lambda(a \overset{\gamma}{\rightsquigarrow} z) \in \mathscr{R}_n$ to be $\Phi(1)$. (Cf. [6] §2, [16] §1; we here follow Hain's path convention in loc. cit.) In the case $a$ being the tangential base point $\overrightarrow{01}$, we interpret $\Lambda(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z)$ in a suitable manner introduced in [2], [17, §3.2].

Let $\mathsf{M}_n$ be the set of all monomials (words) in $X_n$ and $Y_{i,n}$ $(0 \le i < n)$. Then, we can expand

$$\Lambda(a \overset{\gamma}{\rightsquigarrow} z) = 1 + \sum_{w \in \mathsf{M}_n} \mathsf{Li}_w(a \overset{\gamma}{\rightsquigarrow} z) \cdot w \tag{6}$$

in $\mathscr{R}_n$. If $w = X_n^{a_0} Y_{i_1,n} X_n^{a_1} \cdots Y_{i_k,n} X_n^{a_k}$, then

$$\mathsf{Li}_w(a \overset{\gamma}{\rightsquigarrow} z) = \int_{a,\gamma}^{z} \underbrace{\frac{dz}{z} \cdots \frac{dz}{z}}_{a_0} \cdot \frac{dz}{z - \zeta_n^{i_1}} \cdots \frac{dz}{z - \zeta_n^{i_k}} \cdot \underbrace{\frac{dz}{z} \cdots \frac{dz}{z}}_{a_k}, \tag{7}$$

the iterated integral along $\gamma$.

**Definition 2.** For a word $w = X_n^{a_0} Y_{i_1,n} X_n^{a_1} \cdots Y_{i_k,n} X_n^{a_k}$, we define its $X$-weight by

$$\mathsf{wt}_X(w) = a_0 + \cdots + a_k.$$

Let the cyclic cover

$$\pi_{rn,r} : V_{rn} \longrightarrow V_r \tag{8}$$

be given by $\pi_{rn,r}(z) = z^n$. Then, we have

$$\Big[\mathrm{Id} \otimes (\pi_{rn,r})_*\Big](\omega(V_{rn})) = \Big[(\pi_{rn,r})^* \otimes \mathrm{Id}\Big](\omega(V_r)).$$

This implies that the induced map from $(\pi_{rn,r})_*$ on complete tensor algebras (denoted by the same symbol):

$$\begin{array}{ccc} \hat{T}(\Omega^1_{\log}(V_{rn})^*) & \longrightarrow & \hat{T}(\Omega^1_{\log}(V_r)^*) \\ \| & & \| \\ \mathbb{C}\langle\langle X_{rn}, Y_{j,rn} \mid 0 \le j < rn \rangle\rangle & \longrightarrow & \mathbb{C}\langle\langle X_r, Y_{i,r} \mid 0 \le i < r \rangle\rangle \end{array}$$

preserves the associated power series:

$$(\pi_{rn,r})_* \left( \Lambda(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z) \right) = \Lambda(\overrightarrow{01} \overset{\pi_{rn,r}(\gamma)}{\rightsquigarrow} z^n). \tag{9}$$

Note that

$$\begin{cases} (\pi_{rn,r})_*(X_{rn}) = nX_r, \\ (\pi_{rn,r})_*(Y_{j,rn}) = Y_{i,r} \quad (i \equiv j \bmod r). \end{cases} \tag{10}$$

**Definition 3.** For $w \in \mathsf{M}_{rn}$, we mean by $(w \bmod r)$ the word in $\mathsf{M}_r$ obtained by replacing each letter $X_{rn}$, $Y_{j,rn}$ $(0 \le j < rn)$ appearing in $w$ by $X_r$, $Y_{i,r}$ (where $i$ is an integer with $0 \le i < r$, $i \equiv j \bmod r$) respectively. If $r$ is a common divisor of $m$ and $n$, $w \in \mathsf{M}_m$, $w' \in \mathsf{M}_n$ and $(w \bmod r) = (w' \bmod r)$, then we shall write

$$w \equiv w' \mod r.$$

**Theorem 4.** *Notations being as above, let $\gamma$ be a path on $V_{rn}$ from $\overrightarrow{01}$ to a point $z$. Then, for any word $w \in \mathsf{M}_r$, we have the distribution relation*

$$\mathsf{Li}_w(\overrightarrow{01} \overset{\pi_{rn,r}(\gamma)}{\rightsquigarrow} z^n) = n^{\mathsf{wt}_X(w)} \sum_{\substack{u \in \mathsf{M}_{rn} \\ u \equiv w \bmod r}} \mathsf{Li}_u(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z).$$

*Proof.* The theorem follows immediately from the formula (9): Write $\Lambda(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z) = 1 + \sum_{u \in \mathsf{M}_{rn}} \mathsf{Li}_u(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z) \cdot u$ in $\mathscr{R}_{rn}$. Applying (9), we obtain

$$1 + \sum_{w \in \mathsf{M}_r} \mathsf{Li}_w(\overrightarrow{01} \overset{\pi_{rn,r}(\gamma)}{\rightsquigarrow} z^n) \cdot w = 1 + \sum_{u \in \mathsf{M}_{rn}} \mathsf{Li}_u(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z) \cdot (\pi_{rn,r})_*(u).$$

Given any specific $w \in \mathsf{M}_r$ in LHS, collect from RHS all the coefficients of $(\pi_{rn,r})_*(u)$ for those $u$ satisfying $(u \bmod r) = w$. Noting that $(\pi_{rn,r})_*(u) = n^{\mathsf{wt}_X(u)} w = n^{\mathsf{wt}_X(w)} w$ for them, we settle the assertion of the theorem. $\square$

The above theorem generalizes the distribution relation (1) for the classical polylogarithm $Li_k(z)_\gamma$ along the path $\gamma : \overrightarrow{01} \rightsquigarrow z$. Indeed, in the notation above, since $\frac{dz}{1-z} = -\frac{dz}{z-1}$, we may identify

$$Li_k(z)_\gamma = -\mathsf{Li}_{YX^{k-1}}(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z).$$

Applying the theorem to the special case $\pi_{n,1} : V_n \to V_1$, $w = YX^{k-1}$ where $Y = Y_{0,1}$, $X = X_1$, we obtain

$$\int_{\overrightarrow{01}, \pi_{n,1}(\gamma)}^{z^n} \frac{dz}{z-1} \cdot \underbrace{\frac{dz}{z} \cdots \frac{dz}{z}}_{k-1} = n^{k-1} \sum_{\zeta \in \mu_n} \int_{\overrightarrow{01}, \gamma}^{z} \frac{dz}{z-\zeta} \cdot \underbrace{\frac{dz}{z} \cdots \frac{dz}{z}}_{k-1}.$$

Each term of RHS turns out to be $Li_k(\zeta z)$ along the path $\delta_\zeta \cdot \mathscr{J}_\zeta(\gamma) : \overrightarrow{01} \rightsquigarrow \zeta\overrightarrow{01} \rightsquigarrow \zeta z$, after integrated by substitution $z \to \zeta z$. Noting that the integration here over $\delta_\zeta : \overrightarrow{01} \rightsquigarrow \zeta\overrightarrow{01}$ vanishes (cf. [17] §3), we obtain (1) with path system specified as follows:

$$Li_k(z^n)_{\pi_{n,1}(\gamma)} = n^{k-1} \sum_{\zeta \in \mu_n} Li_k(\zeta z)_{\delta_\zeta \cdot \mathscr{J}_\zeta(\gamma)}. \tag{11}$$

## 3. $\ell$-adic case (general)

We shall look at the $\ell$-adic analogue of the previous section by recalling the following construction which essentially dates back to [18]. Let $K \subset \mathbb{C}$ and consider

$$\pi_1^\ell(V_n \otimes \overline{K}, \overrightarrow{01}),$$

the pro-$\ell$ (completion of the étale) fundamental group of $V_n \otimes \overline{K}$. It is easy to see the loops $x_n, y_{1,n}, \ldots, y_{n-1,n}$ introduced in §1 form a free generator system of the pro-$\ell$ fundamental group. Consider the (multiplicative) Magnus embedding into the ring of non-commutative power series

$$\iota_{\mathbb{Q}_\ell} : \pi_1^\ell(V_n \otimes \overline{K}, \overrightarrow{01}) \longrightarrow \mathbb{Q}_\ell\langle\langle X_n, Y_{i,n} \mid 0 \le i \le n-1 \rangle\rangle \tag{12}$$

defined by $\iota_{\mathbb{Q}_\ell}(x_n) = \exp(X_n)$, $\iota_{\mathbb{Q}_\ell}(y_{i,n}) = \exp(Y_{i,n})$ (cf. [21, 15.1]). For simplicity, we often identify elements of $\pi_1^\ell(V_n \otimes \overline{K}, \overrightarrow{01})$ with their images by $\iota_{\mathbb{Q}_\ell}$. Let us write again $\mathsf{M}_n$ for the set of monomials in $X_n, Y_{i,n}$ ($i = 0, \ldots, n-1$) (although variables have different senses from the previous section where they were duals of differential forms). We shall also employ the usage '$\mathsf{wt}_X(w)$' and '$w \equiv w' \bmod r$' by following the same manners as Definitions 2 and 3.

Recall that we have a canonical Galois action $G_K$ on (étale) paths on $V_n \otimes \overline{K}$ with both ends at $K$-rational (tangential) points. Given a path $\gamma$ from such a point $a$ to a point $z \in V_n(K)$, we set, for any $\sigma \in G_K$,

$$\mathfrak{f}_\sigma^\gamma := \gamma \cdot \sigma(\gamma)^{-1} \in \pi_1^\ell(V_n \otimes \overline{K}, a), \tag{13}$$

where the RHS is understood to be the image in the pro-$\ell$ quotient. When $a = \overrightarrow{01}$, we expand $\mathfrak{f}_\sigma^\gamma \in \pi_1^\ell(V_n \otimes \overline{K}, \overrightarrow{01})$ in the form

$$\mathfrak{f}_\sigma^\gamma = 1 + \sum_{w \in \mathsf{M}_n} \mathsf{Li}_w(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z)(\sigma) \cdot w \tag{14}$$

in $\mathbb{Q}_\ell\langle\langle X_n, Y_{i,n} \mid 0 \le i \le n-1 \rangle\rangle$, and associating the coefficient

$$\mathsf{Li}_w(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z)(\sigma) := \mathsf{Coeff}_w(\mathfrak{f}_\sigma^\gamma)$$

of $w \in \mathsf{M}_n$ to $\sigma \in G_K$, we define the $\ell$-adic Galois 1-cochain

$$\mathsf{Li}_w(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z) \left(= \mathsf{Li}_w^{(\ell)}(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z)\right) : G_K \to \mathbb{Q}_\ell$$

for every monomial $w \in \mathsf{M}_n$. We call each $\mathsf{Li}_w(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z)$ the $\ell$-adic iterated integral associated to $w \in \mathsf{M}_n$ and to the path $\gamma$ on $V_n$.

**Remark 5.** The above naming '$\ell$-adic iterated integral' is intended to be an analog of the iterated integral appearing in the complex case (6), (7). They represent general coefficients of the associator in the Magnus expansion. Conceptually, the associator lies in the pro-unipotent hull of the fundamental group and the monodromy information encoded in the total set of them is equivalent to that encoded in the general coefficients with respect to any fixed Hall basis of the corresponding Lie algebra. This line of formulation was, in fact, taken up, e.g., in [18] §5. However for the purpose of pursuing the distribution formulas in the present paper, the simple form of trace properties (9), (10) along the cyclic coverings $\pi_{rn,r} : V_{rn} \longrightarrow V_r$ is most essential. This is why we start with Magnus expansions $\mathfrak{f}_\sigma^\gamma$ in $\mathbb{Q}_\ell\langle\langle X_n, Y_{i,n}\rangle\rangle_i$ rather than with Lie expansions of $\log \mathfrak{f}_\sigma^\gamma$ with respect to a Hall basis in $\mathrm{Lie}\langle\langle X_n, Y_{i,n}\rangle\rangle_i$. But we shall discuss their relations in the polylogarithmic part of $n = 1$ in §4.

Now, as in §2, let us consider the morphism $\pi_{rn,r} : V_{rn} \to V_r$ given by $\pi_{rn,r}(z) = z^n$ for $n, r > 0$, and let $\gamma$ be a path on $V_{rn}$ from $\overrightarrow{01}$ to a $K$-rational point $z$. By our construction, the $\ell$-adic analogue of the equality (9) holds, i.e., $\pi_{rn,r}$ preserves the $\ell$-adic associators:

$$(\pi_{rn,r})_*(\mathfrak{f}_\sigma^\gamma) = \mathfrak{f}_\sigma^{\pi_{rn,r}(\gamma)} \quad (\sigma \in G_K). \tag{15}$$

However, unlike the complex case (10), $\pi_{rn,r}$ does not preserve the expansion coefficients homogeneously, i.e., it maps as

$$\begin{cases} (\pi_{rn,r})_*(X_{rn}) &= nX_r, \\ (\pi_{rn,r})_*(Y_{j,rn}) &= \exp(kX_r) Y_{i,r} \exp(-kX_r) \qquad (j = i + kr,\ 0 \leq i < r). \end{cases} \tag{16}$$

*Proof of* (16). Note that the cyclic projections $\pi_{rn,r}$ identify $\{\pi_1(V_n)\}_n$ as a sequence of subgroups of $\pi_1(V_1)$ as in (4), and regard $x_{rn} = {x_r}^n = x^{rn}$, $y_{j,rn} = x^j y x^{-j} = (x^r)^k x^i y x^{-i} (x^r)^{-k} = x_r^k y_{i,r} x_r^{-k}$. Although $\pi_{rn,r}$ does not keep injectivity on the complete envelops, it does induce a functorial homomorphism on them. The formula follows then from $x_n = \exp(X_n)$, $y_{s,n} = \exp(Y_{s,n})$. □

This causes generally (lower degree) error terms to appear in distribution relations for $\ell$-adic iterated integrals.

Still, if we restrict ourselves to the words whose $X$-weights are zero, we have the following

**Proposition 6.** *Notations being as above, if $w \in \mathsf{M}_r$ is a word with $\mathsf{wt}_X(w) = 0$, i.e., of the form $w = Y_{i_k,r} \cdots Y_{i_1,r}$, then it holds that*

$$\mathsf{Li}_w(\overrightarrow{01} \overset{\pi_{rn,r}(\gamma)}{\rightsquigarrow} z^n)(\sigma) = \sum_{\substack{u \in \mathsf{M}_{rn} \\ u \equiv w \bmod r}} \mathsf{Li}_u(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z)(\sigma) \qquad (\sigma \in G_K).$$

*Proof.* In the expansion of $(\pi_{rn,r})_*(\mathfrak{f}_\sigma^\gamma) = \mathfrak{f}_\sigma^{\pi_{rn,r}(\gamma)}$, the contributions to the coefficient of $w$ come only from the first '$Y$-only' term of each $u \in \mathsf{M}_{rn}$ with $u \equiv w \bmod r$. The proposition follows from this observation. □

**Remark 7.** In the $\ell$-adic Galois case, the distribution relations of Proposition 6 are used in [23] to construct measures on $\mathbb{Z}_\ell^r$ which generalize the measure on $\mathbb{Z}_\ell$ in [12]. The general distribution formula analogous to Theorem 4 for arbitrary words in $\mathsf{M}_r$ hold only up to lower degree terms in the $\ell$-adic Galois case. More generally, any covering maps between smooth algebraic varieties will give some kind of distribution relations.

## 4. $\ell$-adic polylogarithms (review)

Henceforth, we shall closely look at the case of $\ell$-adic polylogarithm where $r=1$ and only those words $w \in \mathsf{M}_1$ involving $Y_{0,1}$ only once are concerned, in the setting of the previous section. For simplicity, we write $x := x_1$, $y := y_{0,1}$ and $X := \log(x)$, $Y := \log(y)$, and will be concerned with those coefficients of the words $YX^{k-1}$ of $\mathfrak{f}_\sigma^\gamma$.

Let us recall some basic facts from [12], [13]. We introduced, for any path $\gamma : \overrightarrow{01} \rightsquigarrow z$ on $V_1 = \mathbf{P}^1 - \{0,1,\infty\}$, the *$\ell$-adic polylogarithms*

$$\ell i_m(z,\gamma) : G_K \to \mathbb{Q}_\ell \tag{17}$$

(with regard to the fixed free generator system $\{x,y\}$ of $\pi_1^\ell(V_1 \otimes \overline{K}, \overrightarrow{01})$) to be the Lie expansion coefficients of the associator $\mathfrak{f}_\sigma^\gamma = \gamma \cdot \sigma(\gamma)^{-1}$ for $\sigma \in G_K$ modulo the ideal $I_Y$ of Lie monomials including $Y$ twice or more:

$$\log(\mathfrak{f}_\sigma^\gamma)^{-1} \equiv \rho_z(\sigma)X + \sum_{m=1}^{\infty} \ell i_m(z,\gamma)(\sigma)\mathrm{ad}(X)^{m-1}(Y) \mod I_Y. \tag{18}$$

Here, $\rho_z : G_K \to \mathbb{Z}_\ell(1)$ designates the Kummer 1-cocycle for power roots of $z$ along $\gamma$. Note, however, that the other coefficients $\ell i_m(z,\gamma)(\sigma) \in \mathbb{Q}_\ell$ are generally not valued in $\mathbb{Z}_\ell$ due to applications of log respectively to $x$, $y$ and $\mathfrak{f}_\sigma^\gamma \in \pi_1^\ell(V_1 \otimes \overline{K}, \overrightarrow{01})$. In fact, we can bound the denominators of $\ell i_m(z,\gamma)(\sigma)$ by relating them with more explicitly defined $\mathbb{Z}_\ell$-valued 1-cochains called the *$\ell$-adic polylogarithmic characters*

$$\tilde{\chi}_m^z (= \tilde{\chi}_m^{z,\gamma}) : G_K \to \mathbb{Z}_l \qquad (m \geq 1) \tag{19}$$

defined by the Kummer properties for $n \geq 1$:

$$\zeta_{\ell^n}^{\tilde{\chi}_m^z(\sigma)} = \sigma\left(\prod_{a=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{\chi(\sigma)^{-1}a} z^{1/\ell^n})^{\frac{a^{m-1}}{\ell^n}}\right) \Big/ \prod_{a=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{a+\rho_z(\sigma)} z^{1/\ell^n})^{\frac{a^{m-1}}{\ell^n}}, \tag{20}$$

where $(1-\zeta_{\ell^n}^\alpha z^{1/\ell^n})^{\frac{\beta}{\ell^n}}$ means the $\beta$-th power of a carefully chosen $\ell^n$-th root of $(1-\zeta_{\ell^n}^\alpha z^{1/\ell^n})$ along $\gamma$. It is shown in [12, p.293 Corollary] that, for each $\sigma \in G_K$, the $\ell$-adic polylogarithm $\ell i_m(z,\gamma)(\sigma) \in \mathbb{Q}_\ell$ can be expressed by the Kummer- and $\ell$-adic polylogarithmic characters $\rho_z(\sigma), \tilde{\chi}_m^z(\sigma) \in \mathbb{Z}_\ell$ as follows:

$$\ell i_m(z,\gamma)(\sigma) = (-1)^{m+1} \sum_{k=0}^{m-1} \frac{B_k}{k!}(-\rho_z(\sigma))^k \frac{\tilde{\chi}_{m-k}^z(\sigma)}{(m-k-1)!} \qquad (m \geq 1). \tag{21}$$

One has then the following relations among $\ell i_m(z,\gamma)(\sigma) \in \mathbb{Q}_\ell$ (17), $\tilde{\chi}_m^z(\sigma) \in \mathbb{Z}_\ell$ (19) and $\mathsf{Li}_{YX^{m-1}}(\overrightarrow{01} \overset{\gamma}{\rightsquigarrow} z)(\sigma) \in \mathbb{Q}_\ell$ (§3):

**Proposition 8.** *(i) Notations being as above, we have*

$$\tilde{\chi}_m^z(\sigma) = (-1)^{m+1}(m-1)! \sum_{k=1}^{m} \frac{\rho_z(\sigma)^{m-k}}{(m+1-k)!} \ell i_k(z,\gamma)(\sigma) \quad (m \geq 1).$$

*(ii) Moreover, the expansion of $\mathfrak{f}_\sigma^\gamma$ in $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ partly looks like*

$$\mathfrak{f}_\sigma^\gamma = 1+\sum_{i=1}^{\infty}\frac{(-\rho_z(\sigma))^i}{i!}X^i-\sum_{i=0}^{\infty}\frac{\tilde{\chi}_{i+1}^z(\sigma)}{i!}YX^i+\ldots(\textit{other terms}).$$

*In particular, we have*

$$\mathsf{Li}_{YX^{m-1}}(\overrightarrow{01}\overset{\gamma}{\rightsquigarrow}z)(\sigma)=-\frac{\tilde{\chi}_m^z(\sigma)}{(m-1)!}\quad(m\geq 1).$$

*Proof.* (i) follows immediately from inductively reversing the formula (21). (ii) also follows easily from discussions in [13, p.284-285]: Suppose $\mathfrak{f}_\sigma^\gamma$ has monomial expansion as

$$\mathfrak{f}_\sigma^\gamma = 1+\sum_{i=1}^{\infty}c_i\frac{X^i}{i!}-\sum_{i=0}^{\infty}d_{i+1}YX^i+\ldots(\text{other terms}).$$

First, from (18), we see that $\mathfrak{f}_\sigma^\gamma\equiv e^{cX}$ modulo $Y=0$ with a constant $c:=-\rho_z(\sigma)$, hence that $c_i=c^i$. Next, to look at the coefficients of monomials of the forms $X^i$, $YX^i$ ($i=0,1,2,\dots$) closely, we take reduction modulo the ideal $J_Y:=\langle XY,Y^2\rangle$ of $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$. Observe then the congruence:

$$\begin{aligned}\log(\mathfrak{f}_\sigma^\gamma)&\equiv(\mathfrak{f}_\sigma^\gamma-1)\left\{1-\frac{1}{2}(\mathfrak{f}_\sigma^\gamma-1)+\frac{1}{3}(\mathfrak{f}_\sigma^\gamma-1)^2-+\cdots\right\}\\&\equiv\left(-\sum_{i=0}^{\infty}d_{i+1}YX^i\right)\left\{1-\frac{1}{2}(e^{cX}-1)+\frac{1}{3}(e^{cX}-1)^2-+\cdots\right\}\\&\equiv\left(-\sum_{i=0}^{\infty}d_{i+1}YX^i\right)\left\{\sum_{k=0}^{\infty}\frac{B_k}{k!}c^kX^k\right\}\qquad(\mathrm{mod}\ J_Y)\end{aligned}$$

and find that the coefficient of $YX^{m-1}$ in $\log(\mathfrak{f}_\sigma^\gamma)$ is

$$-\sum_{k=0}^{m-1}\frac{B_k}{k!}c^kd_{m-k}\tag{$*$}$$

for $m\geq 1$. On the other hand, the formula (18) combined with (21) calculates the same coefficient, which is $(-1)^{m-1}$-multiple of that of $\mathrm{ad}(X)^{m-1}(Y)$, as to be

$$(-1)^{m-1}\ell i_m(z,\gamma)=\sum_{k=0}^{m-1}\frac{B_k}{k!}(-\rho_z(\sigma))^k\frac{\tilde{\chi}_{m-k}^z(\sigma)}{(m-k-1)!}\tag{$**$}$$

for $m\geq 1$. Comparing those $(*)$ and $(**)$ inductively on $m\geq 1$, we conclude our desired identities $d_{i+1}=-\tilde{\chi}_{i+1}(\sigma)/i!$ $(i\geq 0)$. $\square$

**Remark 9.** The $\ell$-adic polylogarithm was constructed as a certain lisse $\mathbb{Q}_\ell$-sheaf on $V_1=\mathbf{P}^1-\{0,1,\infty\}$ as in [1], [7], [5] and [22]. The fiber over a point $z\in V_1(K)$ forms a polylogarithmic quotient torsor of $\ell$-adic path classes from $\overrightarrow{01}$ to $z$. We have the $G_K$-action on the path space whose specific coefficients are the $\ell$-adic (Galois) polylogarithms in our sense (17), viz., realized as $\mathbb{Q}_\ell$-valued 1-cochains on $G_K$. See also, e.g., [11] §3 for a concise account from the viewpoint of non-abelian cohomology in a mixed Tate category.

Note that there are misprints in [13, p.284] where exponents $\circledast=2,3$ of $(e^{(\log z)X}-1)^{\circledast}$ should read $\circledast=1,2$ respectively in the 2nd and 3rd terms in line $-11$.

## 5. Distribution relations for $\tilde{\chi}_m^z$

Suppose now that $\mu_n \subset K \subset \mathbb{C}$ and that we are given a point $z \in V_n(K)$ together with a(n étale) path $\gamma : \overrightarrow{01}_n \rightsquigarrow z$ on $V_n \otimes \overline{K} = \mathbf{P}^1_{\overline{K}} - \{0, \mu_n, \infty\}$. We consider the $\ell$-adic polylogarithmic characters $\tilde{\chi}_m^{z^n}$, $\tilde{\chi}_m^{\zeta z} : G_K \to \mathbb{Z}_\ell$ ($\zeta \in \mu_n$) along the paths $\pi_n(\gamma) : \overrightarrow{01} \rightsquigarrow z^n$ and $\delta_\zeta \mathscr{J}_\zeta(\gamma) : \overrightarrow{01} \rightsquigarrow \zeta z$ respectively. In this section, we shall show the following $\ell$-adic analog of the distribution formula:

**Theorem 10.** *Notations being as above, we have*

$$\tilde{\chi}_k^{z^n}(\sigma) = \sum_{d=1}^{k} \binom{k-1}{d-1} n^{d-1} \sum_{s=0}^{n-1} (s\chi(\sigma))^{k-d} \tilde{\chi}_d^{\zeta_n^s z}(\sigma) \qquad (\sigma \in G_K,\ \zeta_n = e^{\frac{2\pi i}{n}},\ 0^0 = 1).$$

Consider now the $\ell$-adic Lie algebras $L_{\mathbb{Q}_\ell}(\overrightarrow{01}_n)$ and $L_{\mathbb{Q}_\ell}(\overrightarrow{01})$ associated to $\pi_1^\ell(V_n, \overrightarrow{01}_n)$ and $\pi_1^\ell(V_1, \overrightarrow{01}_1)$ respectively, and set specific elements of them by $X_n := \log x_n$, $Y_{i,n} := \log y_{i,n}$ ($i = 0, \dots, n-1$), $X := \log x$ and $Y := \log y$.

In the following of this section, we shall fix $\sigma \in G_K$ and frequently omit mentioning $\sigma$ that is potentially appearing in each term of our functional equation. In particular, the quantities $\chi$, $\rho_z$ designate the values $\chi(\sigma)$, $\rho_z(\sigma)$ at $\sigma \in G_K$ respectively. For our fixed $\sigma \in G_K$, let us determine the polylogarithmic part of the Galois transformation $\mathfrak{f}_\sigma^\gamma := \gamma \cdot \sigma(\gamma)^{-1}$ of the path $\gamma : \overrightarrow{01}_n \rightsquigarrow z$ in the form:

$$\begin{aligned} \log(\mathfrak{f}_\sigma^\gamma)^{-1} &\equiv C X_n + \sum_{s=0}^{n-1} \sum_{m=1}^{\infty} C_{s,m}\, \mathrm{ad}(X_n)^{m-1}(Y_{s,n}) \\ &\equiv C X_n + \sum_{s=0}^{n-1} \mathsf{C}_s(\mathrm{ad} X_n)(Y_{s,n}) \quad \mod I_{Y_*}, \end{aligned} \tag{22}$$

where, $I_{Y_*}$ represents the ideal generated by those terms including $\{Y_{0,n}, \dots, Y_{n-1,n}\}$ twice or more, and $\mathsf{C}_s(t) = \sum_{m=1}^\infty C_{s,m} t^{m-1} \in \mathbb{Q}_\ell[[t]]$ ($s = 0, \dots, n-1$).

We determine the above coefficients $C$, $C_{s,m}$ by applying the morphisms $\mathscr{J}_\zeta$ ($\zeta \in \mu_n$). Let us set

$$\begin{aligned} &\mathsf{L}^{(\zeta)}(t) := \mathsf{L}_1(\zeta z) + \mathsf{L}_2(\zeta z) t + \mathsf{L}_3(\zeta z) t^2 + \cdots \quad (\zeta \in \mu_n); \\ &\mathsf{L}^{(n)}(t) := \mathsf{L}_1(z^n) + \mathsf{L}_2(z^n) t + \mathsf{L}_3(z^n) t^2 + \cdots, \end{aligned}$$

with

$$\begin{cases} \mathsf{L}_0(\zeta z) := \rho_{\zeta z} = \rho_z + \frac{s}{n}(\chi - 1) \\ \mathsf{L}_1(\zeta z) := \rho_{1-\zeta z}, \\ \mathsf{L}_k(\zeta z) := \frac{\tilde{\chi}_k^{\zeta z}(\sigma)}{(k-1)!} \quad (k \geq 2); \\ (\zeta = e^{2\pi i s/n},\ s = 0, 1, \dots, n-1), \end{cases} \qquad \begin{cases} \mathsf{L}_0(z^n) & := \rho_{z^n} = n\rho_z, \\ \mathsf{L}_1(z^n) & := \rho_{1-z^n} = \sum_{\zeta \in \mu_n} \rho_{1-\zeta z}, \\ \mathsf{L}_k(z^n) & := \frac{\tilde{\chi}_k^{z^n}(\sigma)}{(k-1)!} \quad (k \geq 2). \end{cases}$$

Then,

**Lemma 11.**

(1) $C = \mathsf{L}_0(z) = \rho_z$.

(2) $\mathsf{C}_0(t) = \mathsf{L}^{(1)}(-t)\dfrac{\rho_z t}{e^{\rho_z t}-1}$.

(3) $\mathsf{C}_s(t) = \mathsf{L}^{(\zeta)}(-t)e^{((\frac{s}{n}-1)\chi-\frac{s}{n})t}\dfrac{\rho_z t}{e^{\rho_z t}-1} \qquad (s=1,\dots,n-1;\ \zeta = e^{-\frac{2\pi i s}{n}})$.

The proof of this lemma will be given later in this section.

**Proof of Theorem 10 assuming Lemma 11:**

We apply the morphism $\pi_n : V_n \to V_1$ to $\log(\mathfrak{f}_\sigma^\gamma)^{-1}$. We first observe that $\pi_n(X_n) = nX$, $\pi_n(Y_{s,n}) = x^s Y x^{-s} = \sum_{k=0}^\infty \frac{s^k}{k!}(\mathrm{ad}X)^k(Y) = e^{s\cdot \mathrm{ad}X}(Y)$ for $s = 0,\dots,n-1$. Hence,

$$\pi_n(\log(\mathfrak{f}_\sigma^\gamma)^{-1}) = CnX + \sum_{s=0}^{n-1}\mathsf{C}_s(n\,\mathrm{ad}X)\left(\sum_{k=0}^\infty \frac{s^k}{k!}(\mathrm{ad}X)^k\right)(Y). \tag{23}$$

The above LHS equals to

$$\log(\mathfrak{f}_\sigma^{\pi_n(\gamma)})^{-1} = \rho_{z^n}X + \sum_{k=1}^\infty \ell i_k(z^n,\pi_n(\gamma))(\mathrm{ad}X)^{k-1}(Y). \tag{24}$$

From the formula (21) we see that

$$\sum_{k=1}^\infty \ell i_k(z^n,\pi_n(\gamma))t^k = t\,\mathsf{L}^{(n)}(-t)\frac{\rho_z nt}{e^{\rho_z nt}-1}, \tag{25}$$

hence that the equality of RHSs of (23) and (24) results in:

$$\sum_{s=0}^{n-1}\mathsf{C}_s(nt)e^{st} = \mathsf{L}^{(n)}(-t)\frac{\rho_z nt}{e^{\rho_z nt}-1}. \tag{26}$$

Substituting $\mathsf{C}_s(t)$ $(s=0,\dots,n-1)$ by Lemma 11 (2), (3), the above left side equals

$$\left(\mathsf{L}^{(1)}(-nt) + \sum_{s=1}^{n-1}\mathsf{L}^{(\zeta)}(-nt)e^{((\frac{s}{n}-1)\chi-\frac{s}{n})nt}e^{st}\right)\frac{\rho_z nt}{e^{\rho_z nt}-1} \tag{27}$$

where, in the summation $\sum_s$, we understand $\zeta = e^{-\frac{2\pi i s}{n}}$. As $((\frac{s}{n}-1)\chi - \frac{s}{n})nt + st = -(n-s)\chi t$, the replacement of $\zeta$ by $\zeta_n^s = e^{\frac{2\pi i s}{n}}$ enables us to collect the sum as $\sum_{s=0}^{n-1}\mathsf{L}^{(\zeta)}(-nt)e^{-s\chi t}$. Finally, substituting $t$ for $-t$, we obtain

$$\mathsf{L}^{(n)}(t) = \sum_{s=0}^{n-1}\mathsf{L}^{(\zeta)}(nt)e^{s\chi t} \qquad (\zeta = e^{\frac{2\pi i s}{n}}). \tag{28}$$

Theorem 10 follows from comparing the coefficients of the above equation. $\square$

We prepare the following combinatorial lemma concerning the Baker-Campbell-Hausdorff sum: $S \underset{\mathsf{CH}}{\oplus} T = \log(e^S e^T)$. Let

$$\beta(t) = \frac{t}{e^t-1} = \sum_{n=0}^\infty B_n\frac{t^n}{n!}$$

be the generating function for Bernoulli numbers.

**Lemma 12.** *Let $K$ be a field of characteristic 0 and let $\alpha, \ell_0, \ell_1, \dots \in K$. Let $\ell(X,Y) = \ell_0 X + \ell_+(\mathrm{ad}X)(Y) = \ell_0 X + \sum_{k=1}^{\infty} \ell_k \, (\mathrm{ad}X)^{k-1}(Y)$ be an arbitrary element of the formal Lie series ring $\mathrm{Lie}_K\langle\langle X,Y\rangle\rangle$ with $\ell_+(t) \in K[\![t]\!]$. Then, we have the following congruence formulas modulo $I_Y$.*

$$(i) \quad \ell(X,Y) \underset{\mathsf{CH}}{\oplus} \alpha X \equiv (\alpha+\ell_0)X + \left(\frac{\beta((\alpha+\ell_0)\mathrm{ad}X)}{\beta(\alpha\,\mathrm{ad}X)}\ell_+(\mathrm{ad}X)\right)(Y);$$

$$(ii) \quad \alpha X \underset{\mathsf{CH}}{\oplus} \ell(X,Y) \equiv (\alpha+\ell_0)X + \left(\frac{\beta((\alpha+\ell_0)\mathrm{ad}X)}{\beta(\ell_0\,\mathrm{ad}X)}\ell_+(\mathrm{ad}X)e^{\alpha\,\mathrm{ad}X}\right)(Y).$$

*Proof.* Both formulas follow from the polylogarithmic BCH formula and with a representation of the core generating function. See [13, Prop. 5.9 and (5.8)]. □

**Proof of Lemma 11:**

Apply the morphisms $\mathscr{J}_\zeta$ ($\zeta \in \mu_n$) to determine the coefficients $C_{m,s}$ of the polylogarithmic terms of $\log \mathfrak{f}_\sigma^\gamma$ in (22).

**Case** $\zeta = 1$: Observe that $\mathscr{J}_1(X_n) = X$, $\mathscr{J}_1(Y_{0,n}) = Y$ and $\mathscr{J}_1(Y_{i,n}) = 0$ ($i \neq 0$). Then, it follows from (22) that

$$\mathscr{J}_1(\log(\mathfrak{f}_\sigma^\gamma)^{-1}) = \log(\mathfrak{f}_\sigma^{\mathscr{J}_1(\gamma)})^{-1} \equiv CX + \big(\mathsf{C}_0(\mathrm{ad}X)\big)(Y) \quad \mod I_Y.$$

We immediately see that the first coefficient $C$ is given by

$$C = \rho_z = \mathsf{L}_0(z), \tag{29}$$

and that the other polylogarithmic coefficients are given by (21) as follows:

$$\mathsf{C}_0(t) = \sum_{k=1}^{\infty} \ell i_k(z, \mathscr{J}_1(\gamma))t^{k-1} = \mathsf{L}^{(1)}(-t)\frac{\rho_z t}{e^{\rho_z t}-1}. \tag{30}$$

**Case** $\zeta \neq 1$: Assume $\zeta = e^{-\frac{2\pi i s}{n}}$ ($s = 1, \dots, n-1$). We observe in this case that $\delta_\zeta \, \mathscr{J}_\zeta(X_n) \, \delta_\zeta^{-1} = X$, $\delta_\zeta \, \mathscr{J}_\zeta(Y_{s,n}) \, \delta_\zeta^{-1} = xYx^{-1} = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}X)^k(Y)}{k!} = e^{\mathrm{ad}X}(Y)$ and $\mathscr{J}_\zeta(Y_{i,n}) = 0$ ($i \neq 0$). Therefore, it follows from (22) that

$$\delta_\zeta \cdot \mathscr{J}_\zeta(\log(\mathfrak{f}_\sigma^\gamma)^{-1}) \cdot \delta_\zeta^{-1} \equiv CX + \big(\mathsf{C}_s(\mathrm{ad}X)e^{\mathrm{ad}X}\big)(Y) \quad \mod I_Y. \tag{31}$$

On the other side, since $\mathfrak{f}_\sigma^{\delta_\zeta \mathscr{J}_\zeta(\gamma)} = \delta_\zeta \mathfrak{f}_\sigma^{\mathscr{J}_\zeta(\gamma)} \delta_\zeta^{-1} \mathfrak{f}_\sigma^{\delta_\zeta}$ by (13), we have

$$\begin{aligned} &\delta_\zeta \cdot \mathscr{J}_\zeta(\log(\mathfrak{f}_\sigma^\gamma)^{-1}) \cdot \delta_\zeta^{-1} = \delta_\zeta \cdot \log(\mathfrak{f}_\sigma^{\mathscr{J}_\zeta(\gamma)})^{-1} \cdot \delta_\zeta^{-1} \\ &= \left(-\log(\mathfrak{f}_\sigma^{\delta_\zeta})^{-1}\right) \underset{\mathsf{CH}}{\oplus} \left(\log(\mathfrak{f}_\sigma^{\delta_\zeta \mathscr{J}_\zeta(\gamma)})^{-1}\right) \\ &\equiv \left(-\frac{n-s}{n}(\chi-1)X\right) \underset{\mathsf{CH}}{\oplus} \left(\ell i_0(\zeta z)X + \sum_{k=1}^{\infty} \ell i_k(\zeta z)(\mathrm{ad}X)^{k-1}(Y)\right) \end{aligned} \tag{32}$$

mod $I_Y$, where $\ell i_k(\zeta z)$ ($k \geq 0$) are taken along the path $\delta_\zeta \mathscr{J}_\zeta(\gamma)$. Note here that $\ell i_0(\zeta z) = \mathsf{L}_0(\zeta z) = \rho_z + \frac{n-s}{n}(\chi - 1)$ and that (21) implies

$$\sum_{k=1}^{\infty} \ell i_k(\zeta z, \delta_\zeta \mathscr{J}_\zeta(\gamma))t^{k-1} = \mathsf{L}^{(\zeta)}(-t)\,\beta(\mathsf{L}_0(\zeta z)\,t) = \mathsf{L}^{(\zeta)}(-t)\frac{\mathsf{L}_0(\zeta z)t}{e^{\mathsf{L}_0(\zeta z)t}-1}. \tag{33}$$

Putting this into (32) and using Lemma 12(ii), we find

$$\delta_\zeta \cdot \mathscr{J}_\zeta(\log(\mathfrak{f}_\sigma^\gamma)^{-1}) \cdot \delta_\zeta^{-1} \equiv \rho_z X + \left(\mathsf{L}^{(\zeta)}(-\mathrm{ad}X) e^{-\frac{n-s}{n}(\chi-1)\mathrm{ad}X} \frac{\rho_z \mathrm{ad}X}{e^{\rho_z \mathrm{ad}X}-1}\right)(Y) \tag{34}$$

mod $I_Y$. Comparing this with (31), we obtain

$$\mathsf{C}_s(t) = \mathsf{L}^{(\zeta)}(-t) e^{(-\frac{n-s}{n}(\chi-1)-1)t} \frac{\rho_z t}{e^{\rho_z t}-1} \qquad (s = 1, \dots, n-1;\ \zeta = e^{-\frac{2\pi i s}{n}}). \tag{35}$$

Thus, the proof of Lemma 11 is completed. □

**Remark 13.** In [13, Theorem 5.7], we gave a general tensor criterion to have a functional equation of (complex and $\ell$-adic) polylogarithms from a collection of morphisms $\{f_i : X \to \mathbf{P}^1 - \{0, 1, \infty\}\}_{i\in I}$ and their formal sum $\sum_{i\in I} c_i[f_i]$. In our above case, it holds that the collection $\{\pi_n, \mathscr{J}_0, \dots, \mathscr{J}_{n-1} : V_n \to V_1\}$ satisfies the criterion with coefficients $1, -n^{k-1}, \dots, -n^{k-1}$ (as observed already in [4, (1.9) (iii)]). Explicit evaluation of the error terms $\mathsf{E}_k := \mathsf{E}_k(\sigma, \gamma)$ discussed in [13] (that explains part of lower degree inhomogeneous terms of our functional equation) can be obtained a posteriori from (25), (30), (33) and (28) as:

$$\sum_{k=1}^{\infty} \mathsf{E}_k t^k = \frac{\rho_z n t^2}{e^{\rho_z n t}-1} \sum_{s=1}^{n-1} \mathsf{L}^{(\zeta_n^s)}(-nt)(e^{-s\chi t} - e^{-s(\chi-1)t}).$$

Note that the lower degree terms other than $\mathsf{E}_k$ are explained by the Roger type normalization (difference from $\ell i_k$ and $\tilde{\chi}_k$) and the effects from compositions of paths $\overrightarrow{01} \rightsquigarrow \zeta\overrightarrow{01} \rightsquigarrow \zeta z$ of Baker-Campbell-Hausdorff type.

**Remark 14.** Replacing $\mathsf{L}^{(n)}(t)$, $\mathsf{L}^{(\zeta)}(nt)$ in (28) by those generating functions for $\ell i_k(z^n, \pi_n(\gamma))$, $\ell i_k(\zeta z, \delta_\zeta \mathscr{J}_\zeta(\gamma))$ by (25), (30) and (33), we obtain an equation

$$\begin{aligned}&\sum_{k=1}^{\infty} \ell i_k(z^n, \pi_n(\gamma)) t^{k-1}\\ &= \frac{\rho_z n t}{e^{\rho_z n t}-1} \sum_{s=0}^{n-1} e^{s\chi t} \left(\frac{e^{-\mathsf{L}_0(\zeta_n^s z) n t}-1}{-\mathsf{L}_0(\zeta_n^s z) n t}\right) \sum_{k=1}^{\infty} \ell i_k(\zeta_n^s z, \mathscr{J}_{\zeta_n^s}(\gamma))(-nt)^{k-1}\end{aligned}$$

in $\mathbb{Q}_\ell[\![t]\!]$. From this, for every fixed $k \geq 1$, one may express $\ell i_k(z^n, \pi_n(\gamma))$ as a linear combination of the $\ell i_d(\zeta_n^s z, \delta_{\zeta_n^s} \mathscr{J}_{\zeta_n^s}(\gamma))$ $(s = 0, \dots, n-1,\ d = 1, \dots, k)$. However, those coefficients are apparently more complicated than those in Theorem 10 where the polylogarithmic characters $\tilde{\chi}_k^{z^n}$, $\tilde{\chi}_d^{\zeta z}$ are treated.

## 6. **Homogeneous form**

We keep the notations in §5 with assuming $\mu_n \subset K$. Let $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n)$ denote the *$\ell$-adic pro-unipotent fundamental group* of $V_n \otimes \overline{K}$ based at $\overrightarrow{01}_n$ which is by definition the pro-unipotent hull of the image of the Magnus embedding (12) consisting of all the group-like elements of the complete Hopf algebra $\mathbb{Q}_\ell\langle\!\langle X_n, Y_{i,n} \mid 0 \leq i \leq n-1 \rangle\!\rangle$. We also define the *$\ell$-adic pro-unipotent path space* (or *$\mathbb{Q}_\ell$-path space* for short) $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n, v)$ for a $K$-(tangential) point $v$ on $V_n$ to be the $\mathbb{Q}_\ell$-rational extension of the path torsor $\pi_1^\ell(V_n \otimes \overline{K}, \overrightarrow{01}_n, v)$ via $\pi_1^\ell(V_n \otimes \overline{K}, \overrightarrow{01}_n) \subset \pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n)$. Note that both $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n)$ and $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n, v)$ have natural actions by $G_K$ compatible with identification

$$\pi_1^\ell(V_n\otimes\overline{K},\overrightarrow{01}_n)\subset\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n),\quad \pi_1^\ell(V_n\otimes\overline{K},\overrightarrow{01}_n,v)\subset\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n,v).$$

Let us introduce rational modifications of the loops $y_{s,n}$ $(s=0,\dots,n-1)$ and the paths $\delta_\zeta$ $(\zeta\in\mu_n)$ respectively as follows. For $s=0,\dots,n-1$ and $\zeta=e^{2\pi is/n}$, set

$$\tilde{y}_{s,n}:=x_n^{-\frac{s}{n}}y_{s,n}x_n^{\frac{s}{n}}\in\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n),$$
$$\varepsilon_\zeta:=x^{-\frac{s}{n}}\cdot\delta_\zeta\in\pi_{\mathbb{Q}_\ell}(\overrightarrow{01},\zeta\overrightarrow{01}).$$

Note that, in the case $n=1$, we have $x=x_1$, $y=\tilde{y}_{0,1}$ by definition.

The following lemma is the key to homogenize the $\ell$-adic distribution formula.

**Lemma 15.** *(i) For every $\sigma\in G_K$ and $\zeta\in\mu_n$, we have $\sigma(\varepsilon_\zeta)=\varepsilon_\zeta$. Moreover, for any path $\gamma$ from $\zeta\overrightarrow{01}$ to a $K$-point $w$ on $V_1$, we have $\varepsilon_\zeta\,\mathfrak{f}_\sigma^\gamma\,\varepsilon_\zeta^{-1}=\mathfrak{f}_\sigma^{\varepsilon_\zeta\gamma}$.*

*(ii) The natural extensions of the homomorphisms on $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_*)$ induced by $\mathscr{J}_\zeta:V_n\to V_1$, $\pi_{rn,r}:V_{rn}\to V_r$ (denoted by the same symbols) map the loops $x_n$, $\tilde{y}_{s,n}$ $(s=0,\dots,n-1)$ as follows.*

$$(a)\quad \varepsilon_\zeta\,\mathscr{J}_\zeta(x_n)\,\varepsilon_\zeta^{-1}=x.$$
$$(b)\quad \pi_{rn,r}(x_{rn})=x_r{}^n.$$
$$(c)\quad \varepsilon_\zeta\,\mathscr{J}_\zeta(\tilde{y}_{s,n})\,\varepsilon_\zeta^{-1}=\begin{cases}y & (\zeta=e^{-2\pi is/n}),\\ 1 & (\zeta\neq e^{-2\pi is/n}).\end{cases}$$
$$(d)\quad \pi_{rn,r}(\tilde{y}_{j,rn})=\tilde{y}_{i,r}\qquad (0\le i<r,\ 0\le j<rn,\ i\equiv j \bmod r).$$

*Proof.* (i): Let $\zeta=e^{2\pi is/n}$ $(s=0,\dots,n-1)$. By the assumption $\mu_n\subset K$, we have $\chi(\sigma)\equiv 1 \bmod n$ for $\sigma\in G_K$. The first assertion follows immediately from the formula

$$\sigma(\delta_\zeta)=x^{\frac{s}{n}(\chi(\sigma)-1)}\delta_\zeta,$$

which can be easily seen from an argument similar to the proof of [12, Prop. 1] with replacement of $\bar{F}(\!(t-z)\!)$ by $\bar{F}\{\{\zeta t\}\}$. The second claim follows easily from the definition (13): $\mathfrak{f}_\sigma^p=p\cdot\sigma(p)^{-1}$ for any path $p:a\rightsquigarrow b$.

(ii): (a), (b) and the case $\zeta\neq e^{-2\pi is/n}$ of (c) are trivial. (d) follows from (b) and the fact $\pi_{rn,r}(y_{j,rn})=x_r^k y_{i,r}x_r^{-k}$ with $j=i+kr$, $0\le i<r$ (16). It remains to prove (c) in the case $\zeta=e^{-2\pi is/n}$. Suppose first that $\zeta$ is different from 1, i.e., $\zeta=e^{-2\pi is/n}$ for any fixed $s=1\dots n-1$. Then $\varepsilon_\zeta=x^{-\frac{n-s}{n}}\cdot\delta_\zeta$. Since $\delta_\zeta\,\mathscr{J}_\zeta(y_{s,n})\,\delta_\zeta^{-1}=xyx^{-1}$, (a) implies $\delta_\zeta\mathscr{J}_\zeta(\tilde{y}_{s,n})\delta_\zeta^{-1}=x^{-\frac{s}{n}}xyx^{-1}x^{\frac{s}{n}}=x^{\frac{n-s}{n}}yx^{-\frac{n-s}{n}}$. It follows then that $\varepsilon_\zeta\mathscr{J}_\zeta(\tilde{y}_{s,n})\varepsilon_\zeta^{-1}=y$. Next, suppose $\zeta=1$ (i.e., $s=0$). Then, it is easy to settle this case by $\mathscr{J}_1(y_{0,n})=y$. We thus complete the proof of (c). $\square$

Now, we embed $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}_n)$ and its Lie algebra $L_{\mathbb{Q}_\ell}(\overrightarrow{01}_n)$ into the non-commutative power series ring $\mathbb{Q}_\ell\langle\!\langle\mathcal{X}_n,\mathcal{Y}_{s,n}\mid 0\le s<n\rangle\!\rangle$ by setting $\mathcal{X}_n:=X_n=\log x_n$, $\mathcal{Y}_{s,n}:=\log\tilde{y}_{s,n}$, and denote by $\mathscr{M}_n$ the set of monomials in $\mathcal{X}_n,\mathcal{Y}_{s,n}$ $(s=0,\dots,n-1)$. For $w\in\mathscr{M}_n$, let $\mathsf{wt}_X(w)$ denote the number of $\mathcal{X}_n$ appearing in $w$. We shall also employ the monomial congruence '$w\equiv w' \bmod r$' by following the same manner as Definition 3 after replacing $X_n$, $Y_{i,n}$ by $\mathcal{X}_n$, $\mathcal{Y}_{i,n}$ $(n\in\mathbb{Z}_{>0},\ 0\le i<n)$ respectively. For the case $n=1$, we will also simply write $\mathcal{X}=\mathcal{X}_1$, $\mathcal{Y}=\mathcal{Y}_{0,1}$.

**Definition 16.** Let $z$ be a point in $V_n(K)$. Given a $\mathbb{Q}_\ell$-path $p \in \pi_{\mathbb{Q}_\ell}(\overrightarrow{01}, z)$ and any $\sigma \in G_K$, we set $\mathfrak{f}^p_\sigma := p \cdot \sigma(p)^{-1}$ and expand it in the form

$$\mathfrak{f}^p_\sigma = 1 + \sum_{w \in \mathscr{M}_n} \mathscr{L}i_w(\overrightarrow{01} \overset{p}{\rightsquigarrow} z)(\sigma) \cdot w$$

in $\mathbb{Q}_\ell\langle\langle \mathcal{X}_n, \mathcal{Y}_{i,n} \mid 0 \le i \le n-1 \rangle\rangle$. (Recall that, in (14), another (non-commutative) expansion of $\mathfrak{f}^\gamma_\sigma$ for $\gamma \in \pi^\ell_1(V_n \otimes \overline{K}, \overrightarrow{01}, z)$ was considered by using a different set of variables.) We call the above coefficient character

$$\mathscr{L}i_w(\overrightarrow{01} \overset{p}{\rightsquigarrow} z) \left( = \mathscr{L}i^{(\ell)}_w(\overrightarrow{01} \overset{p}{\rightsquigarrow} z) \right) : G_K \to \mathbb{Q}_\ell$$

the *$\ell$-adic iterated integral* associated to the word $w \in \mathscr{M}_n$ and to the $\mathbb{Q}_\ell$-path $p$ on $V_n$.

**Theorem 17.** *Let $p$ be a $\mathbb{Q}_\ell$-path on $V_{rn}$ from $\overrightarrow{01}$ to a point $z \in V_{rn}(K)$. Then, for any word $w \in \mathscr{M}_r$, we have the distribution relation*

$$\mathscr{L}i_w(\overrightarrow{01} \overset{\pi_{rn,r}(p)}{\rightsquigarrow} z^n)(\sigma) = n^{\mathsf{wt}_X(w)} \sum_{\substack{u \in \mathscr{M}_{rn} \\ u \equiv w \mod r}} \mathscr{L}i_u(\overrightarrow{01} \overset{p}{\rightsquigarrow} z)(\sigma)$$

*for $\sigma \in G_K$.*

*Proof.* The assertion follows in the same way as Theorem 4 after the above Lemma 15 (b), (d). □

Next, let us concentrate on the polylogarithmic part on $V_1$. Recall that both $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01})$ and its Lie algebra $L_{\mathbb{Q}_\ell}(\overrightarrow{01})$ are embedded in $\mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$, where $X = \mathcal{X}_1$ and $Y = \mathcal{Y}_{0,1}$.

**Definition 18.** Let $z$ be a point in $V_1(K) = \mathbf{P}^1(K) - \{0, 1, \infty\}$ and $p : \overrightarrow{01} \rightsquigarrow z$ a $\mathbb{Q}_\ell$-path. Consider the associator $\mathfrak{f}^p_\sigma := p \cdot \sigma(p)^{-1} \in \pi_{\mathbb{Q}_\ell}(\overrightarrow{01})$ for $\sigma \in G_K$, and define

$$\boldsymbol{\rho}_{z,p} : G_K \to \mathbb{Q}_\ell, \quad \ell i_m(z, p) : G_K \to \mathbb{Q}_\ell$$

by the non-commutative expansion corresponding to (18):

$$\log(\mathfrak{f}^p_\sigma)^{-1} \equiv \boldsymbol{\rho}_{z,p}(\sigma) X + \sum_{m=1}^{\infty} \ell i_m(z, p)(\sigma)(\mathrm{ad}X)^{m-1}(Y) \mod I_Y,$$

where $I_Y$ represents the ideal generated by those terms including $Y$ twice or more. Using these, we also define

$$\tilde{\boldsymbol{\chi}}^{z,p}_m : G_K \to \mathbb{Q}_\ell$$

for $m \ge 1$ by the equation extending Proposition 8 (i):

$$\tilde{\boldsymbol{\chi}}^{z,p}_m(\sigma) = (-1)^{m+1}(m-1)! \sum_{k=1}^{m} \frac{\boldsymbol{\rho}_{z,p}(\sigma)^{m-k}}{(m+1-k)!} \ell i_k(z, p)(\sigma). \tag{36}$$

Since $\mathbb{Q}_\ell$-paths generally do not give bijection systems between fibers of endpoints on finite étale covers, no simple interpretation is available for $\boldsymbol{\rho}_{z,p}$ or $\tilde{\boldsymbol{\chi}}^{z,p}_m$ by Kummer properties: For example, the above $\tilde{\boldsymbol{\chi}}^{z,p}_m(\sigma)$ ($\sigma \in G_K$) generally has a denominator in $\mathbb{Q}_\ell$, i.e., may not be valued in $\mathbb{Z}_\ell$. This makes it difficult to understand $\tilde{\boldsymbol{\chi}}^{z,p}_m(\sigma)$ in terms of Kummer properties at finite levels of an arithmetic sequence like (20).

Once $\boldsymbol{\rho}_{z,p}$, $\ell i_m(z,p)$ and $\tilde{\boldsymbol{\chi}}_m^{z,p} : G_K \to \mathbb{Q}_\ell$ are defined as in the above Definition, the identities as in Proposition 8 (ii) and (21) can be extended in obvious ways for them by formal transformations of generating functions. In the same way, it holds that

$$-\frac{\tilde{\boldsymbol{\chi}}_m^{z,p}(\sigma)}{(m-1)!} = \mathscr{L}i_{YX^{m-1}}(\overrightarrow{01} \overset{p}{\rightsquigarrow} z)(\sigma) \tag{37}$$

for $p \in \pi_{\mathbb{Q}_\ell}(\overrightarrow{01}, z)$ and $\sigma \in G_K$.

**Theorem 19.** *Suppose $\mu_n \subset K \subset \mathbb{C}$ and let $p$ be a $\mathbb{Q}_\ell$-path on $V_n$ from $\overrightarrow{01}$ to a point $z \in V_n(K)$. Then,*

$$\ell i_k(z^n, \pi_n(p))(\sigma) = n^{k-1} \sum_{\zeta\in\mu_n} \ell i_k(\zeta z, \varepsilon_\zeta \mathscr{J}_\zeta(p))(\sigma)$$

*holds for $\sigma \in G_K$.*

*Proof.* We first put the Lie expansion of $\log(\mathfrak{f}_\sigma^p)^{-1}$ in $\mathcal{X}_n = X_n = \log x_n$, $\mathcal{Y}_{s,n} = \log \tilde{y}_{s,n}$ $(s = 0, \dots, n-1)$ in the Lie algebra $L_{\mathbb{Q}_\ell}(\overrightarrow{01}_n)$ as:

$$\begin{aligned} \log(\mathfrak{f}_\sigma^\gamma)^{-1} &\equiv DX_n + \sum_{s=0}^{n-1}\sum_{m=1}^{\infty} D_{s,m}\,(\mathrm{ad}\mathcal{X}_n)^{m-1}(\mathcal{Y}_{s,n}) \\ &\equiv DX_n + \sum_{s=0}^{n-1} \left(\mathsf{D}_s(\mathrm{ad}\mathcal{X}_n)\right)(\mathcal{Y}_{s,n}) \quad \mod I_{\mathcal{Y}_*}, \end{aligned} \tag{38}$$

where, $I_{\mathcal{Y}_*}$ represents the ideal generated by those terms including $\{\mathcal{Y}_{0,n}, \dots, \mathcal{Y}_{n-1,n}\}$ twice or more, and $\mathsf{D}_s(t) = \sum_{m=1}^{\infty} D_{s,m}\, t^{m-1} \in \mathbb{Q}_\ell[[t]]$ $(s = 0, \dots, n-1)$. We shall determine those coefficients $D$ and $\mathsf{D}_{s,m}$ by applying the morphisms $\mathscr{J}_\zeta$. For any fixed $\zeta = \zeta_n^{-s}$ $(s = 0, \dots, n-1)$, by Lemma 15 (i), we obtain $\mathfrak{f}_\sigma^{\varepsilon_\zeta \mathscr{J}_\zeta(p)} = \varepsilon_\zeta \cdot \mathscr{J}_\zeta(p \cdot \sigma(p)^{-1}) \cdot \sigma(\varepsilon_\zeta)^{-1} = \varepsilon_\zeta \cdot \mathscr{J}_\zeta(\mathfrak{f}_\sigma^p) \cdot \varepsilon_\zeta^{-1}$, hence

$$\varepsilon_\zeta \cdot \mathscr{J}_\zeta(\log(\mathfrak{f}_\sigma^p)^{-1}) \cdot \varepsilon_\zeta^{-1} = \log(\mathfrak{f}_\sigma^{\varepsilon_\zeta \mathscr{J}_\zeta(p)})^{-1}.$$

As the right hand side comes from the associator for the path $\varepsilon_\zeta \mathscr{J}_\zeta(p) : \overrightarrow{01} \rightsquigarrow \zeta z$, it should coincide, by definition, with

$$\boldsymbol{\rho}_{\zeta z, \varepsilon_\zeta \mathscr{J}_\zeta(p)}(\sigma) X + \sum_{k=1}^{\infty} \ell i_k(\zeta z, \varepsilon_\zeta \mathscr{J}_\zeta(p))(\sigma)\,(\mathrm{ad}X)^{k-1}(Y),$$

while, the left hand side can be calculated after Lemma 15 (ii) (a), (c) to equal to

$$DX + \sum_{k=1}^{\infty} D_{s,k}\,(\mathrm{ad}X)^{k-1}(Y)$$

with $s$ given by $\zeta = e^{-2\pi i s/n}$. Therefore, we conclude

$$D = \boldsymbol{\rho}_{\zeta z, \varepsilon_\zeta \mathscr{J}_\zeta(p)}(\sigma), \tag{39}$$

$$D_{s,k} = \ell i_k(\zeta z, \varepsilon_\zeta \mathscr{J}_\zeta(p))(\sigma) \tag{40}$$

for $\zeta = e^{-2\pi i s/n}$ $(s = 0, \ldots, n-1)$. Now, apply the projection morphism $\pi_n := \pi_{n,1} : V_n \to V_1$ and interpret the both sides of equality $\pi_n(\log(\mathfrak{f}_\sigma^p)^{-1}) = \log(\mathfrak{f}_\sigma^{\pi_n(p)})^{-1}$. Then, we obtain

$$DnX + \sum_{s=0}^{n-1}\sum_{k=1}^{\infty} D_{s,k}\,(n\,\mathrm{ad}X)^{k-1}(Y) = \boldsymbol{\rho}_{z^n,\pi_n(p)}X + \sum_{k=1}^{\infty} \ell i_k(z^n, \pi_n(p))\,\mathrm{ad}(X)^{k-1}(Y).$$

Comparing the coefficient of $(\mathrm{ad}X)^{k-1}(Y)$ in the above and (40), we conclude the proof of the theorem. $\square$

In the above proof, for a given $\mathbb{Q}_\ell$-path $p : \overrightarrow{01} \rightsquigarrow z$ on $V_n$, we considered the collection of $\mathbb{Q}_\ell$-paths

$$\mathscr{P}_n := \{\varepsilon_\zeta \mathscr{J}_\zeta(p) : \overrightarrow{01} \rightsquigarrow \zeta z \mid \zeta = \zeta_n^s \in \mu_n\ (s = 0, 1, \ldots, n-1)\}$$

on $V_1 = \mathbf{P}^1 - \{0, 1, \infty\}$ . Note that each $\varepsilon_\zeta \mathscr{J}_\zeta(p)$ can also be written as the composite of paths on $V_1$:

$$\varepsilon_\zeta \cdot [\zeta p] = x^{-\frac{s}{n}} \cdot \delta_\zeta \cdot [\zeta p] : \tag{41}$$

$$\overrightarrow{01} \overset{x^{-\frac{s}{n}}}{\rightsquigarrow} \overrightarrow{01} \overset{\delta_\zeta}{\rightsquigarrow} \zeta\overrightarrow{01} \overset{[\zeta p]}{\rightsquigarrow} \zeta z$$

where $[\zeta p] : \zeta\overrightarrow{01} \rightsquigarrow \zeta z$ means a path obtained by "rotating" $p : \overrightarrow{01} \rightsquigarrow z$ by the automorphism of $\mathbf{P}^1 - \{0, \infty\}$ with multiplication by $\zeta$.

**Corollary 20.** *Notations being as above, the maps $\boldsymbol{\rho}_{\zeta z,p}(\sigma) : G_K \to \mathbb{Q}_\ell$ are all the same for the $\mathbb{Q}_\ell$-paths $[p : \overrightarrow{01} \rightsquigarrow \zeta z] \in \bigcup_{n=1}^{\infty} \mathscr{P}_n$.*

*Proof.* As seen in (39), we have the common $D$ upon applying $\mathscr{J}_\zeta$ to the first term of $\log(\mathfrak{f}_\sigma^p)^{-1}$. The assertion follows from this and the fact that $\mathscr{P}_n$ contains $\mathscr{P}_1 \neq \emptyset$. $\square$

From this corollary, we immediately see that the above theorem also gives homogeneous functional equations for the rationally extended $\ell$-adic polylogarithmic characters.

**Corollary 21.** *Notations being as in Theorem 19, let $\tilde{\boldsymbol{\chi}}_k^{z^n,\pi_n(p)}$ and $\tilde{\boldsymbol{\chi}}_k^{\zeta z, \varepsilon_\zeta \mathscr{J}_\zeta(p)}$ $(\zeta \in \mu_n)$ be the extended $\ell$-adic polylogarithmic characters. Then, we have*

$$\tilde{\boldsymbol{\chi}}_k^{z^n,\pi_n(p)}(\sigma) = n^{k-1} \sum_{\zeta \in \mu_n} \tilde{\boldsymbol{\chi}}_k^{\zeta z, \varepsilon_\zeta \mathscr{J}_\zeta(p)}(\sigma) \qquad (\sigma \in G_K).$$

*Proof.* The assertion follows from Theorem 19 by applying Corollary 20 to the definition of $\ell$-adic polylogarithmic characters for $\mathbb{Q}_\ell$-paths (Definition 18). $\square$

## 7. Translation in Kummer-Heisenberg measure

Let $\gamma : \overrightarrow{01} \rightsquigarrow z$ be an $\ell$-adic path in $\pi_1^\ell(\mathbf{P}^1_{\overline{K}} - \{0, 1, \infty\}; \overrightarrow{01}, z)$ and $p := x^{-\frac{s}{n}}\gamma$ be the pro-unipotent path in $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01}, z)$ produced by the composition with $x^{-\frac{s}{n}}$ for any fixed $s \in \mathbb{Z}_\ell$ and $n \in \mathbb{N}$. By definition we have $\mathfrak{f}_\sigma^p = x^{-\frac{s}{n}} \mathfrak{f}_\sigma^\gamma x^{\frac{s}{n}\chi(\sigma)}$ for $\sigma \in G_K$. Since $x^{-\frac{s}{n}} = \exp(\frac{-s}{n}X) \equiv 1$ modulo the right ideal $X{\cdot}\mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$, it follows from Proposition 8 (ii) that

$$\begin{aligned} -\frac{\tilde{\boldsymbol{\chi}}_k^{z,p}(\sigma)}{(k-1)!} &= \mathsf{Coeff}_{YX^{k-1}}(\mathfrak{f}_\sigma^p) = \mathsf{Coeff}_{YX^{k-1}}\left(1 \cdot \mathfrak{f}_\sigma^\gamma \cdot \exp\left(\frac{s\,\chi(\sigma)}{n}X\right)\right) \\ &= \sum_{i=0}^{k-1} \mathsf{Coeff}_{YX^i}(\mathfrak{f}_\sigma^\gamma) \cdot \frac{\left(\frac{s}{n}\chi(\sigma)\right)^{k-i-1}}{(k-i-1)!} = -\sum_{i=0}^{k-1} \frac{\tilde{\chi}_{i+1}^{z,\gamma}(\sigma)}{i\,!} \cdot \frac{\left(\frac{s}{n}\chi(\sigma)\right)^{k-i-1}}{(k-i-1)!}. \end{aligned}$$

Thus we obtain

$$\tilde{\boldsymbol{\chi}}_k^{z,p}(\sigma)=\sum_{i=0}^{k-1}\binom{k-1}{i}\left(\frac{s}{n}\chi(\sigma)\right)^{k-i-1}\tilde{\chi}_{i+1}^{z,\gamma}(\sigma)\qquad(\sigma\in G_K).\tag{42}$$

Recall then that, in [12], introduced is a certain $\mathbb{Z}_\ell$-valued measure (called the Kummer-Heisenberg measure) $\boldsymbol{\kappa}_{z,\gamma}(\sigma)$ on $\mathbb{Z}_\ell$ for every path $\gamma:\overrightarrow{01}\rightsquigarrow z$ and $\sigma\in G_K$, which is characterized by the integration properties:

$$\tilde{\chi}_k^{z,\gamma}(\sigma)=\int_{\mathbb{Z}_\ell}a^{k-1}d\boldsymbol{\kappa}_{z,\gamma}(\sigma)(a)\qquad(k\geq 1).\tag{43}$$

Putting this into (42), we may rewrite the RHS to get

$$\tilde{\boldsymbol{\chi}}_k^{z,p}(\sigma)=\int_{\mathbb{Z}_\ell}\left(a+\frac{s}{n}\chi(\sigma)\right)^{k-1}d\boldsymbol{\kappa}_{z,\gamma}(\sigma)(a).\tag{44}$$

Note that $\frac{s}{n}+\mathbb{Z}_\ell=\frac{s}{n}\chi(\sigma)+\mathbb{Z}_\ell$ as a subset of $\mathbb{Q}_\ell$ when $\mu_n\subset K$. Comparison of (43) and (44) leads us to introduce the following

**Definition 22.** Suppose $\mu_n\subset K$, and let $\sigma\in G_K$ and $p=x^{-\frac{s}{n}}\gamma\in\pi_{\mathbb{Q}_\ell}(\overrightarrow{01},z)$ be as above. Define a $\mathbb{Z}_\ell$-valued measure $\boldsymbol{\kappa}_{z,p}(\sigma)$ on the coset $\frac{s}{n}+\mathbb{Z}_\ell(\subset\mathbb{Q}_\ell)$ by the property:

$$\tilde{\boldsymbol{\chi}}_k^{z,p}(\sigma)=\int_{\frac{s}{n}+\mathbb{Z}_\ell}a^{k-1}d\boldsymbol{\kappa}_{z,p}(\sigma)(a)\qquad(k\geq 1).$$

A verification of this new notion of the extended measure $\boldsymbol{\kappa}_{z,p}(\sigma)$ is that our distribution relations in Corollary 21 can be summarized into a single relation of measures:

**Theorem 23.** *For $s\in\mathbb{Z}_\ell$, let $[n]:\frac{s}{n}+\mathbb{Z}_\ell\to\mathbb{Z}_\ell$ $(a\mapsto na)$ denote the continuous map of multiplication by $n\in\mathbb{N}$, and denote by $[n]_*\boldsymbol{\kappa}$ the push-forward measure on $\mathbb{Z}_\ell$ obtained from any measure $\boldsymbol{\kappa}$ on $\frac{s}{n}+\mathbb{Z}_\ell$ by $U\mapsto\boldsymbol{\kappa}([n]^{-1}(U))$ for the compact open subsets $U$ of $\mathbb{Z}_\ell$. Then,*

$$\boldsymbol{\kappa}_{z^n,\pi_n(\gamma)}(\sigma)=\sum_{\zeta\in\mu_n}[n]_*\boldsymbol{\kappa}_{\zeta z,\varepsilon_\zeta\mathscr{I}_\zeta(\gamma)}(\sigma)\qquad(\sigma\in G_K).$$

*Proof.* The formula follows immediately from Corollary 21 and the characteristic property (44) of the Kummer-Heisenberg measure. □

**Question 24.** In the above discussion, we defined $\boldsymbol{\kappa}_{z,p}(\sigma)$ only for $\mathbb{Q}_\ell$-paths $p:\overrightarrow{01}\rightsquigarrow z$ of the form $p=x^\alpha\gamma$ with $\alpha\in\mathbb{Q}_\ell$ and $\gamma:\overrightarrow{01}\rightsquigarrow z$ being $\ell$-adic (i.e., $\mathbb{Z}_\ell$-integral) paths. It is natural to conjecture existence of a suitable measure $\boldsymbol{\kappa}_{z,p}(\sigma)$ for a more general $\mathbb{Q}_\ell$-path $p:\overrightarrow{01}\rightsquigarrow z$ satisfying the property of Definition 22. The support of this measure should be a parallel transport $R(p,\sigma)$ of $\mathbb{Z}_\ell$ in $\mathbb{Q}_\ell$ such that $x^{R(p,\sigma)}\subset x^{\mathbb{Q}_\ell}$ is the image of $\pi_1^\ell(V_1\otimes\overline{K};\overrightarrow{01},z)\cdot\sigma(p)^{-1}$ via the projection $\pi_{\mathbb{Q}_\ell}(\overrightarrow{01})\twoheadrightarrow x^{\mathbb{Q}_\ell}$.

## 8. **Inspection of special cases**

In this section, we shall closely look at special cases of the $\ell$-adic distribution formula. Let us first consider dilogarithms, i.e., for the case of $k=2$. By Theorem 10, we have

**Corollary 25.** *Let $\mu_n \subset K$ and $\gamma : \overrightarrow{01} \rightsquigarrow z \in V_n(K)$ be an $\ell$-adic path which induces paths $\pi_n(\gamma) : \overrightarrow{01} \rightsquigarrow z^n$ and $\delta_\zeta \mathscr{J}_\zeta(\gamma) : \overrightarrow{01} \rightsquigarrow \zeta z$ ($\zeta = \zeta_n^s \in \mu_n$) on $V_1 = \mathbf{P}^1 - \{0, 1, \infty\}$. Along these paths, we have the following $\mathbb{Z}_\ell$-valued functional equation*

$$\tilde{\chi}_2^{z^n}(\sigma) = n \sum_{s=0}^{n-1} \tilde{\chi}_2^{\zeta_n^s z}(\sigma) + \sum_{s=1}^{n-1} s\chi(\sigma)\rho_{1-\zeta_n^s z}(\sigma) \qquad (\sigma \in G_K),$$

*where $\rho_{1-\zeta_n^s z}$ is the same as the 1st polylogarithmic character $\tilde{\chi}_1^{\zeta_n^s z} : G_K \to \mathbb{Z}_\ell$.* □

In particular when $n = 2$, the above formula is specialized to the following.

**Corollary 26.** *For $\gamma : \overrightarrow{01} \rightsquigarrow z$ on $V_2 = \mathbf{P}^1 - \{0, \pm 1, \infty\}$, let $\pi_2(\gamma) : \overrightarrow{01} \rightsquigarrow z^2$, $\mathscr{J}_1(\gamma) : \overrightarrow{01} \rightsquigarrow z$ and $\delta_{-1}\mathscr{J}_{-1}(\gamma) : \overrightarrow{01} \rightsquigarrow -z$ be the induced paths on $\mathbf{P}^1 - \{0, 1, \infty\}$. Note here that $\delta_{-1} : \overrightarrow{01} \rightsquigarrow -\overrightarrow{01}$ is the positive half rotation. Along these paths, we have a functional equation of the $\ell$-adic polylogarithmic characters*

$$\tilde{\chi}_2^{z^2}(\sigma) = 2(\tilde{\chi}_2^{z}(\sigma) + \tilde{\chi}_2^{-z}(\sigma)) + \chi(\sigma)\rho_{1+z}(\sigma) \qquad (\sigma \in G_K). \quad \square$$

Putting $z = \overrightarrow{10}$ in the above, and recalling $\tilde{\chi}_{2k}^{\overrightarrow{10}}(\sigma) = \frac{B_{2k}}{2(2k)}(\chi(\sigma)^{2k} - 1)$ ($\sigma \in G_{\mathbb{Q}}$) from [NW2] Proposition 5.13, we immediately obtain

**Corollary 27.** *Along the path $\gamma_{-1} : \overrightarrow{01} \rightsquigarrow (z=1) \rightsquigarrow (z=-1)$ induced by the positive half arc on the unit circle on $\mathbf{P}^1 - \{0, 1, \infty\}$, we have the following $\mathbb{Z}_\ell$-valued equation:*

$$\tilde{\chi}_2^{z=-1}(\sigma) = -\frac{\chi(\sigma)^2 - 1}{48} - \frac{1}{2}\chi(\sigma)\rho_2(\sigma) \qquad (\sigma \in G_{\mathbb{Q}}). \quad \square$$

This result is an $\ell$-adic analog of the classical result $Li_2(-1) = -\frac{\pi^2}{12}$ ([Le]), and is compatible with [13, Remark 5.14 and Remark after (6.31)].

To confirm validity of our above narrow stream of geometrical arguments toward Corollary 27, we here present an alternative direct proof in a purely arithmetic way as below:

**Arithmetic proof of Corollary 27**. We (only) make use of the characterization of $\tilde{\chi}_m^z$ by the Kummer properties (20). Applying it to our case $m = 2$, $z = -1$ where $\rho_z(\sigma) = \frac{1}{2}(\chi(\sigma) - 1)$, we obtain

$$\zeta_{\ell^n}^{\tilde{\chi}_2^{z=-1}(\sigma)} = \sigma \left( \prod_{a=0}^{\ell^n - 1} (1 - \zeta_{2\ell^n}^{2\chi(\sigma)^{-1}a+1})^{\frac{a}{\ell^n}} \right) \Big/ \prod_{a=0}^{\ell^n - 1} (1 - \zeta_{2\ell^n}^{2a+\chi(\sigma)})^{\frac{a}{\ell^n}}. \qquad (*)$$

We evaluate both the denominator and numerator of the above right hand side, first by pairing two factors indexed by $a$ and $a' = -\chi(\sigma) - a$ and by simplifying their product by

$$\left(1 - \zeta_{2\ell^n}^{-2a-\chi(\sigma)}\right)^{\frac{1}{\ell^n}} = \left(1 - \zeta_{2\ell^n}^{2a+\chi(\sigma)}\right)^{\frac{1}{\ell^n}} \cdot \zeta_{2\ell^n}^{l^n - \langle 2a+\chi(\sigma)\rangle}$$

with $0 \le \langle 2a + \chi(\sigma) \rangle \le 2\ell^n$ being the unique residue of $2a + \chi(\sigma)$ mod $2\ell^n$. Pick a disjoint decomposition of the index set $S := \{0 \le a \le \ell^n - 1\}$ into $S_+ \cup S_- \cup S_0$ so that, for all $a \in S$,

(i) $a \in S_+$ iff $\langle -\chi(\sigma) - a \rangle \in S_-$;
(ii) $a \in S_0$ iff $a \equiv -a - \chi(\sigma)$ mod $\ell^n$.

Then, one finds:

$$\prod_{a\in S-S_0}(1-\zeta_{2\ell^n}^{2\chi(\sigma)^{-1}a+1})^{\frac{a}{\ell^n}}=\prod_{a\in S_\pm}(1-\zeta_{2\ell^n}^{2\chi(\sigma)^{-1}a+1})^{\frac{-\chi(\sigma)}{\ell^n}}\zeta_{2\ell^n}^{(\ell^n-\langle 1+2\chi(\sigma)^{-1}a\rangle)(-a-\chi(\sigma))},$$
$$\prod_{a\in S-S_0}(1-\zeta_{2\ell^n}^{2a+\chi(\sigma)})^{\frac{a}{\ell^n}}=\prod_{a\in S_\pm}(1-\zeta_{2\ell^n}^{2a+\chi(\sigma)})^{\frac{-\chi(\sigma)}{\ell^n}}\zeta_{2\ell^n}^{(\ell^n-\langle 2a+\chi(\sigma)\rangle)(-a-\chi(\sigma))}.$$

Noting that $\prod_{a\in S}(1-\zeta_{2\ell^n}^{2a+1})=2$, we obtain the squared sides of $(*)$ as

$$\zeta_{\ell^n}^{2\tilde{\chi}_2^{z=-1}(\sigma)}=\frac{\sigma\Big(2^{\frac{-\chi(\sigma)}{\ell^n}}\prod_{a\in S}\zeta_{2\ell^n}^{(\ell^n-\langle 1+2\chi(\sigma)^{-1}a\rangle)(-a-\chi(\sigma))}\Big)}{2^{\frac{-\chi(\sigma)}{\ell^n}}\prod_{a\in S}\zeta_{2\ell^n}^{(\ell^n-\langle 2a+\chi(\sigma)\rangle)(-a-\chi(\sigma))}}.$$

Here, note that contribution from $S_0$ (which is empty when $\ell=2$) is included into the factor $2^{\frac{-\chi(\sigma)}{\ell^n}}$ both in the numerator and the denominator. Now, choose integers $c,\bar{c}\in\mathbb{Z}$ so that $c\equiv\chi(\sigma)$, $c\bar{c}\equiv 1 \bmod 2\ell^n$. Then, we obtain the following congruence equation mod $\ell^n$:

$$\begin{aligned}2\tilde{\chi}_2^{z=-1}(\sigma)&\equiv-\chi(\sigma)\rho_2(\sigma)+\frac{1}{2}\sum_{a\in S}\chi(\sigma)(-a-c)(\ell^n-\langle 1+\bar{c}a\rangle)-(-a-c)(\ell^n-\langle 1+2\bar{c}a\rangle)\\&\equiv-\chi(\sigma)\rho_2(\sigma)+\frac{1}{2}\sum_{a\in S}(-a-c)\left[\frac{\chi(\sigma)-1}{2}+\left\{\frac{2a+c}{2\ell^n}\right\}-c\left\{\frac{1+2\bar{c}a}{2\ell^n}\right\}\right]\\&\equiv-\chi(\sigma)\rho_2(\sigma)+\frac{1}{2}\sum_{b\in S}b\left[c\left\{\frac{1+2\bar{c}b}{2\ell^n}\right\}-\left\{\frac{c+2b}{2\ell^n}\right\}+\frac{1-c}{2}\right].\end{aligned}$$

By basic properties of the Bernoulli polynomial $B_2(X)=X^2-X+\frac{1}{6}$ (cf. [8]), the last sum is congruent modulo $\frac{\ell^n}{48}\mathbb{Z}$ to

$$\begin{aligned}&\sum_{b\in S}\frac{\ell^n}{2}\left[c^2B_2\left(\left\{\frac{1+2\bar{c}b}{2\ell^n}\right\}\right)-B_2\left(\left\{\frac{2b+c}{2\ell^n}\right\}\right)\right]\\&=\frac{1}{2}(\chi(\sigma)^2-1)B_2(\frac{1}{2})=-\frac{1}{24}\chi(\sigma)^2-1.\end{aligned}$$

Summing up, we find the congruence relations

$$2\tilde{\chi}_2^{z=-1}(\sigma)\equiv-\chi(\sigma)\rho_2(\sigma)-\frac{1}{24}(\chi(\sigma)^2-1)\quad\bmod\frac{\ell^n}{48}\mathbb{Z}$$

for all $n$, hence the equality in $\mathbb{Z}_\ell$. This concludes the proof of the corollary. □

Turning to Theorem 10, by specialization to the case $n=2$ (but for general $k$), we obtain:

**Corollary 28.** *Along the paths from $\overrightarrow{01}$ to $\pm z, z^2$ on $\mathbf{P}^1-\{0,1,\infty\}$ used in Corollary 26, it holds that*

$$\tilde{\chi}_k^{z^2}(\sigma)=2^{k-1}\tilde{\chi}_k^{z}(\sigma)+\sum_{d=1}^{k}\binom{k-1}{d-1}2^{d-1}\chi(\sigma)^{k-d}\tilde{\chi}_d^{-z}(\sigma)\qquad(k\geq 1,\ \sigma\in G_K).$$

Upon observing special cases of the above formula, we find that $\tilde{\chi}_4^{z=-1}$ does not factor through $\mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q})$, because it involves a nontrivial term from $\tilde{\chi}_3^{\overrightarrow{10}}(\sigma)$ which does not vanish on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty}))$ by Soulé [So].

With regard to the classical formula $Li_{2k}(-1) = (-1)^{k+1}(1-2^{2k-1})B_{2k}\frac{\pi^{2k}}{(2k)!}$ ([Le]), we should rather figure out its $\ell$-adic analog in terms of the "$\mathbb{Q}_\ell$-adic" polylogarithmic characters introduced in Definition 18. In fact,

**Corollary 29.** *Let $\gamma_{-1} : \overrightarrow{01} \rightsquigarrow (z=-1)$ be the path in Corollary 27. Then, along the $\mathbb{Q}_\ell$-path $x^{-\frac{1}{2}}\gamma_{-1} : \overrightarrow{01} \rightsquigarrow (z=-1)$, it holds that*

$$\tilde{\boldsymbol{\chi}}_{2k}^{z=-1}(\sigma) = \frac{(1-2^{2k-1})}{2^{2k}} \frac{B_{2k}}{2k}(\chi(\sigma)^{2k}-1) \qquad (\sigma \in G_{\mathbb{Q}}).$$

*Proof.* Applying Corollary 21 to the case where $n=2$ and $p : \overrightarrow{01} \rightsquigarrow \overrightarrow{10}$ is the straight path on $V_2 = \mathbf{P}^1 - \{0, \pm 1, \infty\}$, we obtain

$$\tilde{\boldsymbol{\chi}}_k^{\overrightarrow{10},\pi_2(p)}(\sigma) = 2^{k-1}(\tilde{\boldsymbol{\chi}}_k^{\overrightarrow{10},\mathscr{I}_1(p)}(\sigma) + \tilde{\boldsymbol{\chi}}_k^{z=-1,\gamma_{-1}}(\sigma)).$$

Since $\pi_2(p)$ and $\mathscr{I}_1(p)$ are the same standard path $\overrightarrow{01} \rightsquigarrow \overrightarrow{10}$ on $V_1$, the values $\tilde{\boldsymbol{\chi}}_k^{\overrightarrow{10},\pi_2(p)}(\sigma)$ and $\tilde{\boldsymbol{\chi}}_k^{\overrightarrow{10},\mathscr{I}_1(p)}(\sigma)$ coincide with the (extended) Soulé value $\tilde{\chi}_k^{\overrightarrow{10}}(\sigma)$ (cf. [12, Remark 2]). The desired formula follows then from a basic formula from [13, Proposition 5.13]: $\tilde{\chi}_{2k}^{\overrightarrow{10}}(\sigma) = \frac{B_{2k}}{2(2k)}(\chi(\sigma)^{2k}-1)$ $(\sigma \in G_{\mathbb{Q}})$. $\square$

Unlike the $\mathbb{Z}_\ell$-integral analog stated in Corollary 27, the above right hand side generally has denominators in $\mathbb{Q}_\ell$. This is due to the concern of $x^{-\frac{1}{2}} \in \pi_{\mathbb{Q}_l}(\overrightarrow{01})$ which does not lie in $\pi_1^\ell(V_1 \otimes \overline{K}, \overrightarrow{01})$ when $\ell = 2$.

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## References

[1] A.Beilinson, P.Deligne, *Interprétation motivique de la conjecture de Zagier reliant polylogarithms et régulateurs*, Proc. Symp. in Pure Math. (AMS) **55-2** (1994), 97–121.

[2] P.Deligne, *Le Groupe Fondamental de la Droite Projective Moins Trois Points*, in "Galois group over $\mathbb{Q}$" (Y.Ihara, K.Ribet, J.-P.Serre eds.), MSRI Publ. Vol. 16 (1989), 79–297.

[3] J.-C. Douai, Z. Wojtkowiak, Descent for $l$-adic polylogarithms, *Nagoya Math. J.* **192** (2008), 59–88.

[4] H.Gangl, *Families of Functional Equations for Polylogarithms*, Comtemp. Math. (AMS) **199** (1996), 83–105.

[5] A. Huber, G. Kings *Polylogarithms for families of commutative group schemes*, ArXive: 1505.04574.

[6] R.Hain, *On a generalization of Hilbert's 21st problem*, Ann. Sci. Éc. Norm. Sup., **19** (1986), 609–627.

[7] A. Huber, J. Wildeshaus, *Classical motivic polylogarithm according to Beilinson and Deligne*, Doc. Math. **3** (1998), 27-133; Correction Doc. Math. **3** (1998), 297299.

[8] S.Lang, *Cyclotomic Fields I and II*, Graduate Texts in Math. **121**, Springer 1990.

[9] L.Lewin, *Polylogarithms and associated functions*, North Holland, 1981.

[10] J.Milnor, *On polylogarithms, Hurwitz zeta functions, and the Kubert identities*, L'Enseignement Math. **29** (1983), 281-322.

[11] H. Nakamura, K. Sakugawa and Z. Wojtkowiak: *Polylogarithmic analogue of the Coleman-Ihara formula, II*, RIMS Kôkyûroku Bessatsu (to appear). [available at `https://sites.google.com/site/kenjisakugawashomepage/reserch-articles` ]

[12] H.Nakamura, Z.Wojtkowiak, *On explicit formulae for l-adic polylogarithms*, Proc. Symp. Pure Math. (AMS) **70** (2002) 285–294.

[13] H.Nakamura, Z.Wojtkowiak, *Tensor and homotopy criterions for functional equations of $\ell$-adic and classical iterated integrals*, in "Non-abelian Fundamental Groups and Iwasawa Theory" (J.Coates et.al. eds.), London Math. Soc. Lect. Note Ser., **393** (2012), 258–310.

[14] Ch. Soulé, *On higher $p$-adic regulators*, Springer Lecture Notes in Math., **854** (1981), 372–401.

[15] Z.Wojtkowiak, *A note on functional equations of the $p$-adic polylogarithms*, Bull. Soc. Math. France, **119** (1991), 343–370.

[16] Z.Wojtkowiak, *Functional equations of iterated integrals with regular singularities*, Nagoya Math. J., **142** (1996), 145–159.

[17] Z.Wojtkowiak, *Monodromy of iterated integrals and non-abelian unipotent periods*, in "Geometric Galois Actions II", London Math. Soc. Lect. Note Ser. **243** (1997) 219–289.

[18] Z.Wojtkowiak, *On $\ell$-adic iterated integrals, I – Analog of Zagier Conjecture*, Nagoya Math. J., **176** (2004), 113–158.

[19] Z.Wojtkowiak, *A note on functional equations of $\ell$-adic polylogarithms*, J. Inst. Math. Jussieu, **3** (2004), 461–471.

[20] Z.Wojtkowiak, *On $\ell$-adic iterated integrals, II – Functional equations and $\ell$-adic polylogarithms*, Nagoya Math. J., **177** (2005), 117–153.

[21] Z.Wojtkowiak, *On $\ell$-adic iterated integrals, III – Galois actions on fundamental groups*, Nagoya Math. J., **178** (2005), 1–36.

[22] Z.Wojtkowiak, *On $l$-adic iterated integrals V, linear independence, properties of $l$-adic polylogarithms, $l$-adic sheave*, in "The Arithmetic of Fundamental Groups, PIA 2010", (Jacob Stix ed.), pp. 339–374. Springer-Verlag 2012.

[23] Z.Wojtkowiak, *On $\ell$-adic Galois $L$-functions*, in "Algebraic Geometry and Number Theory, Summer School, Galatasaray University, Istanbul, 2014", Progress in Math. Birkhauser, **321**, pp. 161–209, Springer International Publishing, 2017. (arXiv:1403.2209v1 [math.NT], 10 March, 2014.)



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