Math. J. Okayama Univ. **46** (2004), 39–75

EIGENLOCI OF 5 POINT CONFIGURATIONS ON THE RIEMANN SPHERE AND THE GROTHENDIECK-TEICHMÜLLER GROUP

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Contents

1. Introduction

Let $G_{\mathbb{Q}}$ be the absolute Galois group of the rational number field $\mathbb{Q}$. In this paper we closely study the action of $G_{\mathbb{Q}}$ on an element of the Teichmüller modular group which can be viewed simply as the order 5 rotation of the Riemann sphere marked at the 5th-roots of unity. Especially, we explicitly compute the conjugating factor of the action in terms of the Galois representation in $\pi_1(\mathbf{P}^1-\{0,1,\infty\},\overrightarrow{01})$. The overall meaning of this computation can be explained most naturally from the point of view, or the framework, of Grothendieck-Teichmüller theory. In this introduction we will content ourselves with recalling the least necessary background, relying on references for technical detail. We postpone until §6 of the present paper a short discussion of the why and what for.

Let $M_{0,n}$ (resp. $M_{0,[n]}$) be the fine moduli space of sphere with n labeled (resp. unlabeled) marked points, viewed as a $\mathbb{Q}$-scheme (resp. stack). Let Γ_0^n (resp. $\Gamma_0^{[n]}$) be the topological (resp. orbifold) fundamental group of $M_{0,n}$ (resp. $M_{0,[n]}$) as a complex manifold (resp. orbifold), regarding $\overline{\mathbb{Q}}$ as embedded in $\mathbb{C}$. Finally, let $\hat{\Gamma}_0^n$ and $\hat{\Gamma}_0^{[n]}$ be the profinite completions of these groups, which one can regard as the geometric fundamental groups of $M_{0,n}$ and $M_{0,[n]}$ respectively. There is a canonical outer action of $G_{\mathbb{Q}}$ on $\hat{\Gamma}_0^{[n]}$, which preserves the pure subgroup $\hat{\Gamma}_0^n$.

In this situation, V.Drinfeld introduced the Grothendieck-Teichmüller group $\widehat{GT}$, whose action on $\hat{\Gamma}_0^n$ extends that of $G_{\mathbb{Q}}$ as proved by Ihara, Ihara-Matsumoto ([Dr], [I1], [IM]). In particular $M_{0,4}$ is isomorphic to $\mathbf{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$, so that its topological fundamental group Γ_0^4 is isomorphic to F_2, the free group on two generators. More precisely we regard $M_{0,4}$ as the punctured projective t-line $\mathbf{P}^1_t$ and let x (resp. y, z) denote a loop based at the tangential base point $\overrightarrow{01}_t$ and circling once counterclockwise around 0 (resp. 1, ∞). We then write $F_2 = \langle x, y, z \mid xyz = 1\rangle$; This defines an action of $G_{\mathbb{Q}}$ and $\widehat{GT}$ on the geometric fundamental group of $M_{0,4}$ ($\simeq \hat{F}_2$) based at $\overrightarrow{01}_t$, and we get inclusions: $G_{\mathbb{Q}} \subset \widehat{GT} \subset \mathrm{Aut}(\hat{F}_2)$.

An element σ of $\widehat{GT}$ has defining parameters $(\lambda_\sigma, f_\sigma) \in \hat{\mathbb{Z}}^* \times \hat{F}_2$ such that $\sigma(x) = x^{\lambda_\sigma}$ and f_σ describes the action of σ on the path p between $\overrightarrow{01}_t$ and $\overrightarrow{10}_t$ (essentially the open interval $(0,1)$); in fact, by the definition of f_σ, we have $\sigma(p) = f_\sigma^{-1} p$ (cf. the Appendix). These parameters are subject to the following equations:

(I) $f(x,y)f(y,x) = 1$ in $\hat{F}_2$,

(II) $f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$ in $\hat{F}_2$, where $m = (\lambda-1)/2$,

(III) $f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})f(x_{12},x_{23}) = 1$ in $\hat{\Gamma}_0^5$,

where x_{ij} denotes the standard generator of Γ_0^5 which braids only the strands i and j, and $f(x_{ij}, x_{kl})$ denotes the image of $f = f(x,y)$ under the homomorphism $\hat{F}_2 \to \hat{\Gamma}_0^5$ mapping $x \mapsto x_{ij}$, $y \mapsto x_{kl}$; we refer to [Dr], [LS1,2], [N] for details and references.

For $\sigma \in G_{\mathbb{Q}}$, we have $\lambda_\sigma = \chi(\sigma)$ (the cyclotomic character), and we write λ_σ or $\chi(\sigma)$ indifferently; we also occasionally drop the mention of σ whenever it is clear from the context, writing simply λ, f for λ_σ, f_σ (and the same for analogous parameters which we introduce below). See [I1], [LS1-2], [N-I,II], [F], and several other places for detail on the above standard situation. The Appendix below explains compatibility of conventions/symbols of most references.

The automorphism groups $\mathrm{Aut}(M_{0,4})$ and $\mathrm{Aut}(M_{0,5})$ (all automorphism groups of the moduli spaces are intended over $\mathbb{Q}$), especially the latter one plays an important role in this paper. As is well-known, $\mathrm{Aut}(M_{0,n}) \simeq S_n$ for $n \geq 5$, where the permutation group S_n describes the permutation of the marked points. For $n = 4$, $\mathrm{Aut}(M_{0,4}) \simeq S_3$, where S_3 can be viewed as permuting the points 0, 1, ∞ after the identification $M_{0,4} \simeq \mathbf{P}^1_t \setminus \{0,1,\infty\}$. The group $\mathrm{Aut}(M_{0,4})$ is generated by the 2- and 3-cycles $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ acting as $\boldsymbol{\theta} : t \mapsto 1-t$ and $\boldsymbol{\omega} : t \mapsto (1-t)^{-1}$, which have been used in [LS2] and [NT]. They induce automorphisms of the geometric fundamental group $\hat{F}_2$ of $M_{0,4}$, which we denote θ, ω. The main point of the present paper is to achieve, in the two-dimensional case of $M_{0,5}$, some results already proved for $M_{0,4}$ (cf. [NT]); this will require substantially more complicated computations. The important new automorphism in this situation is the 5-cycle in $S_5 \simeq \mathrm{Aut}(M_{0,5})$; indeed, S_5 is generated by the 5-cycle together with the stabilizer of any of the 5 points. It turns out to be convenient to use the cube of the standard 5-cycle; we set $\boldsymbol{\rho} = (14253) \in \mathrm{Aut}(M_{0,5})$, and denote by $\rho = \boldsymbol{\rho}_*$ the corresponding automorphism of Γ_0^5 and $\hat{\Gamma}_0^5$.

We may now recall the following

Theorem 1 ([LS2], Theorem 2)**.** *Let $\sigma \in \widehat{GT}$ with parameters $(\lambda, f) = (\lambda_\sigma, f_\sigma)$. Then there exist elements g and $h \in \hat{F}_2$ and $k \in \hat{\Gamma}_0^5$ such that we have the following equalities, of which the first two take place in $\hat{F}_2$ and the third in $\hat{\Gamma}_0^5$:*

$$\text{(I}'\text{)} \qquad f = \theta(g)^{-1} g$$

$$\text{(II}'\text{)} \qquad f x^{\frac{\lambda-1}{2}} = \begin{cases} \omega(h)^{-1} h & \textit{if } \lambda \equiv 1 \textit{ mod } 3, \\ \omega(h)^{-1}\, y^{-1}\, h & \textit{if } \lambda \equiv -1 \textit{ mod } 3; \end{cases}$$

$$\text{(III}'\text{)} \qquad f(x_{12}, x_{23}) = \begin{cases} \rho(k)^{-1} k & \textit{if } \lambda \equiv \pm 1 \textit{ mod } 5, \\ \rho(k)^{-1}\, x_{34} x_{51}^{-1} x_{45} x_{12}^{-1}\, k & \textit{if } \lambda \equiv \pm 2 \textit{ mod } 5. \end{cases}$$

Note that in fact, $x_{34}x_{51}^{-1}x_{45}x_{12}^{-1} = x_{25}x_{45}$ in Γ_0^5. The first expression was emphasized in [LS2] because it uses the same generators $x_{i,i+1}$ as relation (III) above.

The elements g, h and k are closely connected with the $\widehat{GT}$-action on certain paths joining the standard tangential base point to points of the moduli spaces $M_{0,4}$ and $M_{0,5}$ with special automorphism. In the simplest case, looking at $M_{0,4}$, we see that the point $\frac{1}{2}$ is a fixed point of $\boldsymbol{\theta}$ and g actually describes the action of $\widehat{GT}$ on the path joining $\overrightarrow{01}_t$ to that point. We refer to [LS2] for more in this direction.

The problem that then naturally arises is whether it is possible to express the elements g, h and k in terms of f. Put in a more general way: Is the $\widehat{GT}$-action on the groupoid based at the automorphism points (whose very existence is part of Theorem 1) computable in terms of the action based at infinity (which lead to the definition of $\widehat{GT}$ in the first place). For Galois elements σ, this question was answered in the one-dimensional case of $M_{0,4}$, i.e. the elements g and h associated to σ were expressed in terms of f, as follows.

Theorem 2 ([NT])**.** *Let $\hat{B}_3$ be the profinite braid group generated by the symbols τ_1, τ_2 with the defining relation $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$. For an integer $a > 1$, let $\rho_a : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}$ be the Kummer 1-cocycle defined by $(\sqrt[n]{a})^{\sigma-1} = \zeta_n^{\rho_a(\sigma)}$*

($\sigma \in G_{\mathbb{Q}}$, $n \geq 1$, $\zeta_n = \exp(2\pi i/n)$). *Then, the image of* $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ *satisfies the following equations:*

$$(\mathrm{GF}_0) \qquad g(\tau_1^2, \tau_2^2) = \eta^{2\rho_2-\rho_3} f(\tau_1, \eta)\tau_1^{-2\rho_2+3\rho_3},$$

$$(\mathrm{GF}_1) \qquad g(\tau_1^2, \tau_2^2) = f(\tau_1^2, \eta)\tau_1^{4\rho_2},$$

$$(\mathrm{HF}_0) \qquad h(\tau_1^2, \tau_2^2) = (\xi_\pm)^{\rho_2+\frac{\lambda\mp 1-6\rho_3}{4}} f(\tau_1, \xi_\pm)\tau_1^{3\rho_3-2\rho_2-\frac{\lambda\mp 1}{2}},$$

$$(\mathrm{HF}_1) \qquad h(\tau_1^2, \tau_2^2) = (\xi_\pm)^{\frac{\lambda\mp 1-6\rho_3}{4}} f(\tau_1^2, \xi_\pm)\tau_1^{3\rho_3-\frac{\lambda\mp 1}{2}},$$

where, in the first two equations η *denotes* $\tau_1\tau_2\tau_1$, *and in the last two equations,* ξ_+, ξ_- *denote* $\tau_1\tau_2$, $\tau_2\tau_1$ *respectively, and the sign* $\mp$ *is taken according as* $\lambda \equiv \pm 1$ *mod 6 respectively.*

In the statement above, the elements τ_1^2 and τ_2^2 generate a free profinite group (a copy of $\hat{F}_2$) which one can regard as the profinite fundamental group of $M_{0,4}/\overline{\mathbb{Q}}$. Note that putting the relations of Theorem 2 together yields several relations involving f alone; it is not known whether any of these hold true in all of $\widehat{GT}$. For instance, one can insert g and h (as given by (GF_1) and (HF_1)) into their respective defining properties in Theorem 1. This produces the following relations:

Theorem 3 ([NT], Corollary C)**.** *With notation as in Theorem 2, the following equations hold for the image of* $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$*:*
(I′) (Harmonic equation) $\quad f(\tau_1^2, \tau_2^2) = \tau_2^{-4\rho_2} f(\tau_2^2, \eta)^{-1} f(\tau_1^2, \eta)\tau_1^{4\rho_2}.$
(II′) (Equianharmonic equation)

$$f(\tau_1^2, \tau_2^2) = \tau_2^{-3\rho_3-\frac{\lambda-1}{2}} f(\tau_2^2, \tau_1\tau_2)^{-1}(\tau_1\tau_2)^{\frac{\lambda-1}{2}} f(\tau_1^2, \tau_1\tau_2)\tau_1^{3\rho_3-\frac{\lambda-1}{2}}.$$

The main goal of the present paper is to prove analogs of these two theorems in the two-dimensional case, i.e. to express the element $k \in \hat{\Gamma}_0^5$ of theorem 1 in terms of the parameter f, and to use this expression to obtain a new relation on the parameter f. As in dimension 1, we achieve this only in the Galois case, where the element (λ, f) lies in $G_{\mathbb{Q}} \subset \widehat{GT}$, the case for general elements of $\widehat{GT}$ being still unknown. The two-dimensional case is not only substantially more involved, it also requires a new approach. In this paper we will make use of a particular locus (actually a curve) of the sort more generally defined in [L], to which we refer for motivation and more on the subject. We make use of the basic idea that one can use the naturality of the Galois action in order to get relations which may or may not be satisfied by the whole of $\widehat{GT}$. Concretely speaking, and to take a simple but typical case, if $\mathcal{E}$ is a (marked hyperbolic) curve defined over $\mathbb{Q}$ with a morphism $\phi : \mathcal{E} \to M_{g,n}$ defined over the maximal cyclotomic field $\mathbb{Q}^{\mathrm{ab}}$, the equivariance of the outer $G_{\mathbb{Q}^{\mathrm{ab}}}$-action leads to the commutativity condition:

$\sigma \circ \phi_* = \phi_* \circ \sigma$ for any $\sigma \in G_{\mathbb{Q}^{\mathrm{ab}}}$, where ϕ_* is the morphism induced by ϕ on the geometric fundamental groups. If one is willing to make the base points precise, as we indeed will, one gets a commutativity condition on automorphisms of $\hat{\Gamma}_{g,n}$. In this paper, we actually work out the computation over $\mathbb{Q}$; then, the above equivariance has to be refined as $\sigma \circ \phi_* = (\phi^\sigma)_* \circ \sigma$. This leads us not only to deal with more elaborate considerations on braid groups, but also to deal with two-dimensional tangential base points at "symmetric points" on the relevant moduli space (in particular, these points are located "far from infinity"; cf. §3).

Let us now specialize to $g = 0$, $n = 5$ and choose a particular $\mathcal{E}$. In our case ϕ will be generically injective, so let us somewhat informally identify $\mathcal{E}$ with its image in $M_{0,5}$. Our choice of $\mathcal{E}$ will ensure the following properties: First $\mathcal{E}$ is (globally) stable under the action of $\boldsymbol{\rho}$; second the projection of $\mathcal{E}$ to $M_{0,[5]}$ is a projective line with three marked (or deleted) points. The first property implies that one of the marked points corresponds to a marked sphere with 5-cyclic symmetry; the second ensures that we 'know' the Galois action on the fundamental groupoid of the projection. These two key properties are what make it possible for us to prove two-dimensional analogs of theorems 2 and 3 above.

Let us say a few words about how we came up with a curve satisfying these properties, referring to [L] for a broader picture. Note that curves of this type, which map to a projective line minus several points in the ordered moduli space $M_{0,5}$ and descend to a projective line minus three points in $M_{0,[5]}$, have also been studied in some detail in [T]; however that paper concentrates on the case where everything is defined over $\mathbb{Q}$, which is not the case of our locus $\mathcal{E}$, defined over $\mathbb{Q}(\zeta_5)$.

The geometry here is best understood by going up to the Teichmüller space $\mathcal{T}_{0,5}$. Choosing a preimage $\mathcal{U}$ of $\mathcal{E}$, one can show that $\mathcal{U} \subset \mathcal{T}_{0,5}$ is actually a geodesic disk, that is a copy of the unit disk (or the Poincaré upper half-plane) on which the Teichmüller metric coincides with the Poincaré metric. The automorphism $\boldsymbol{\rho}$ acts on $\mathcal{U}$, and it acts geodesically for the Teichmüller metric; since the latter coincides with the Poincaré metric on $\mathcal{U}$ and since $\boldsymbol{\rho}$ has finite order, it is actually a rotation, the center $O \in \mathcal{U}$ corresponding to a marked sphere with 5-symmetry. Finally, $\boldsymbol{\rho}$ also acts linearly on the tangent space to $\mathcal{T}_{0,5}$ at O, and the tangent vector to $\mathcal{U}$ at that point is an *eigenvector* for that linearized action. Thus, forgetting about Teichmüller space, one can think of $\mathcal{E}$ (whose precise definition is given in §2.3 below) as an *eigenlocus*, that is, a certain arithmetic (i.e. defined over a number field) geodesic curve in $M_{0,5}$.

We now have to prepare some notation before stating our main result, giving the value of k_σ in terms of f_σ for any $\sigma \in G_{\mathbb{Q}}$. Throughout this paper, we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ once and for all. We write $\zeta_n = e^{2\pi i/n}$ and $\mu_n = \{1, \zeta_n, \dots, \zeta_n^{n-1}\}$.

Definition. We define a pair of Kummer characters $\chi_{13}, \chi_{45} : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}$ by

$$\{\sigma(\sqrt[n]{u}),\ \sigma(\sqrt[n]{v})\} = \{\zeta_{2n}^{\chi_{13}(\sigma)}\sqrt[n]{u},\ \zeta_{2n}^{\chi_{45}(\sigma)}\sqrt[n]{v}\},$$

where $u = \frac{1}{2}(1-\zeta_5)^5$, $v = \frac{1}{2}(1-\zeta_5^2)^5$ and the n-th roots denote the values which are closest to 1 (principal branches).

Note that the set $\{u, v, -u, -v\}$ of purely imaginary numbers forms a $G_{\mathbb{Q}}$-orbit in $\overline{\mathbb{Q}}$, so that each of their n-th roots can be written uniquely in the form $\zeta_{2n}^a\sqrt[n]{u}$ or $\zeta_{2n}^b\sqrt[n]{v}$ $(a, b \in \mathbb{Z})$. Later, in §5.10 we will discuss natural extensions of χ_{13}, χ_{45} to functions from $\widehat{GT}$ to $\hat{\mathbb{Z}}$ by using Ihara's theory [I2].

Let $\vartheta = \tau_1\tau_2\tau_3\tau_4$ denote a standard order 5 element in $\Gamma_0^{[5]}$. We note that the automorphism ρ of $\Gamma_0^{[5]}$ is realized by conjugation by ϑ^3, namely, $\rho(*) = \vartheta^3(*)\vartheta^{-3}$.

We also need to introduce several specific braids which play important roles in later sections. First we define the two involutive braids ε and ε'

$$\varepsilon = \tau_1\tau_2\tau_1\tau_4^{-1}, \qquad \varepsilon' = \tau_3^{-1}\varepsilon\tau_3.$$

Then, define

$$V_\lambda := \begin{cases} 1 & (\lambda \equiv \pm 1 \bmod 5); \\ V_\sharp := \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\vartheta^2 & (\lambda \equiv \pm 2 \bmod 5), \end{cases}$$

$$\varepsilon_\lambda := \begin{cases} 1 & (\lambda \equiv 1 \bmod 5), \\ \varepsilon & (\lambda \equiv -1 \bmod 5), \\ \varepsilon' & (\lambda \equiv -2 \bmod 5), \\ \varepsilon \cdot \varepsilon' & (\lambda \equiv 2 \bmod 5), \end{cases}$$

Finally, ϑ_λ and Ω_λ are defined by

$$\begin{cases} \vartheta_\lambda & := \vartheta^{3-2\lambda^2}(\tau_2\tau_3)\vartheta^{2\lambda^2-2}(\tau_2\tau_3)^{-1}, \\ \Omega_\lambda & := \vartheta^{\lambda^3+\lambda+3}(\tau_2\tau_3)\vartheta^{\lambda-\lambda^3}(\tau_2\tau_3)^{-1}. \end{cases}$$

This being said, we can finally state

Theorem A. *For $\sigma \in G_{\mathbb{Q}}$ and $k = k(\{x_{ij}\})$ as in theorem 1, we have*

$$k(\{x_{ij}\}) = V_\lambda \varepsilon_\lambda \vartheta_\lambda^{-6\rho_2} f(\tau_{13}\tau_{45}, \vartheta_\lambda^3)\tau_{13}^{\chi_{13}}\tau_{45}^{\chi_{45}} f(x_{12}, x_{13}) x_{12}^{\frac{\lambda-1}{2}}.$$

We write $\lambda = \lambda_\sigma (= \chi(\sigma))$, $\rho_2 = \rho_2(\sigma)$ etc. Since this formula may appear slightly off putting at first sight, it may be useful to graphically isolate its core by stripping it from the cyclotomic and Kummer characters. In fact, for any σ belonging to the large closed subgroup of $G_{\mathbb{Q}}$ defined by $\chi(\sigma) = 1$, $\rho_2(\sigma) = \chi_{13}(\sigma) = \chi_{45}(\sigma) = 0$, we simply get:

$$k_\sigma(\{x_{ij}\}) = f(\tau_{13}\tau_{45}, \vartheta^3) f(x_{12}, x_{13}),$$

where we recall that $\vartheta = \tau_1\tau_2\tau_3\tau_4$ is a standard order 5 rotation. This skeletal form shows the geometric significance of the formula more clearly; indeed, using the curve $\mathcal{E}$, it is not difficult to establish geometrically, as we show in §5 below. The reader might want to concentrate at first reading on this simple case, in which all the local factors are trivial.

Just as Theorem 3 above is deduced from Theorem 2, one can derive from Theorem A a relation involving f only, which is satisfied for any $\sigma \in G_{\mathbb{Q}}$ and which may or may not be satisfied for all $\sigma \in \widehat{GT}$.

Theorem B. *The following "pentaharmonic" equation holds for every element $\sigma \in G_{\mathbb{Q}}$:*

$$\begin{aligned} f(x_{12}, x_{23}) = {} & x_{45}^{\frac{1-\lambda}{2}} f(x_{14}, x_{45}) \tau_{23}^{-\chi_{45}} \tau_{14}^{-\chi_{13}} f(\rho(\vartheta_\lambda^3), \tau_{23}\tau_{14}) \\ & \cdot \Omega_\lambda \cdot f(\tau_{45}\tau_{13}, \vartheta_\lambda^3)\tau_{13}^{\chi_{13}}\tau_{45}^{\chi_{45}} f(x_{12}, x_{13}) x_{12}^{\frac{\lambda-1}{2}}. \end{aligned}$$

In some sense, the above formula comes from a decomposition of the standard pentagon into 5 pieces which are permuted under the action of $\boldsymbol{\rho}$. Since relation (III) in the original definition of $\widehat{GT}$ comes directly from the simple connectedness of that pentagon, the equation above deserves to be denoted $(\frac{1}{5}\mathrm{III})$; indeed, taking its five versions (under the action of $\boldsymbol{\rho}$) together does yield the original relation (III).

Theorem B is a direct corollary of Theorem A and Theorem 1 (III′) above which gives f in terms of k. One also has to make use of the following braid-theoretic lemma which we include here for frequent reference throughout the rest of the article. We skip the proof, which is an easy matter of checking the identities algebraically, or alternatively, by braiding actual strands.

Lemma (1.1).

(1) $\rho(V_\sharp)\vartheta^{-1}V_\sharp^{-1} = x_{25}x_{45} = x_{34}x_{51}^{-1}x_{45}x_{12}^{-1}$.

(2) $\varepsilon^2 = \varepsilon'^2 = 1, \quad (\varepsilon\varepsilon')^2 \neq 1, \quad \varepsilon' = \tau_2\tau_3\vartheta^2 = \tau_2\vartheta^2\tau_1,$

(3) $\varepsilon\vartheta\varepsilon = \vartheta^{-1}, \quad \varepsilon\tau_{13}\varepsilon = x_{12}\tau_{13}x_{12}^{-1}, \quad \varepsilon\tau_{45}\varepsilon = \tau_{45}, \quad \varepsilon'\tau_{13}\varepsilon' = \tau_{45}.$

(4) $\vartheta_\lambda^5 = 1;$ *in fact,* $\vartheta_\lambda = \begin{cases} \vartheta & (\lambda \equiv \pm 1 \ mod\ 5); \\ \varepsilon'\vartheta\varepsilon' & (\lambda \equiv \pm 2 \ mod\ 5). \end{cases}$

(5) $\Omega_\lambda = \rho(\varepsilon_\lambda)^{-1}\vartheta^{3\lambda^2-3}\varepsilon_\lambda;$ *in particular,* $\vartheta = \rho(\varepsilon)^{-1}\varepsilon$. $\square$

2. A D_5-covering of the projective line and eigenloci in $M_{0,5}$

(2.1) In this section, we consider the covering map $\beta : \mathbf{P}^1_t \to \mathbf{P}^1_u$ of the projective lines defined by

$$u = \beta(t) = \frac{4t^5}{(t^5+1)^2}.$$

Over $\mathbb{C}$, this is a Galois cover with Galois group isomorphic to the dihedral group D_5 of order 10. It is ramified only over $u = 0, 1, \infty$ with preimages $\{0, \infty\}$, μ_5, $-\mu_5$ whose ramification indices are $5, 2, 2$ respectively. The restriction of β to $\mathcal{L}_t := \mathbf{P}^1 - \mu_5$ allows us to regard $\mathcal{L}_u := \mathbf{P}^1_u - \{1\}$ as an orbifold quotient "$\mathbf{P}^1_{(5,\infty,2)}$" of $\mathcal{L}_t$ which has fundamental group isomorphic to the triangle group

$$\Delta(5, \infty, 2) = \langle x_u, y_u, z_u \mid x_u y_u z_u = x_u^5 = z_u^2 = 1 \rangle.$$

(2.2) Now consider $\beta : \mathbf{P}^1_t \to \mathbf{P}^1_u$ as a $\mathbb{Q}$-morphism between the projective lines $\mathbf{P}^1_t$ and $\mathbf{P}^1_u$. Let p be the path from $\overrightarrow{01}_u$ to $\overrightarrow{10}_u$ along the real axis on $\mathcal{L}_u$ and q be the path from $\overrightarrow{01}_t$ to $\overrightarrow{10}_t$ along the real axis on $\mathcal{L}_t$. By taking the Taylor expansions of $\beta(t)$ at $t = 0$ and $t = 1$, we find that the tangential base points are mapped to:

$$\beta(\overrightarrow{01}_t) = \frac{1}{4}\overrightarrow{01}_u, \quad \beta(\overrightarrow{10}_t) = \frac{4}{25}\overrightarrow{10}_u.$$

Generally, for any positive rational number α, if δ denotes the infinitesimal real segment connecting $\alpha\overrightarrow{01}_u$ to $\overrightarrow{01}_u$, then for all $\sigma \in G_\mathbb{Q}$, it is easy to check that $\sigma(\delta) = \delta x_u^{\rho_\alpha}$, where $\rho_\alpha = \rho_\alpha(\sigma)$ is the Kummer character on positive roots of α; similarly, if δ goes from $\alpha\overrightarrow{10}_u$ to $\overrightarrow{10}_u$, we have $\sigma(\delta) = \delta p^{-1} y_u^{\rho_\alpha} p$. In our situation, write δ_1 for the real segment from $\frac{1}{4}\overrightarrow{01}_u$ to $\overrightarrow{01}_u$ and δ_2 from $\frac{4}{25}\overrightarrow{10}_u$ to $\overrightarrow{10}_u$. Then, writing $p' = \beta(q) = \delta_1 p \delta_2^{-1}$ and using $\sigma(p) = f(y_u, x_u)p$, we find that

$$\sigma(p')p'^{-1} = \delta_1 x_u^{-2\rho_2} f(y_u, x_u) y_u^{-2\rho_2+2\rho_5} \delta_1^{-1} \tag{2.2.1}$$

as an equality between loops in $\pi_1(\mathcal{L}_u, \frac{1}{4}\overrightarrow{01}_u)$. We refer to [N-I,II],[NT] for more details on this sort of computation.

(2.3) Let $M_{0,5}$ (resp. $M_{0,[5]}$) be the moduli stack of the projective lines with 5 ordered (resp. unordered) marked points. We shall write a point $(\mathbf{P}^1; a_1, \dots, a_5)$ of $M_{0,5}$ (resp. a point $(\mathbf{P}_1; \{a_1, \dots, a_5\})$ of $M_{0,[5]}$) simply as $(a_1, \dots, a_5)$ (resp. $\{a_1, \dots, a_5\}$). The obvious symmetrization of the marked points gives an etale cover (in the sense of stacks) $M_{0,5} \to M_{0,[5]}$.

We wish to fit the basic D_5-cover $\mathcal{L}_t \to \mathcal{L}_u$ inside in this cover $M_{0,5} \to M_{0,[5]}$, by starting with the locus

$$\mathcal{E}_1 = (\zeta^4 + \zeta^{-4}t, 1 + t, \zeta + \zeta^{-1}t, \zeta^2 + \zeta^{-2}t, \zeta^3 + \zeta^{-3}t)$$

in $M_{0,5}$, where $\zeta = \zeta_5 = \exp(2\pi i/5)$. This locus is globally invariant under the action of the dihedral group $D_5 \subset S_5 = \mathrm{Aut}(M_{0,5})$ generated by (12345) and (13)(45). Over $\mathbb{C}$, it is isomorphic to a copy of $\mathbf{P}^1_{\mathbb{C}} - \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$ parametrized by t; when t takes the five values $1, \zeta, \zeta^2, \zeta^3, \zeta^4$, the locus meets a point at infinity of maximal degeneration in $M_{0,5}$ as illustrated as the following table.

t |	1	ζ	ζ^2	ζ^3	ζ^4
|	(13)(45)	(14)(23)	(24)(51)	(25)(34)	(35)(12)

Here, $(ij)(kl)$ means the maximal degeneration point of the stable 5-pointed $\mathbf{P}^1$-tree such that the i-th and j-th marked points (resp. k-th and l-th marked points) coincide. The locus $\mathcal{E}_1$ is self-crossing at $t = 0, \infty$. It is a cover of degree 10 of its image in the unordered moduli space $M_{0,[5]} = M_{0,5}/S_5$.

(2.4) Since the symmetric functions of the coordinates of points of $\mathcal{E}_1$ are in $\mathbb{Q}(t)$, the image of $\mathcal{E}_1$ in $M_{0,[5]}$ is invariant under the action of $G_{\mathbb{Q}}$. However, $\mathcal{E}_1$ itself is not invariant under the action of $G_{\mathbb{Q}}$. It is mapped to the loci

$$\begin{aligned}
\mathcal{E}_1 =& (\zeta^4 + \zeta^{-4}t, 1 + t, \zeta + \zeta^{-1}t, \zeta^2 + \zeta^{-2}t, \zeta^3 + \zeta^{-3}t,);\\
\mathcal{E}_{-1} =& (\zeta^{-4} + \zeta^{4}t, 1 + t, \zeta^{-1} + \zeta t, \zeta^{-2} + \zeta^{2}t, \zeta^{-3} + \zeta^{3}t);\\
\mathcal{E}_2 =& (\zeta^3 + \zeta^{-3}t, 1 + t, \zeta^2 + \zeta^{-2}t, \zeta^4 + \zeta^{-4}t, \zeta + \zeta^{-1}t);\\
\mathcal{E}_{-2} =& (\zeta^{-3} + \zeta^{3}t, 1 + t, \zeta^{-2} + \zeta^{2}t, \zeta^{-4} + \zeta^{4}t, \zeta^{-1} + \zeta t)
\end{aligned}$$

according to whether $\lambda_\sigma \equiv 1, -1, 2, -2 \bmod 5$ respectively. Let us denote by ψ_i the morphism $\mathcal{L}_t \to \mathcal{E}_i \subset M_{0,5}$ for $i = \pm 1, \pm 2$ so that ${}^\sigma\psi_1 = \psi_{\lambda_\sigma (\mathrm{mod}\ 5)}$

($\sigma \in G_{\mathbb{Q}}$). Their compositions with the projection $\pi : M_{0,5} \to M_{0,[5]}$ all coincide, and the four identical maps $\pi \circ \psi_i$ factor through β, so that they can be written $\psi \circ \beta$ for a certain morphism $\psi : \mathcal{L}_u \to M_{0,[5]}$. In summary, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_t & \xrightarrow{\psi_i} & \mathcal{E}_i \subset M_{0,5} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{L}_u & \xrightarrow{\psi} & M_{0,[5]} \end{array}$$

for each $i \in \{\pm 1, \pm 2\}$. All the morphisms here except for ψ_i are defined over $\mathbb{Q}$.

For each $i \in \{\pm 1, \pm 2\}$, let π_*^i denote the homomorphism

$$\pi_*^i : \pi_1(M_{0,5}, \psi_i(\overrightarrow{01}_t)) \to \pi_1(M_{0,5}, \psi(\frac{1}{4}\overrightarrow{01}_u))$$

corresponding to the projection π. Because $\beta(\overrightarrow{01}_t) = \frac{1}{4}\overrightarrow{01}_u$, the above diagram translates into the top square of the following diagram of fundamental groups:

$$\begin{array}{ccc} \pi_1(\mathcal{L}_t, \overrightarrow{01}_t) & \xrightarrow{(\psi_i)_*} & \pi_1(M_{0,5}, \psi_i(\overrightarrow{01}_t)) \\ {\scriptstyle\beta_*}\downarrow & & \downarrow{\scriptstyle\pi_*^i} \\ \pi_1(\mathcal{L}_u, \frac{1}{4}\overrightarrow{01}_u) & \xrightarrow{\psi_*} & \pi_1(M_{0,[5]}, \psi(\frac{1}{4}\overrightarrow{01}_u)) \\ {\scriptstyle\mathrm{inn}(\delta_1)}\downarrow & & \downarrow{\scriptstyle\mathrm{inn}(\psi(\delta_1))} \\ \pi_1(\mathcal{L}_u, \overrightarrow{01}_u) & \xrightarrow{\widetilde{\psi}} & \pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u)) \end{array}$$

where

$$\mathrm{inn}(\delta_1) : \pi_1(\mathcal{L}_u, \tfrac{1}{4}\overrightarrow{01}_u) \to \pi_1(\mathcal{L}_u, \overrightarrow{01}_u))$$

$$\text{(resp.} \qquad \mathrm{inn}(\psi(\delta_1)) : \pi_1(M_{0,[5]}, \psi(\tfrac{1}{4}\overrightarrow{01}_u)) \to \pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u)) \quad)$$

is the obvious isomorphism obtained by composing each loop with the path δ_1 from $\frac{1}{4}\overrightarrow{01}_u$ to $\overrightarrow{01}_u$ (resp. the path $\psi(\delta_1)$ from $\psi(\frac{1}{4}\overrightarrow{01}_u)$ to $\psi(\overrightarrow{01}_u)$), and $\widetilde{\psi}$ is defined to be $\mathrm{inn}(\delta_1^{-1}) \circ \psi_* \circ \mathrm{inn}(\psi(\delta_1))$.

Writing $\widetilde{\beta} = \beta_* \circ \mathrm{inn}(\delta_1)$, $\widetilde{\pi}_i = \pi_*^i \circ \mathrm{inn}(\psi(\delta_1))$ and (by a slight abuse) ψ_i instead of $(\psi_i)_*$, we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\mathcal{L}_t, \overrightarrow{01}_t) & \xrightarrow{\psi_i} & \pi_1(M_{0,5}, \psi_i(\overrightarrow{01}_t)) \\ {\scriptstyle\widetilde{\beta}}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}_i} \\ \pi_1(\mathcal{L}_u, \overrightarrow{01}_u) & \xrightarrow{\widetilde{\psi}} & \pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u)) \end{array}$$

which we will use to place the equality (2.2.1) inside the fundamental group $\pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u))$.

Lemma (2.5). *Fix $\sigma \in G_{\mathbb{Q}}$ and take $i \in \{\pm 1, \pm 2\}$ congruent to λ_σ mod 5. Then*

$$\widetilde{\pi}_i\big(\sigma(\psi_1(q)) \cdot \psi_i(q)^{-1}\big) = \widetilde{\psi}(x_u)^{-2\rho_2} f(\widetilde{\psi}(x_u), \widetilde{\psi}(y_u))^{-1} \widetilde{\psi}(y_u)^{2\rho_5 - 2\rho_2}$$

holds as an equality of loops of $\pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u))$.

Proof. We have $p' = \beta(q)$, so the loop $\sigma(p')p'^{-1} \in \pi_1(\mathcal{L}_u, \frac{1}{4}\overrightarrow{01}_u)$ on the left-hand side of (2.2.1) can be written $\sigma(\beta(q))\beta(q)^{-1} = \beta_*\big(\sigma(q)q^{-1}\big)$, since σ commutes with β_*. Thus, (2.2.1) becomes the equality

$$\beta_*\big(\sigma(q)q^{-1}\big) = \delta_1 x_u^{-2\rho_2} f(y_u, x_u) y_u^{-2\rho_2 + 2\rho_5} \delta_1^{-1}$$

in $\pi_1(\mathcal{L}_u, \frac{1}{4}\overrightarrow{01}_u)$. Applying $\mathrm{inn}(\delta_1)$ to both sides yields

$$\widetilde{\beta}\big(\sigma(q)q^{-1}\big) = x_u^{-2\rho_2} f(y_u, x_u) y_u^{-2\rho_2 + 2\rho_5}$$

in $\pi_1(\mathcal{L}_u, \overrightarrow{01}_u)$. Applying $\widetilde{\psi}$ to both sides to map the equality into $\pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u))$, and noting that σ commutes with $\widetilde{\psi}$ and that by the commutative diagram, $\widetilde{\beta}\widetilde{\psi} = \psi_i \widetilde{\pi}_i$, we obtain

$$\widetilde{\pi}_i\big(\psi_i(\sigma(q)q^{-1})\big) = \widetilde{\psi}(y_u)^{2\rho_2 - 2\rho_5} f(\widetilde{\psi}(x_u), \widetilde{\psi}(y_u)) \widetilde{\psi}(x_u)^{2\rho_2}.$$

The right-hand side is as in the statement, and using $\psi_i(\sigma(q)) = \sigma(\psi_1(q))$, the left-hand side is equal to $\widetilde{\pi}_i\big(\sigma(\psi_1(q))\psi_i(q)^{-1}\big)$, which proves the lemma. □

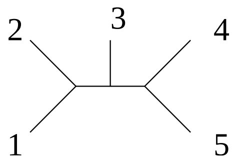
(2.6) For later applications, we need to know more about the starting and endpoints of the paths $\psi_i(q)$ ($i = \pm 1, \pm 2$) in $M_{0,5}(\mathbb{C})$. We saw that these paths are different lifts of the same image in $M_{0,[5]}$, whose endpoints are $\psi(\overrightarrow{01}_u)$ and $\psi(\overrightarrow{10}_u)$. As points of $M_{0,5}(\mathbb{C})$, we have $\psi_1(0) = \psi_{-1}(0)$ and $\psi_2(0) = \psi_{-2}(0)$; the former standard 5-cyclic point we denote by $Q_1 := (1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ and the latter "antipode" 5-cyclic point by $Q_2 := (1, \zeta^3, \zeta, \zeta^4, \zeta^2)$.

The paths $\psi_1(q)$ and $\psi_{-1}(q)$ start at the same point Q_1 but with different directions $\psi_1(\overrightarrow{01}_t)$, $\psi_{-1}(\overrightarrow{01}_t)$. This has to be precisely estimated especially when the ramification in $M_{0,5} \to M_{0,[5]}$ is involved in arguments (see (3.2) below). When $t \to 1$, both paths approach the same maximal degeneration point (13)(45), but the endpoints $\psi_1(\overrightarrow{10}_t)$ and $\psi_{-1}(\overrightarrow{10}_t)$ differ from each other as tangential base points near (13)(45); we discuss this in detail in §4. Similarly, the paths $\psi_2(q)$ and $\psi_{-2}(q)$ start from the same point Q_2 with different directions $\psi_2(\overrightarrow{01}_t)$, $\psi_{-2}(\overrightarrow{01}_t)$. They also approach the maximal degeneration point (13)(45) when $t \to 1$, but again as shown in §4, they end with different tangential base points $\psi_2(\overrightarrow{10}_t)$ and $\psi_{-2}(\overrightarrow{10}_t)$.

The tangential directions $\psi_i(\overrightarrow{01}_t)$, $\psi_i(\overrightarrow{10}_t)$ can be explicitly computed using the explicit expressions of the $\mathcal{E}_i$, cf. (3.2). See the figure in (3.5) for a visualization of these base points.

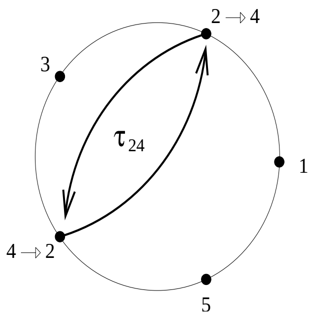
3. Definitions of necessary paths in $M_{0,5}$

(3.1) In this section, we shall consider the images of the fundamental groups $\pi_1(\mathcal{E}_i)$ $(i = \pm 1, \pm 2)$ in $\pi_1(M_{0,5})$ by fixing base points and paths connecting them. We employ the standard tangential base point $\mathcal{A}$ near the maximal degenerate point (12)(45) on the moduli space $M_{0,5}$ introduced by Ihara; this base point corresponds to the planar tree



and can be represented by the point $\mathcal{A} = (1 - \epsilon, 1, \infty, 0, \delta)$, where ϵ and δ are small real numbers.

We introduce the standard braid $\tau_{ij}(i, j \in \mathbb{Z}/5\mathbb{Z}, i \neq j)$ of $\pi_1(M_{0,[5]}(\mathbb{C}), \mathcal{A})$, which interchanges the marked points a_i and a_j counterclockwise on $\mathbf{P}^1(\mathbb{R})$ (in the figure below, the circle represents $\mathbf{P}^1(\mathbb{R})$ and its interior represents the upper half-plane). Note in particular that the $\tau_i := \tau_{i,i+1}$ $(i = 1, \ldots, 4)$ generate $\pi_1(M_{0,[5]}(\mathbb{C}), \mathcal{A})$.



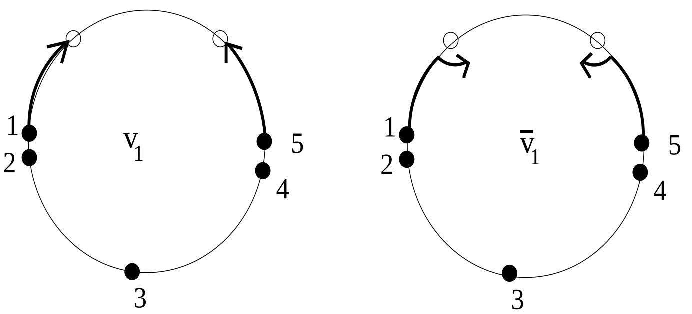
A generating system of the pure part $\pi_1(M_{0,5}(\mathbb{C}), \mathcal{A})$ is given by the collection of braids $x_{ij} = \tau_{ij}^2$ $(i, j \in \mathbb{Z}/5\mathbb{Z})$ which coincide with those given in [N] §3. The generating system here is slightly different from that used in [LS1]. We will make their compatibilities clear in the Appendix.

(3.2) The points of maximal degeneration listed in the table of §2 are cyclically transformed into each other by applications of the automorphism

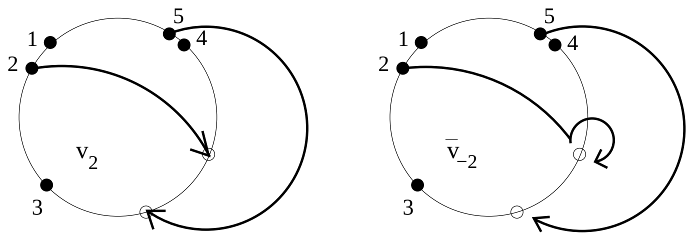
$$\boldsymbol{\rho} = (14253) \in \mathrm{Aut}(M_{0,5}).$$

The point of maximal degeneration (12)(45) and its images under $\boldsymbol{\rho}$ lie on the rim of the real 2-dimensional "pentagonal" region of $M_{0,5}$ consisting of the points corresponding to spheres $(\mathbf{P}^1; a_1, \ldots, a_5)$ with $a_1, \ldots, a_5$ lying on $\mathbb{R} \cup \{\infty\}$ in that cyclic order. The 5-cyclic point $Q_1 = (1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ lies in the center of this pentagon.

Letting v_1 be the unique (up to homotopy) path from $\mathcal{A}$ to Q_1 lying in this simply connected pentagon, conjugating by v_1 gives an isomorphism of $\pi_1(M_{0,5}, Q_1)$ with $\pi_1(M_{0,5}, \mathcal{A})$. As a movement of points on the sphere (actually sliding without crossing along the real axis), 'the' path v_1 from $\mathcal{A}$ to Q_1 is illustrated in the left-hand figure below. In this figure and all the following ones, the circle can be viewed either as the unit circle or as the real axis $\mathbf{P}^1(\mathbb{R})$ on $\mathbf{P}^1(\mathbb{C})$, seen from the north pole. Either way, $\mathbf{P}^1(\mathbb{C})$ equipped with the marked points shown on the circle corresponds to the same tangential base point or 5-cycle) point on $M_{0,5}$. Now, because Q_1 is a special orbifold point of $M_{0,[5]}$, when we wish to consider the image of v_1 as a path of the fundamental groupoid in the sense of stacks, we need to choose a tangential direction from which v_1 approaches Q_1. We take a "shortening" $\bar{v}_1$ of the path v_1, connecting $\mathcal{A}$ to the tangential base point $\psi_1(\overrightarrow{01}_t)$ at Q_1, as illustrated in the right-hand figure below.



Since the other 5-cyclic point Q_2 does not lie in the pentagon, we do not have a canonical choice of a path from $\mathcal{A}$ to $Q_2 = (\zeta^2, 1, \zeta^3, \zeta, \zeta^4)$. We choose such path v_2 given by motion of points on $\mathbf{P}^1$ as in the left-hand figure below, and also a "shortening" $\bar{v}_{-2}$ which starts at $\mathcal{A}$ and tangentially approaches the point Q_2 in the direction $\psi_{-2}(\overrightarrow{01}_t)$.



Further descriptions of $\bar{v}_1$ and $\bar{v}_{-2}$ will be given in the proof of Lemma (3.4).

(3.3) At this stage, we review the definition of $k_\sigma(\{x_{ij}\}) \in \hat{\Gamma}_0^5$ for $\sigma \in G_{\mathbb{Q}}$ which was introduced in [LS2] by the Galois transforms of the path v_1 in the fundamental groupoid of $M_{0,5}$. In this paper we employ another system of convention for paths and generators (cf. Appendix), so the definition looks slightly different from the original one in [LS2]. Firstly, if σ fixes the point Q_1 (i.e. if $\lambda_\sigma \equiv \pm 1 \bmod 5$), then we define k_σ by

$$\sigma(v_1) = k_\sigma(\{x_{ij}\})^{-1} \cdot v_1.$$

Next, if σ maps Q_1 to Q_2 (i.e. if $\lambda_\sigma \equiv \pm 2 \bmod 5$), then $\sigma(v_1)$ is a pro-path from $\mathcal{A}$ to Q_2. So in this case, we define k_σ by

$$\sigma(v_1) = k_\sigma(\{x_{ij}\})^{-1} \cdot v_2.$$

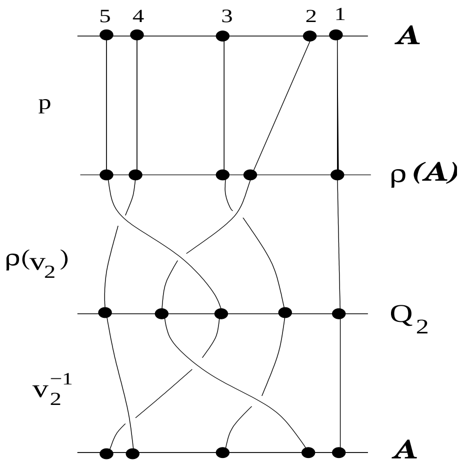
Now, we shall check that these k_σ satisfy property (III$'$) of Theorem 1. In the case $\lambda_\sigma \equiv \pm 1 \bmod 5$, it is a simple consequence of applying σ to the homotopy equivalence $p = v_1 \boldsymbol{\rho}(v_1)^{-1}$, where we recall that $\boldsymbol{\rho}$ is the automorphism $(14253) \in S_5$ of $M_{0,5}$, $\rho = \boldsymbol{\rho}_*$, and p denotes the standard path (edge of the pentagon) from $\mathcal{A}$ to $\boldsymbol{\rho}(\mathcal{A})$. Indeed, we have

$$f_\sigma(x_{12}, x_{23}) = p\,\sigma(p)^{-1} = p\,\boldsymbol{\rho}(\sigma(v_1))\sigma(v_1)^{-1} = p\boldsymbol{\rho}(k_\sigma^{-1})p^{-1}k_\sigma = \rho(k_\sigma)^{-1}k_\sigma.$$

Similarly, applying σ to $p = v_1\boldsymbol{\rho}(v_1)^{-1}$ when $\lambda_\sigma \equiv \pm 2 \bmod 5$ leads to

$$f_\sigma(x_{12}, x_{23}) = \rho(k_\sigma)^{-1}(p\,\boldsymbol{\rho}(v_2)v_2^{-1})k_\sigma.$$

It remains only to identify the loop $p\,\boldsymbol{\rho}(v_2)v_2^{-1}$ as an element of $\Gamma_0^5 = \pi_1(M_{0,5}(\mathbb{C}), \mathcal{A})$. Using the definition-drawing of v_2 given in (3.2), but drawing it "kinematically" as a braid moving downward with time, we find the following illustration of p, followed by $\boldsymbol{\rho}(v_2)$ followed by v_2^{-1} (cf. [LS2] p.592):



This braid is easily seen to be $x_{25}x_{45} = x_{34}x_{15}^{-1}x_{45}x_{12}^{-1}$. Thus, for all $\sigma \in G_{\mathbb{Q}}$, our $k_\sigma(\{x_{ij}\})$ satisfies the same property (III$'$) of Theorem 1 as the original k_σ of [LS2].

We next compute the Galois action on the path $\bar{v}_1$.

Lemma (3.4). *There exists a unique path s_1 (resp. s_2) from $\psi_1(\overrightarrow{01}_t)$ to $\psi_{-1}(\overrightarrow{01}_t)$ (resp. $\psi_{-2}(\overrightarrow{01}_t)$ to $\psi_2(\overrightarrow{01}_t)$) such that, for $\sigma \in G_{\mathbb{Q}}$, we have*

$$\sigma(\bar{v}_1) = \begin{cases} k_\sigma^{-1}\bar{v}_1 & (\lambda_\sigma \equiv 1 \ mod\ 5); \\ k_\sigma^{-1}\bar{v}_1 s_1 & (\lambda_\sigma \equiv -1 \ mod\ 5); \\ k_\sigma^{-1}\bar{v}_{-2} & (\lambda_\sigma \equiv -2 \ mod\ 5); \\ k_\sigma^{-1}\bar{v}_{-2} s_2 & (\lambda_\sigma \equiv 2 \ mod\ 5). \end{cases}$$

Proof. Introduce affine coordinates u, v of the structure ring of $M_{0,5}$ by $(0, u, 1, v^{-1}, \infty) \in M_{0,5}$. Then, on the locus near $\psi_i(\overrightarrow{01}_t)$ on $\mathcal{E}_i$ ($i \in \{\pm1, \pm2\} = (\mathbb{Z}/5\mathbb{Z})^\times$), these are expanded as $u = a_i + tf_i(t)$, $v = b_i + tg_i(t)$ in the ring $\mathbb{Q}(\zeta)[[t]]$, where $a_i, b_i \in \mathbb{Q}(\sqrt{5})$, $f_i(t), g_i(t) \in \mathbb{Q}(\sqrt{5})[[t]]$. For each i, the homomorphism

$$\begin{aligned} \mathbb{Q}(\sqrt{5})[[u - a_i, v - b_i]] &\longrightarrow \mathbb{Q}(\zeta)[[t]] : \\ u - a_i &\mapsto tf_i(t), \\ v - b_i &\mapsto tg_i(t) \end{aligned}$$

determines the location of $\psi_i(\overrightarrow{01}_t)$ near the local ring of Q_1 or Q_2 according to $i = \pm1$ or ±2. We then have a path l_i from the $\mathbb{Q}(\sqrt{5})$-rational point Q_1 or Q_2 to the $\mathbb{Q}(\zeta)$-rational tangential base point $\psi_i(\overrightarrow{01}_t)$ as the one corresponding to the specialization homomorphism $\mathbb{Q}(\zeta)[[t]] \to \mathbb{Q}(\zeta)$ ($t \mapsto 0$) for each $i \in (\mathbb{Z}/5\mathbb{Z})^\times$. From this definition, one sees that $\sigma(l_i) = l_{\lambda_\sigma i}$ for

$\sigma \in G_{\mathbb{Q}}$. The paths $\bar{v}_1$ and $\bar{v}_{-2}$ described in (3.2) are then formally defined by $\bar{v}_1 := v_1 \cdot l_1$, $\bar{v}_{-2} = v_2 \cdot l_{-2}$. Thus, for $\lambda_\sigma \equiv \pm 1$, we have

$$\sigma(\bar{v}_1) = \sigma(v_1)\sigma(l_1) = k_\sigma^{-1} v_1 l_{\lambda_\sigma},$$

and for $\lambda_\sigma \equiv \pm 2$, we have

$$\sigma(\bar{v}_1) = \sigma(v_1)\sigma(l_1) = k_\sigma^{-1} v_2 l_{\lambda_\sigma}.$$

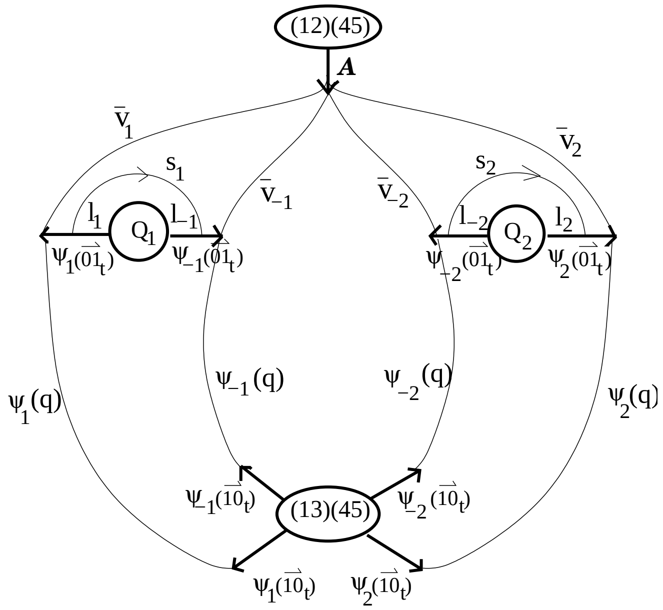
Thus, setting $s_1 := l_1^{-1} \cdot l_{-1}$ and $s_2 := l_{-2}^{-1} \cdot l_2$ give the desired properties. □

With regard to the above lemma, we make the following

Definition (3.5). Set $\bar{v}_{-1} := \bar{v}_1 s_1$, $\bar{v}_2 := \bar{v}_{-2} s_2$, so that

$$\sigma(\bar{v}_1) = k_\sigma^{-1} \bar{v}_i \quad \text{where} \quad i \equiv \lambda_\sigma \mod 5.$$

The paths $\bar{v}_1$, $\bar{v}_{-1}$, $\bar{v}_2$, $\bar{v}_{-2}$, s_1, s_2, l_1, l_{-1}, l_2, l_{-2}, as well as the four paths $\psi_i(q)$ with each of their tangential endpoints $\psi_i(\overrightarrow{01}_t)$ and $\psi_i(\overrightarrow{10}_t)$, are shown in the following figure, which gives a visualization of the identities $s_1 = l_1^{-1} l_{-1}$, $s_2 = l_{-2}^{-1} l_2$, $\bar{v}_{-1} = \bar{v}_1 s_1$, $\bar{v}_2 = \bar{v}_{-2} s_2$ etc.



(3.6) The images of the paths $\bar{v}_1$, $\bar{v}_{-1}$, $\bar{v}_2$, $\bar{v}_{-2}$ on the unordered stack $M_{0,[5]}$ are four different paths from $\mathcal{A}$ to the same endpoint $\psi(\frac{1}{4}\overrightarrow{01}_u)$; by composing with the infinitesimal path $\mathrm{inn}(\psi(\delta_1))$ of (2.4), we consider them as paths from $\mathcal{A}$ to $\psi(\overrightarrow{01}_u)$ (without adding further notation). They induce four different homomorphisms of (orbifold) fundamental groups

$$\pi_1(\mathcal{L}_u, \overrightarrow{01}_u) \longrightarrow \pi_1(M_{0,[5]}, \mathcal{A}) \qquad (i = \pm 1, \pm 2)$$

via $\gamma \mapsto \bar{v}_i \widetilde{\psi}(\gamma) \bar{v}_i^{-1}$, where $\widetilde{\psi} : \pi_1(\mathcal{L}_u, \overrightarrow{01}_u) \to \pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u))$ is as defined in (2.4). We are particularly interested in the braids associated to the images $\widetilde{\psi}(x_u)$ and $\widetilde{\psi}(y_u)$ of the generators x_u and y_u of $\pi_1(\mathcal{L}_u, \overrightarrow{01}_u)$; i.e. we want to identify the braids associated to the elements

$$x_{u,i} := \bar{v}_i\, \widetilde{\psi}(x_u)\, \bar{v}_i^{-1}, \quad y_{u,i} := \bar{v}_i\, \widetilde{\psi}(y_u)\, \bar{v}_i^{-1} \quad (i = \pm 1, \pm 2)$$

of $\pi_1(M_{0,[5]}, \psi(\overrightarrow{01}_u)) \simeq \Gamma_0^{[5]}$.

Proposition (3.7). *The table below shows how to identify $x_{u,i}, y_{u,i}$ as braids in $\Gamma_0^{[5]}$.*

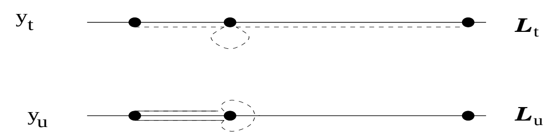
i	1	-1	-2	2
$x_{u,i}$	ϑ^3	$\varepsilon\vartheta^3\varepsilon$	$V_\sharp \vartheta^3 V_\sharp{}^{-1}$	$V_\sharp \varepsilon \vartheta^3 \varepsilon V_\sharp{}^{-1}$
$y_{u,i}$	$\tau_{13}\tau_{45}$	$\varepsilon\tau_{13}\tau_{45}\varepsilon$	$V_\sharp \tau_{13}\tau_{45} V_\sharp{}^{-1}$	$V_\sharp \varepsilon \tau_{13}\tau_{45} \varepsilon V_\sharp{}^{-1}$

where $\vartheta = \tau_1\tau_2\tau_3\tau_4$, $V_\sharp = \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\vartheta^2$, $\varepsilon = \tau_1\tau_2\tau_1\tau_4^{-1}$ as in Lemma (1.1).

Proof. In all of the cases to be proved, we work by lifting the loops x_u and y_u in $\mathcal{L}_u$ to paths x_t and y_t on $\mathcal{L}_t$ and then map these paths to $M_{0,5}$ via ψ_i and conjugate the results by $\bar{v}_1$ and $\bar{v}_2$. This gives paths in $M_{0,5}$ which map down to loops on $M_{0,[5]}$; by "kinematic" parametrization, we are able to determine these paths explicitly.

We approximate the small loop x_u by the parametrization $\epsilon\, \mathrm{e}^{2\pi \mathrm{i} s}$ with $s \in [0,1]$ for very small real ϵ. This means that upstairs in the space $\mathcal{L}_t$, the parameter t runs through one-fifth of the little circle, which we call x_t; it can be parametrized on $\mathcal{L}_t$ by $t = \epsilon\zeta^s$, $s \in [0,1]$.

The path y_u in $\mathcal{L}_u$ lifts to the path y_t on $\mathcal{L}_t$ given in the following figure:



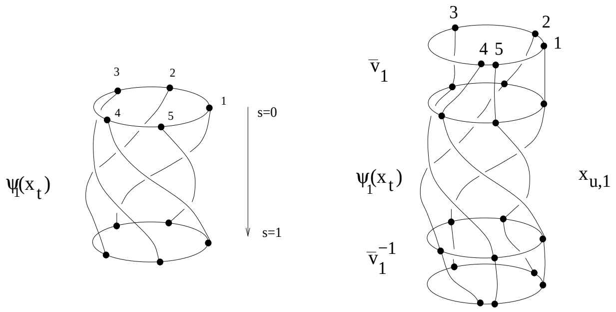
Let us begin with $x_{u,1}$. The image of the little one-fifth circle x_t in $\mathcal{L}_t$ maps under ψ_1 to

$$\left(\zeta^4 + \zeta^{-4+s}\epsilon, 1 + \zeta^s\epsilon, \zeta + \zeta^{-1+s}\epsilon, \zeta^2 + \zeta^{-2+s}\epsilon, \zeta^3 + \zeta^{-3+s}\epsilon\right)$$

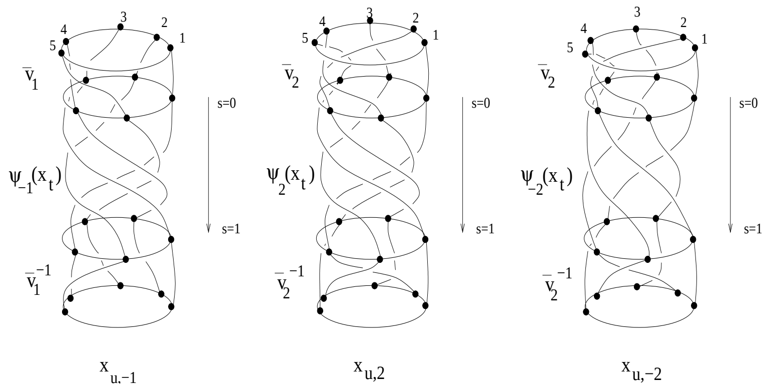
in $\mathcal{E}_1 \subset M_{0,5}$. For the purpose of visualization, we multiply each component by ζ^{2s}, which does not change the point in moduli space; for each $s \in [0,1]$, the same point is now given by

$$\big(\zeta^{4+2s}+\zeta^{-4+3s}\epsilon, \zeta^{2s}+\zeta^{3s}\epsilon, \zeta^{1+2s}+\zeta^{-1+3s}\epsilon, \zeta^{2+2s}+\zeta^{-2+3s}\epsilon, \zeta^{3+2s}+\zeta^{-3+3s}\epsilon\big).$$

Drawing the motion of each of the five points on the sphere as s varies from 0 to 1 yields the left-hand picture of $\psi_1(x_t)$; we read off from the figure that this braid is simply ϑ^3. To compute $x_{u,1}$, this braid must be conjugated by the path $\bar{v}_1$ pictured in (3.2). Viewed as a braid, this path consists in sliding the points of the tangential base point $\mathcal{A}$ apart from each other to a tangential base point at the 5-cycle point Q_1, so composing by it does not change the braid (right-hand figure); therefore we obtain $x_{u,1} = \vartheta^3$.



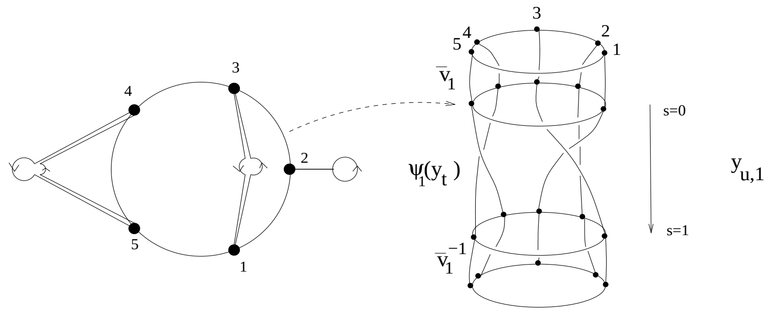
For $i = -1, 2, -2$, we proceed similarly. For $x_{u,-1}$ (resp. $x_{u,-2}$, $x_{u,2}$) we again parametrize the one-fifth circle x_t by $t = \epsilon\zeta^s$, plug this into the expression for $\mathcal{E}_{-1}$ (resp. $\mathcal{E}_{-2}$, $\mathcal{E}_2$) and multiply the result by ζ^{2s} exactly as for $i = 1$; the resulting braids, conjugated by $\bar{v}_1$ and $\bar{v}_2$ respectively, are shown in the following figure.



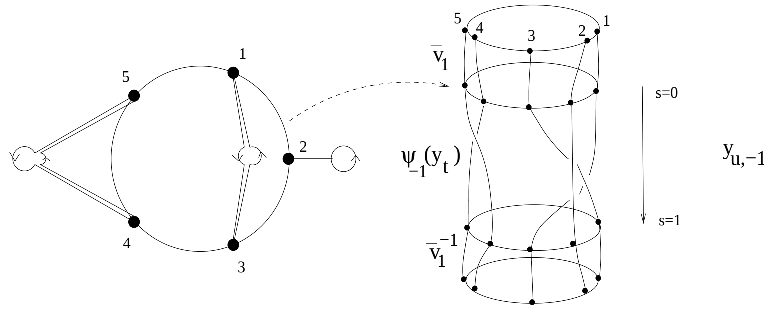
We read the desired braids directly off from this figure. First, clearly $x_{u,-1} = \vartheta^2$; note that this is equal to the expression $\varepsilon\vartheta^3\varepsilon$ given in the statement of the proposition, since $\varepsilon^2 = 1$ and $\varepsilon\vartheta\varepsilon = \vartheta^{-1}$ by Lemma 1.1 (2) and (3). Next, as v_2 is identified with the braid $\tau_4^{-1}\tau_2^{-1}\tau_3^{-1}$ (see the lower half of the figure in (3.2)) and $V_\sharp = \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\vartheta^2$, we obtain $x_{u,-2} = V_\sharp\vartheta^3V_\sharp^{-1}$ and

$x_{u,2} = V_\sharp \vartheta^2 V_\sharp^{-1} = V_\sharp \varepsilon \vartheta^3 \epsilon V_\sharp^{-1}$. This completes the proof of the first line of the table.

We use a similar procedure for the $y_{u,i}$, except that it is actually easier as no reparametrization is needed. Let us begin with $i = 1$. The path y_t in $\mathcal{L}_t$ maps into $M_{0,5}$ as in the following figure; the left-hand side shows it as movements of points on the sphere, and the right-hand side as the middle section of a braid, which is conjugated by $\bar{v}_1$. In the right-hand figure, we have not drawn the points exactly where they should lie with respect to the equator, i.e. very near the equidistant points of Q_1; we have shifted them a little towards the back, so as to be able to read off more easily that the braid $y_{u,1}$ is exactly $\tau_{13}\tau_{45} = \tau_4\tau_1^{-1}\tau_2\tau_1$.



It remains to compute $y_{u,-1}$, $y_{u,-2}$ and $y_{u,2}$. For $y_{u,-1}$, the movement of points and the braid (conjugated by $\bar{v}_1$) representing $\psi_{-1}(y_t)$ are as follows:

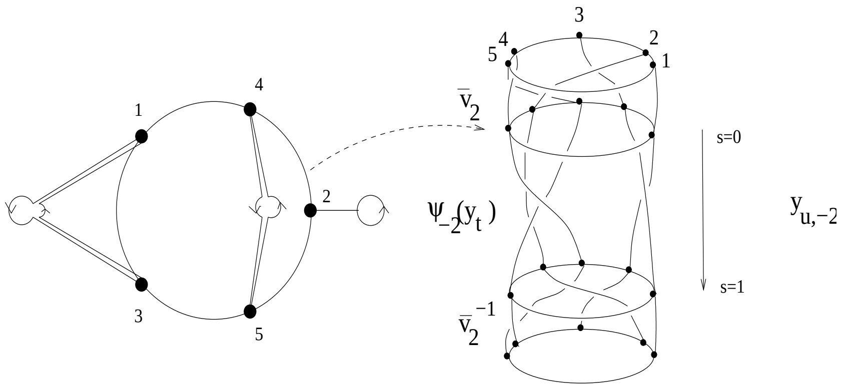


We read directly off the right-hand figure that $y_{u,-1}$ is given by $\tau_4\tau_1\tau_2\tau_1^{-1}$. But since $\varepsilon = \tau_4^{-1}\tau_1\tau_2\tau_1$ and $\tau_2\tau_1^2\tau_2\tau_1^2 = \tau_4^2$, we see that

$$y_{u,-1} = \varepsilon y_{u,1} \varepsilon = \varepsilon \tau_{13}\tau_{45}\varepsilon,$$

as in the table. For $y_{u,-2}$ and $y_{u,2}$, we proceed similarly, but taking care to conjugate by $\bar{v}_2$ rather than $\bar{v}_1$; the figures corresponding to $\psi_{-2}(y_t)$ and

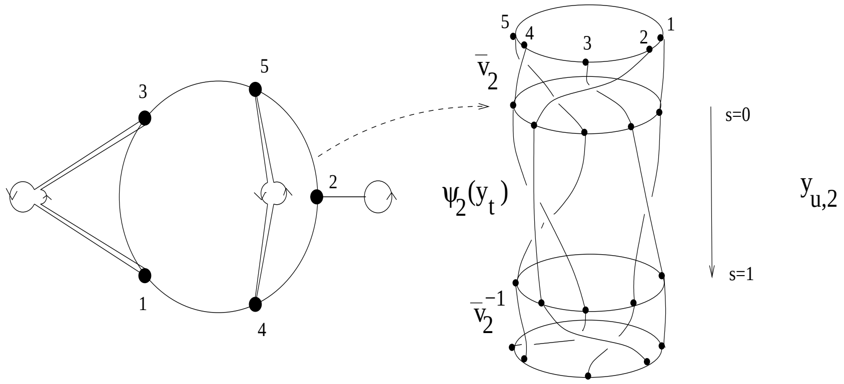
$\psi_2(y_t)$ are as follows:



from which, recalling that $V_\sharp = \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\vartheta^2$ and noting that $\vartheta^2\tau_{13}\tau_{45}\vartheta^{-2} = \tau_{12}\tau_{35}$, we read off

$$y_{u,-2} = \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\tau_{12}\tau_{35}\tau_3\tau_2\tau_4 = V_\sharp\tau_{13}\tau_{45}V_\sharp^{-1},$$

and



from which we read off

$$y_{u,2} = \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\tau_4^{-1}\tau_3\tau_4\tau_1\tau_3\tau_2\tau_4.$$

However, for the middle part of the braid, we have

$$\psi_2(y_t) = \tau_4^{-1}\tau_3\tau_4\tau_1 = \tau_3\tau_4\tau_3^{-1}\tau_1 = \vartheta^2\tau_1\tau_2\tau_1^{-1}\tau_4\vartheta^{-2},$$

and thus,

$$y_{u,2} = V_\sharp\tau_1\tau_2\tau_1^{-1}\tau_4V_\sharp^{-1} = V_\sharp\varepsilon\tau^{-1}\tau_2\tau_1\tau_4\varepsilon V_\sharp^{-1} = V_\sharp\varepsilon\tau_{13}\tau_{45}\varepsilon V_\sharp^{-1}$$

by a simple computation of braids using $\varepsilon = \tau_1\tau_2\tau_1\tau_4^{-1}$ and the identity $\tau_1\tau_2^2\tau_1\tau_2^2 = \tau_4^2$. This concludes the proof of the proposition. $\square$

4. Local factors at the tangential base points

(4.1) In this section, we shall look closely at the tangential base points $\psi_i(\overrightarrow{10}_t)$ $(i = \pm 1, \pm 2)$ near the maximal degenerate point $(13)(45)$. We shall take a standard tangential base point $\mathcal{B}$ near the point $(13)(45)$ given by the ring of Puiseux series $\overline{\mathbb{Q}}\{\{q_1, q_2\}\} := \bigcup_N' \overline{\mathbb{Q}}[\![q_1^{1/N}, q_2^{1/N}]\!]$ (cf. [IN]), where the coordinates q_1, q_2 are defined by

$$(a_1, a_2, a_3, a_4, a_5) \sim (q_2^{-1}, 1, \infty, 0, q_1).$$

Here $\sim$ means the equivalence of tuples by fractional transformations of $\mathbf{P}^1$. We shall identify the tangent space at the maximal degenerate point $(13)(45)$ on $M_{0,5}$ with $\mathbb{C}^2$ by the coordinates (q_1, q_2). The tangential base point $\mathcal{B}$ is equivalent to that given by the tangent vector $(1,1) \in \mathbb{C}^2$.

First, one can compute the case $\psi_1(\overrightarrow{10}_t)$ as follows.

$$\begin{aligned}&\big(\zeta^4 + \zeta^{-4}t, 1 + t, \zeta + \zeta^{-1}t, \zeta^2 + \zeta^{-2}t, \zeta^3 + \zeta^{-3}t\big)\\ &\sim \left(\frac{1}{U(1-t)\big(1 + O(1-t)\big)}, 1, \infty, 0, V(1-t)\big(1 + O(1-t)\big)\right),\end{aligned}$$

where

$$\begin{aligned}U &= \frac{5}{(1-\zeta)^5} = -\left(\frac{1+\sqrt{5}}{2}\right)^{5/2} 5^{-1/4} i,\\ V &= \frac{5}{(1-\zeta^2)^5} = \left(\frac{1+\sqrt{5}}{2}\right)^{-5/2} 5^{-1/4} i,\end{aligned}$$

and $1 + O(1-t)$ designates some power series in $\mathbb{Q}(\zeta)(\!(1-t)\!)$ with constant term 1.

Since the other tangential base points $\psi_i(\overrightarrow{10}_t)$ $(i = -1, \pm 2)$ are Galois conjugates of $\psi_1(\overrightarrow{10}_t)$, the corresponding tangent vectors are easy to identify. We list the coordinates of the corresponding tangent vectors in the following table:

	$\psi_1(\overrightarrow{10}_t)$	$\psi_{-1}(\overrightarrow{10}_t)$	$\psi_2(\overrightarrow{10}_t)$	$\psi_{-2}(\overrightarrow{10}_t)$
q_1	V	$-V$	$-U$	U
q_2	U	$-U$	V	$-V$

(4.2) We shall connect these tangent vectors (V, U), $(-V, -U)$, $(-U, V)$, $(U, -V)$ with $(1,1)$ by the straight lines on $\mathbb{C}^2$. These lines give (etale homotopy) paths

$$\begin{cases} r_1: & \psi_1(\overrightarrow{10}_t) \to \mathcal{B}, \\ r_{-1}: & \psi_{-1}(\overrightarrow{10}_t) \to \mathcal{B}, \\ r_2: & \psi_2(\overrightarrow{10}_t) \to \mathcal{B}, \\ r_{-2}: & \psi_{-2}(\overrightarrow{10}_t) \to \mathcal{B} \end{cases}$$

on $M_{0,5}/\overline{\mathbb{Q}}$. These paths determine specialization homomorphisms of $\overline{\mathbb{Q}}\{\{q_1, q_2\}\}$ to $\overline{\mathbb{Q}}\{\{1-t\}\}$ via the principal branches of roots $\sqrt[N]{V}$, $\sqrt[N]{U}$, $\sqrt[N]{-U}$, $\sqrt[N]{-V}$ nearest to 1. For example, r_1 represents the homomorphism sending $q_1^{1/N}$, $q_2^{1/N}$ to $\sqrt[N]{V}(1-t)^{1/N}$, $\sqrt[N]{U}(1-t)^{1/N}$ respectively. The Galois transformation of r_1 is then given by the following

Lemma (4.3). *For $j = 1, 2$, let $X_j \in \pi_1(M_{0,5}, \mathcal{B})$ be the path corresponding to the local monodromy $q_j^{1/n} \mapsto \zeta_n^{-1} q_j^{1/n}$ $(n \geq 1,\ \zeta_n = \exp(2\pi i/n))$. For $\sigma \in G_{\mathbb{Q}}$, define the values $\rho_V(\sigma), \rho_U(\sigma) \in \hat{\mathbb{Z}}$ by the Kummer property along the principal (i.e., nearest to 1) branches of n-th roots of V and U:*

$$\frac{\sigma(\sqrt[n]{V})}{\sqrt[n]{\sigma(V)}} = \zeta_n^{\rho_V(\sigma)}, \quad \frac{\sigma(\sqrt[n]{U})}{\sqrt[n]{\sigma(U)}} = \zeta_n^{\rho_U(\sigma)} \quad (n \geq 1).$$

Then, we have

$$\sigma(r_1) = r_i \cdot X_1^{-\rho_V(\sigma)} X_2^{-\rho_U(\sigma)} \quad (\sigma \in G_{\mathbb{Q}}),$$

where $i \in \{\pm 1, \pm 2\}$ is determined by the condition $i \equiv \lambda_\sigma$ mod 5.

Proof. Put $\sigma(r_1) = \sigma \cdot r_1 \cdot \sigma^{-1} = r_i \cdot X_1^{c_1} X_2^{c_2}$ and apply both sides to the functions $q_1^{1/n}$ separately. The left-hand side gives then $\zeta_n^{\rho_V(\sigma)}(\sigma(V))^{1/n}(1-t)^{1/n}$, while the right-hand side gives $\zeta_n^{-c_1}(\sigma(V)(1-t))^{1/n}$. This concludes $c_1 = -\rho_V(\sigma)$. The same argument for $q_2^{1/n}$ determines the value of c_2. $\square$

(4.4) Set

$$\begin{cases} X_{1,i} & := (\bar{v}_i \psi_i(q) r_i)(X_1)(\bar{v}\psi_i(q) r_i)^{-1}, \\ X_{2,i} & := (\bar{v}_i \psi_i(q) r_i)(X_2)(\bar{v}\psi_i(q) r_i)^{-1} \end{cases}$$

to be the loops based at $\mathcal{A}$ obtained by conjugating the loops X_1 and X_2, based at $\mathcal{B}$, by the paths $\bar{v}_i \psi_i(q) r_i$ from $\mathcal{A}$ to $\mathcal{B}$. Then, in a similar way to (3.6), we have the following expression of $X_{1,i}$ by braids in $\pi_1(M_{0,5}, \mathcal{A})$:

i	1	-1	-2	2
$X_{1,i}$	x_{45}	$\varepsilon x_{45} \varepsilon$	$V_\sharp x_{45} V_\sharp^{-1}$	$V_\sharp \varepsilon x_{45} \varepsilon V_\sharp^{-1}$
$X_{2,i}$	x_{13}	$\varepsilon x_{13} \varepsilon$	$V_\sharp x_{13} V_\sharp^{-1}$	$V_\sharp \varepsilon x_{13} \varepsilon V_\sharp^{-1}$

We note that ε commutes with τ_4 and hence with x_{45}. Comparing the table in (3.6), we note that

$$y_{u,i}^2 = X_{1,i}X_{2,i} \quad (i = \pm 1, \pm 2).$$

Before closing this section, we shall relate the above ρ_U, ρ_V with the functions χ_{13}, χ_{45} introduced in §1.

Lemma (4.5). *As $\hat{\mathbb{Z}}$-valued functions on $G_{\mathbb{Q}}$, we have*

$$\chi_{13} = \begin{cases} 2(\rho_5 - \rho_2 - \rho_U) + \begin{cases} 0, & (\lambda \equiv 1 \ mod\ 5) \\ -1 & (\lambda \equiv -1 \ mod\ 5) \end{cases} \\ 2(\rho_5 - \rho_2 - \rho_V) + \begin{cases} 0, & (\lambda \equiv -2 \ mod\ 5) \\ -1 & (\lambda \equiv 2 \ mod\ 5) \end{cases} \end{cases}$$

$$\chi_{45} = \begin{cases} 2(\rho_5 - \rho_2 - \rho_V) + \begin{cases} 0, & (\lambda \equiv 1 \ mod\ 5) \\ 1 & (\lambda \equiv -1 \ mod\ 5) \end{cases} \\ 2(\rho_5 - \rho_2 - \rho_U) + \begin{cases} 1 & (\lambda \equiv -2 \ mod\ 5), \\ 0 & (\lambda \equiv 2 \ mod\ 5) \end{cases} \end{cases}$$

Proof. The proof is obtained from case-by-case examination of the branches of n-th roots of quantities and of Galois actions on them. We treat the case where $\lambda_\sigma \equiv -2$ mod 5. In this case, one has

$$\frac{\sigma(\sqrt[n]{U})}{\sqrt[n]{V}} = \frac{\sigma(\sqrt[n]{U})}{\sqrt[n]{-\sigma(U)}} = \zeta_n^{\rho_U(\sigma)-\frac{1}{2}}.$$

This implies

$$\frac{\sigma(\sqrt[n]{u})}{\sqrt[n]{v}} = \zeta_n^{\rho_5-\rho_2-\rho_U+\frac{1}{2}}$$

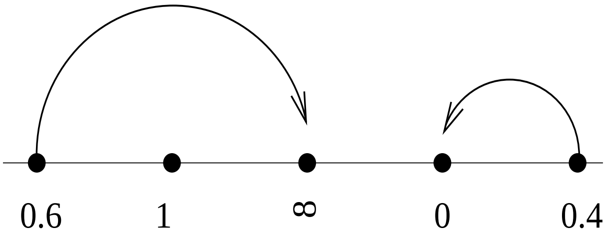
for all $n \geq 1$, hence, $\chi_{45} = 2(\rho_5 - \rho_2 - \rho_U) + 1$. We leave it to the reader to check the other cases. $\square$

5. Path deformations and proof of Theorem A

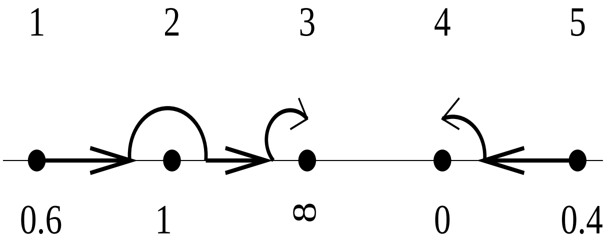
(5.1) Recall that q is the path from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathcal{L}_t$, and $\psi_1 : \mathcal{L}_t \to \mathcal{E}_1 \subset M_{0,5}$. Thus, $\psi_1(q)$ is its image on $M_{0,5}$. We shall homotopically deform the path $\psi_1(q)$ so as to run near to the maximal degenerate point (12)(45) as follows. The starting point of $\psi_1(q)$ is a the tangential base point $\psi_1(\overrightarrow{01}_t)$ neighboring the point $t = 0$ in $\mathcal{E}_1$, which we can normalize as follows:

$$Q_1 = \left(\frac{1}{-\zeta^2 - \zeta^3}, 1, \infty, 0, \frac{1}{1 - \zeta^2 - \zeta^3}\right) \sim (0.6, 1, \infty, 0, 0.4).$$

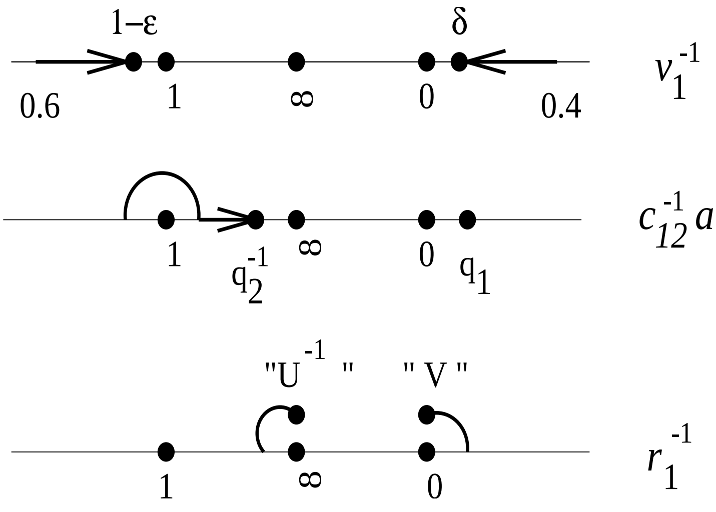
The endpoint of $\psi_1(q)$ is the tangential base point $\psi_1(\overrightarrow{10}_t)$ given by $(1/\varepsilon, 1, \infty, 0, \delta)$ (for small positive real numbers ε and δ) neighboring the maximally degenerate point $(\infty, 1, \infty, 0, 0)$, i.e. (13)(45). Represented on the sphere with marked points, the parametrized path $\psi_1(q)$ is as follows:



We see in this figure that the path ends at a tangential base point which is not one of the standard (real) ones. We deform it to a homotopic path drawn as



We can undo this path into three parts as follows:



The first part is exactly the path $\bar{v}_1^{-1}$; it starts at the starting point of $\psi(q)$, namely at the tangential base point $\psi(\overrightarrow{01}_t)$ near the 5-cycle point Q_1, and ends at the tangential base point represented by $\mathcal{A} = (1-\varepsilon, 1, \infty, 0, \delta)$ for small positive real numbers ε and δ, neighboring the maximally degenerate point (12)(45). The second part takes place in the neighborhood of infinity of the moduli space, and consists in the commutativity move c_{12}^{-1} (a right half twist, see for example [NS]), taking the point to $(1, 1+\varepsilon, \infty, 0, \delta)$ numbered $(2,1,3,4,5)$, followed by the associativity move $a = a_{21,13}$ which brings this point to $\mathcal{B} \sim (1/\varepsilon, 1, \infty, 0, \delta)$ simply by sliding the second point along the real axis from ε to $1/\varepsilon$; the point $\mathcal{B}$ is a tangential base point near (13)(45). The third part is a local path around (13)(45) going from $\mathcal{B}$ to the tangential base point $\psi_1(\overrightarrow{10}_t)$, which by construction is exactly the path r_1^{-1} of (4.2). Thus we obtain the decomposition

$$\psi_1(q) = \bar{v}_1^{-1} \cdot c_{12}^{-1} \cdot a \cdot r_1^{-1}. \tag{5.2}$$

We now come to the proof of the main result of this article.

(5.3) *Proof of Theorem A*: By (5.2), we have an equivalence of paths

$$\bar{v}_1 \cdot \psi_1(q) \cdot r_1 = c_{12}^{-1} \cdot a$$

from $\mathcal{A}$ to $\mathcal{B}$. On the right-hand side, the Galois action on those paths along the 1-dimensional strata at infinity in $\mathcal{M}_{0,5}$ are well-known (cf. [NS]) and may be calculated as:

$$\sigma(c_{12}^{-1}a) = x_{12}^{\frac{1-\lambda_\sigma}{2}} f(x_{12}, x_{13})^{-1} \cdot c_{12}^{-1}a \tag{5.4}$$

for $\sigma \in G_{\mathbb{Q}}$. To consider Galois transformations of the left-hand side, fix $\sigma \in G_{\mathbb{Q}}$ and take $i \in \{\pm 1, \pm 2\}$ so that $i \equiv \lambda_\sigma \bmod 5$. Recall that by Lemma (2.5), we have

$$\sigma\big(\widetilde{\pi}_i(\psi_1(q))\big) = \widetilde{\psi}(x_u)^{-2\rho_2} f\big(\widetilde{\psi}(x_u), \widetilde{\psi}(y_u)\big)^{-1} \widetilde{\psi}(y_u)^{2\rho_5 - 2\rho_2} \widetilde{\pi}_i\big(\psi_i(q)\big).$$

Using this (and dropping the $\widetilde{\pi}_i$ from the notation, as it is merely an inclusion of groups), together with Definition (3.5) and Lemma (4.3), we find that the Galois transformation of the left-hand side can be given as:

$$\begin{aligned} \sigma(\bar{v}_1 \cdot \psi_1(q) \cdot r_1) = {} & k_\sigma(\{x_{ij}\})^{-1} \bar{v}_i \cdot \widetilde{\psi}(x_u)^{-2\rho_2} f\big(\widetilde{\psi}(x_u), \widetilde{\psi}(y_u)\big)^{-1} \\ & \cdot \widetilde{\psi}(y_u)^{2\rho_5 - 2\rho_2} \cdot \psi_i(q) \cdot r_i \cdot X_1^{-\rho_V(\sigma)} X_2^{-\rho_U(\sigma)} \\ = {} & \sigma(c_{12}^{-1}a) \\ = {} & x_{12}^{\frac{1-\lambda_\sigma}{2}} f(x_{13}, x_{12}) c_{12}^{-1} a \end{aligned} \tag{5.5}$$

Summing up, we obtain

$$k_\sigma(\{x_{ij}\}) = F_i X_{1,i}^{-\rho_V(\sigma)} X_{2,i}^{-\rho_U(\sigma)} \mathcal{S}_i f(x_{12}, x_{13}) x_{12}^{\frac{\lambda_\sigma - 1}{2}}, \tag{5.6}$$

with

$$\begin{aligned}
F_i &:= \bar{v}_i \cdot \widetilde{\psi}(x_u)^{-2\rho_2} f\big(\widetilde{\psi}(y_u), \widetilde{\psi}(x_u)\big) \widetilde{\psi}(y_u)^{2\rho_5-2\rho_2} \cdot \bar{v}_i^{-1},\\
X_{1,i} &:= (\bar{v}_i \cdot \psi_i(q) r_i)(X_1)(\bar{v}_i \psi_i(q) r_i)^{-1},\\
X_{2,i} &:= (\bar{v}_i \cdot \psi_i(q) r_i)(X_2)(\bar{v}_i \psi_i(q) r_i)^{-1},\\
\mathcal{S}_i &:= \bar{v}_i \cdot \psi_i(q) \cdot r_i \cdot a^{-1} c_{12}.
\end{aligned}$$

Given that F_i is exactly equal to

$$F_i = x_{u,i}^{-2\rho_2} f(y_{u,i}, x_{u,i}) y_{u,i}^{2\rho_5-2\rho_2},$$

we find, using the table of Proposition (3.6) and the identities $f(\varepsilon\tau_{13}\tau_{45}\varepsilon, \vartheta^2) = \varepsilon f(\tau_{13}\tau_{45}, \vartheta^3)\varepsilon$ and $\varepsilon^2 = 1$, that

$$F_i = \begin{cases} \vartheta^{-6\rho_2} f(\tau_{13}\tau_{45}, \vartheta^3)(\tau_{13}\tau_{45})^{2\rho_5-2\rho_2} & (i=1),\\ \vartheta^{6\rho_2} \varepsilon f(\tau_{13}\tau_{45}, \vartheta^3)(\tau_{13}\tau_{45})^{2\rho_5-2\rho_2}\varepsilon & (i=-1),\\ V_\sharp \vartheta^{-6\rho_2} f(\tau_{13}\tau_{45}, \vartheta^3)(\tau_{13}\tau_{45})^{2\rho_5-2\rho_2} V_\sharp^{-1} & (i=-2),\\ V_\sharp \vartheta^{6\rho_2} \varepsilon f(\tau_{13}\tau_{45}, \vartheta^3)(\tau_{13}\tau_{45})^{2\rho_5-2\rho_2}\varepsilon V_\sharp^{-1} & (i=2). \end{cases}$$

In other words, for $\lambda = \lambda_\sigma$ and heavily using Lemma (1.1) (2), (3) and (4) (especially for $\lambda \equiv \pm 2$), we have

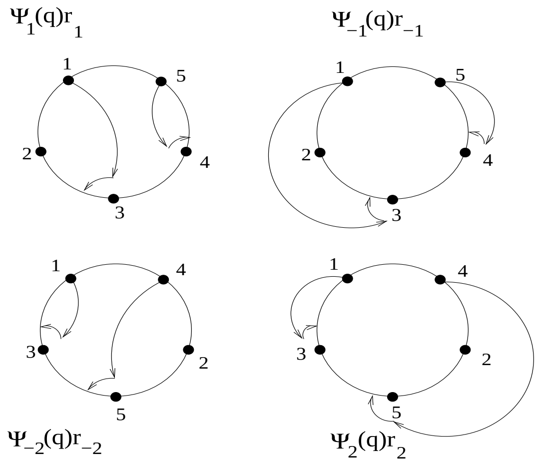
$$(5.7) \qquad F_i = V_\lambda \varepsilon_\lambda \vartheta_\lambda^{-6\rho_2} f(\tau_{13}\tau_{45}, \vartheta_\lambda^3) \cdot \begin{cases} (\tau_{13}\tau_{45})^{2\rho_5-2\rho_2} & (i=1),\\ (\tau_{13}\tau_{45})^{2\rho_5-2\rho_2}\varepsilon & (i=-1),\\ \varepsilon'(\tau_{13}\tau_{45})^{2\rho_5-2\rho_2} V_\sharp^{-1} & (i=-2),\\ \varepsilon'(\tau_{13}\tau_{45})^{2\rho_5-2\rho_2}\varepsilon V_\sharp^{-1} & (i=2). \end{cases}$$

Now, using the table in (4.4), we have

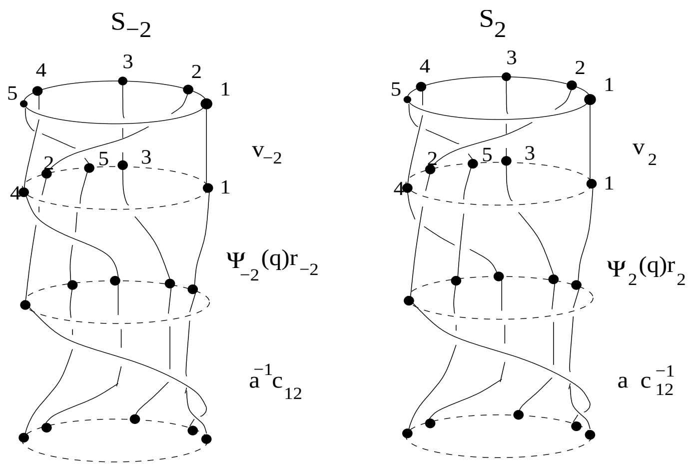
$$(5.8) \qquad X_{1,i}^{-\rho_V} X_{2,i}^{-\rho_U} = \begin{cases} \tau_{13}^{-2\rho_U}\tau_{45}^{-2\rho_V} & (i=1),\\ \varepsilon\tau_{13}^{-2\rho_U}\tau_{45}^{-2\rho_V}\varepsilon & (i=-1),\\ V_\sharp\tau_{13}^{-2\rho_U}\tau_{45}^{-2\rho_V} V_\sharp^{-1} & (i=-2),\\ V_\sharp\varepsilon\tau_{13}^{-2\rho_U}\tau_{45}^{-2\rho_V}\varepsilon V_\sharp^{-1} & (i=2). \end{cases}$$

It remains to compute the $\mathcal{S}_i$. Since $\psi_1(q) = \bar{v}_1^{-1} c_{12}^{-1} a\, r_1^{-1}$, we see that $\mathcal{S}_1 = 1$. In fact, we can treat all of the $\mathcal{S}_i$, $i = \pm 1, \pm 2$ simultaneously by drawing the paths $\psi_i(q) r_i$ with certain components normalized to 0, 1, ∞

and proceeding as for $i = 1$:



From this drawing of $\psi_{-1}(q)$, we see that it is homotopic to $\bar{v}_1^{-1} c_{12} a\, r_{-1}^{-1}$; thus, $\mathcal{S}_{-1} = c_{12} \cdot c_{12} = x_{12}$. For $i = \pm 2$, we compute $\mathcal{S}_i$ by composing the pictured paths as braids, starting with $\bar{v}_2$ (drawn in (3.2)), as follows:



We read the braids immediately from this picture:

$$\mathcal{S}_{-2} = \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\tau_4\tau_3\vartheta^{-1}\tau_1, \quad \mathcal{S}_2 = \tau_4^{-1}\tau_2^{-1}\tau_3^{-1}\tau_4^{-1}\tau_3^{-1}\vartheta^{-1}\tau_1.$$

Using simple braid relations, one can rewrite these expressions using the important involutive braid $\varepsilon' = \tau_2\vartheta^2\tau_1$ (see Lemma (1.1)) which exchanges

τ_{13} and τ_{45} by conjugation. We obtain:

$$\mathcal{S}_i = \begin{cases} 1, & (i=1), \\ x_{12}, & (i=-1), \\ V_\sharp \tau_{13}\varepsilon', & (i=-2), \\ V_\sharp \varepsilon \tau_{45}^{-1}\varepsilon', & (i=2). \end{cases} \tag{5.9}$$

This, together with (3.2) and (5.4), after being collected into (5.5), reduces the proof of Theorem A to checking

$$\tau_{13}^{\chi_{13}}\tau_{45}^{\chi_{45}} = \begin{cases} x_{13}^{\rho_5-\rho_2-\rho_U} x_{45}^{\rho_5-\rho_2-\rho_V} \cdot \begin{cases} 1 & (\lambda \equiv 1 \bmod 5), \\ \tau_{13}^{-1}\tau_{45} & (\lambda \equiv -1 \bmod 5), \end{cases} \\ x_{45}^{\rho_5-\rho_2-\rho_U} x_{13}^{\rho_5-\rho_2-\rho_V} \cdot \begin{cases} \tau_{45} & (\lambda \equiv -2 \bmod 5), \\ \tau_{13}^{-1} & (\lambda \equiv 2 \bmod 5). \end{cases} \end{cases}$$

But this follows from Lemma (4.5). The proof of theorem A is thus completed. □

(5.10) In this subsection, we discuss how to extend the functions χ_{13}, χ_{45} of §1 from $G_{\mathbb{Q}}$ to $\widehat{GT}$ by using Ihara's theory [I2]. We assume basic properties of the profinite free differential calculus. Consider the partial derivative $\frac{\partial f_\sigma(x,y)}{\partial y}$ in the complete group algebra $\hat{\mathbb{Z}}[[\hat{F}_2]]$ for $\sigma \in \widehat{GT}$, and write its image in $\hat{\mathbb{Z}}[x]/(x^5-1)$ (with $y=1$) as:

$$\frac{\partial f_\sigma(x,y)}{\partial y} \equiv -\sum_{a=0}^{4} \kappa_5^a(\sigma)x^{-a} \bmod (x^5-1). \tag{5.10.1}$$

This formula defines the extension to $\widehat{GT}$ of the coefficients κ_5^a which appear as the Soulé characters in the case where $\sigma \in G_{\mathbb{Q}}$. Indeed, by [I2], we know that $\kappa_5^0 = -\rho_5$ on $G_{\mathbb{Q}} \subset \widehat{GT}$ and

$$\zeta_n^{\kappa_5^a(\sigma)} = \frac{\sigma\left(\sqrt[n]{1-\zeta_n^{\lambda_\sigma^{-1}a}}\right)}{\sqrt[n]{1-\zeta_n^a}} \quad (n \geq 1,\ \sigma \in G_{\mathbb{Q}}) \tag{5.10.2}$$

for $a = \pm 1, \pm 2$. Therefore, to extend χ_{13}, χ_{45} to $\widehat{GT}$, it suffices to express them in terms of κ_5^a's (together with ρ_2 whose extension to $\widehat{GT}$ was discussed in [LNS], [NS]). This can be done by looking closely at branches of roots. In fact, we obtain the following equations of $\hat{\mathbb{Z}}$-valued functions on $G_{\mathbb{Q}}$ for which we omit to describe the detailed case-by-case examinations :

$$\chi_{13} + 2\rho_2 = \begin{cases} 10\kappa_5^1 + 2\lambda - 2 & (\lambda \equiv 1 \bmod 5), \\ 10\kappa_5^1 - \lambda - 2 & (\lambda \equiv -1 \bmod 5), \\ 10\kappa_5^1 - 2 & (\lambda \equiv -2 \bmod 5), \\ 10\kappa_5^1 - \lambda - 2 & (\lambda \equiv 2 \bmod 5), \end{cases} \tag{5.10.3}$$

$$\chi_{45} + 2\rho_2 = \begin{cases} 10\kappa_5^2 & (\lambda \equiv 1 \bmod 5), \\ 10\kappa_5^2 - \lambda & (\lambda \equiv -1 \bmod 5), \\ 10\kappa_5^2 - \lambda & (\lambda \equiv -2 \bmod 5), \\ 10\kappa_5^2 + 2\lambda & (\lambda \equiv 2 \bmod 5). \end{cases} \tag{5.10.4}$$

The meaning of this extension is that the equality in theorem A, which holds whenever $(\lambda_\sigma, f_\sigma)$ is associated to an element of $G_{\mathbb{Q}}$, can now be posited as a *relation on elements of* $\widehat{GT}$, which may or may not be satisfied by all elements of $\widehat{GT}$.

6. Interpretation of results and some prospects

We have thus completed the proofs of theorems A,B. In this section, we will add a few remarks about their overall meaning, especially in the framework of Grothendieck-Teichmüller theory. Eigenloci can be defined in general in the moduli stacks $M_{g,n}$ (resp. $M_{g,[n]}$) of curves of genus g with n labeled (resp. unlabeled) marked points; their main defining property is to be stable (though not necessarily pointwise fixed, of course) under the action of a finite subgroup of the modular group Γ_g^n (resp. Γ_g^n) corresponding to the automorphism group of some algebraic curve (see [L] for more detail).

Here we have used a very special case, namely a rational eigencurve $\mathcal{E}$ (dimension 1 eigenlocus) in $M_{0,5}$. Apart from the two key properties of $\mathcal{E}$ mentioned in the Introduction, namely that $\mathcal{E}$ is stable under the action of $\boldsymbol{\rho}$ and that its image in $M_{0,[5]}$ is a copy of $\mathbf{P}^1$ with a missing point and two orbifold points, we have actually used a third geometric property, namely that $\mathcal{E}$ intersects the divisor at infinity of $M_{0,[5]}$ at a point of maximal degeneration. The equation of $\mathcal{E} = \mathcal{E}^{(5)}$ can immediately be generalized to any n. Just write the same formula as in (2.3) above with ζ a primitive n-th root of unity and n entries of the form $z_i(t) = \zeta^i + \zeta^{-i}t$, $i = 0, 1, \ldots, n$. The corresponding eigencurve $\mathcal{E}^{(n)} \subset M_{0,n}$ still enjoys the two key properties, but the third fails for $n \geq 5$, a point to which we will briefly return below.

Let us briefly show that the method we used in this paper could have been used in the lower dimensions $n = 3, 4$ in order to compute the elements g_σ and h_σ in terms of f_σ as was done in Theorem 2 of the Introduction); it turns out that the method and the result is essentially similar to what was done in

[NT]. First, we mention an easy fact, true for general n, which is implicitly used below in the case $n = 3$. One can mark not just the n points $z_i(t)$ on the sphere, but also one or both of the points 0 and/or ∞; these are the two ramification points of the rotational symmetry of order n given by $t \mapsto \zeta_n t$. Adding in one or both of these distinguished points defines eigencurves in $M_{0,n+1}$ and $M_{0,n+2}$ whose automorphism groups decrease accordingly; for instance, adding in the point ∞ decreases the automorphism group from a dihedral group to a cyclic group, as the additional symmetry $t \mapsto 1/t$ no longer preserves the set of marked points.

Let us proceed to g_σ and h_σ. Write $\underline{z} = (z_1, z_2, z_3, z_4)$ for a point on $M_{0,4}$ and use the following definition of the cross-ratio:

$$[z_1, z_2, z_3, z_4] = \frac{z_2 - z_1}{z_3 - z_1} \times \frac{z_3 - z_4}{z_2 - z_4},$$

so that $\underline{z} \sim (0, z, 1, \infty)$ where $\sim$ is equivalence under the PGL_2 action and $z = [z_1, z_2, z_3, z_4]$. Recall that $\boldsymbol{\theta}$ (resp. $\boldsymbol{\omega}$) denotes the order 2 (resp. 3) automorphism of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ given by $z \mapsto 1 - z$ (resp. $z \mapsto 1/(1 - z)$). Take $n = 4$, and let us study the order 2 symmetry $\boldsymbol{\theta}$.

The eigencurve $\mathcal{E}^{(4)}$ parametrized by t reads:

$$\underline{z}(t) = (1 + t, i - it, -1 - t, -i + it). \tag{6.1}$$

Putting this into the standard cross-ratio form $(0, z(t), 1, \infty)$ as above, we find that

$$z(t) = \frac{1}{2} - \frac{it}{1 - t^2}. \tag{6.2}$$

There are four values of t for which the corresponding point of $\mathcal{E}^{(4)}$ is degenerate, namely $t = \pm 1, \pm i$. The corresponding points on $M_{0,4}$ are $z(1) = z(-1) = \infty$, $z(i) = 1$ and $z(-i) = 0$, so the image of the locus $\mathcal{E}^{(4)}$ (minus the degenerate points) in $M_{0,4}$ simply coincides with $M_{0,4}$. The permutation group S_4 acts on $M_{0,4}$ via $S_3 = S_4/V$, where V is the group of products of disjoint transpositions, which does not act effectively; the group S_3 is seen as permuting the points 0, 1 and ∞. Applying the 4-cycle (1234) to $\mathcal{E}^{(4)}$, we find that it acts via the involution $t \mapsto -t$ (its square belongs to V), so that it acts on $z(t)$ via $z(-t) = 1 - z(t) = \boldsymbol{\theta}(z(t))$. Moreover, the transformation $t \mapsto 1/t$ induces the transposition (24), and acts on $z(t)$ via $z(1/t) = z(-t) = \boldsymbol{\theta}(z(t))$. This means that the whole order 8 dihedral subgroup $D_4 \subset S_4$ generated by the 4-cycle (1234) and the transposition (24) (compare with the case $n = 5$ in §2.1) projects to a group of order 2 in S_3 corresponding to the subgroup of the automorphisms of the line generated by $\boldsymbol{\theta}$.

Because $\mathcal{E}^{(4)}$ coincides with $M_{0,4}$, its stabilizer coincides with the whole of S_3. Yet, by analogy with the case $n = 5$ (actually any $n \geq 5$), and following the above, it is natural to single out the group generated by $\boldsymbol{\theta}$ as the 'meaningful' stabilizer. Mimicking the construction in §2.1, one then introduces the u-line with $u = \beta(t) = 4t(1-t)$, which is a Belyi function describing the invariants under $\boldsymbol{\theta}$. We are now precisely in the situation of [NT] (§4) which produces equation (GF_1) of Theorem 2 in the Introduction. In turn, equation (GF_0) of that same theorem is obtained using the whole of S_3 as the stabilizer (which it is!) and the attending invariants.

The case of h_σ, with $n = 3$, can be treated much in the same way. Let ξ denote a primitive third root of unity, and parametrize the eigenlocus $\mathcal{E}^{(3)}$ as follows:

$$\underline{z}(t) = (1+t, \xi + \xi^2 t, \xi^2 + \xi t, \infty), \tag{6.3}$$

where the point ∞ is added as explained above, so as to work in the non-trivial moduli space $M_{0,4}$ rather than the trivial space $M_{0,3}$. In normalized form one gets:

$$z(t) = -\xi \frac{1-\xi^2 t}{1-\xi t}. \tag{6.4}$$

The points $t = 0, \infty$ give the two points of order three, and $t \mapsto \xi^2 t$ corresponds to the action of $\boldsymbol{\omega}$: we have $z(\xi^2 t) = \boldsymbol{\omega}(z(t))$. One also computes that $z(1/t) = 1/z(t)$, corresponding to the transposition (23) of the points 0 and ∞. This time the dihedral group $D_3 \subset S_4$ associated with $\mathcal{E}^{(3)}$ projects to the whole of S_3, and following the same procedure, one is led to look at an order 6 cover describing the S_3 invariants, as in [NT] (§3), that is to equation (HF_0) of Theorem 2. Equation (HF_1) is derived by taking an intermediate cover of order 3, corresponding to the invariants under the 3-cycle ([NT], §5). Logically speaking, in order to retrieve this last equation, one could add the point 0 in the definition of $\mathcal{E}^{(3)}$, getting a curve in $M_{0,5}$ with a stabilizer cyclic of order 3, because the involution $t \mapsto 1/t$ does not apply anymore. However one then has to compute in the fundamental group of $M_{0,[5]}$, which is fairly artificial here.

We thus find that the consideration of the eigencurves $\mathcal{E}^{(3)}$ and $\mathcal{E}^{(4)}$ leads to a computation of the parameters g_σ and h_σ in the Galois case, in terms of f_σ, which essentially coincide with the computations of [NT] given in Theorem 2. The present paper has been devoted to $\mathcal{E}^{(5)}$ and the computation of k_σ. To summarize, one can say that it completes the computation of the Galois action on the automorphisms at the first two levels in terms of the action on the groupoid at infinity. Indeed, g, h and k describe the action of the Galois group on the automorphism groups of the curves on the moduli spaces of dimensions 1 and 2. (This could and should be made a little more precise, as we have considered only the genus 0 spaces ($M_{0,[4]}$ and $M_{0,[5]}$) here, but in fact the genus 1 spaces ($M_{1,1}$ and $M_{1,[2]}$) are similar to the genus

0 spaces, and in particular are uniformized by the same complex Teichmüller spaces.) As for f, it describes the action on the groupoid at infinity of $M_{0,4}$, which is enough to describe the action at infinity on all the spaces $M_{g,n}$ and $M_{g,[n]}$, based at the 'standard tangential points'.

The elements g_σ, h_σ and k_σ carry explicit information about the action of $\widehat{GT}$ on certain torsion elements of the mapping class groups. At present, not much is known about the $\widehat{GT}$ (resp. $G_{\mathbb{Q}}$) action on such elements; in general, what can be said is as follows (cf. [S], [L]). Because we are considering the $\widehat{GT}$ action (and not the outer action), the choice of base point is important: here, we take the standard tangential base point at infinity.

Armed with some material on stacks and their fundamental groups, it is not difficult to prove the following general result (see [LV]): Let γ be a finite order diffeomorphism of a topological surface of type $(g,[n])$, corresponding to a torsion element $\gamma \in \Gamma_g^{[n]}$ (with the same name). One associates with the conjugacy class of the cyclic group $\langle\gamma\rangle$ a nonempty closed integral substack M_γ of the $\mathbb{Q}$-stack $M_{g,[n]}$, called the *special locus* of γ, whose closed points correspond to algebraic curves having an automorphism which acts topologically like γ. This locus M_γ is defined over a finite extension $k = k(\gamma)$ of $\mathbb{Q}$ (where one needs to be a little careful about the notion of field of definition; a formal way to put it is that the (coarse) moduli space associated to the residual gerbe of the generic point of M_γ is isomorphic to $Spec(k)$). The result then says that for $\sigma \in G_k$, σ preserves the conjugacy class of γ. This is a completely general fact, coming from the behavior of stack inertia under the Galois action.

In this context, one can view the work in [NT] and the present paper as computing the conjugating factors of the finite order elements $\eta = \tau_1\tau_2\tau_1$, $\xi = \tau_1\tau_2$ in $\hat{\Gamma}_0^{[4]}$ and $\vartheta = \tau_1\tau_2\tau_3\tau_4$ in $\hat{\Gamma}_0^{[5]}$ under the $G_{\mathbb{Q}}$-action. Namely, for all $\sigma \in G_{\mathbb{Q}}$, we have

$$\eta \longmapsto \tau_1^{-4\rho_2} f(\eta,\tau_1^2)\cdot\eta^\lambda\cdot f(\tau_1^2,\eta)\tau_1^{4\rho_2}; \tag{6.5.1}$$

$$\xi \longmapsto \tau_1^{\frac{1-\lambda}{2}-3\rho_3} f(\xi,\tau_1^2)\cdot\xi^\lambda\cdot f(\tau_1^2,\xi)\tau_1^{3\rho_3+\frac{\lambda-1}{2}}; \tag{6.5.2}$$

$$\begin{aligned}\vartheta \longmapsto\ & x_{12}^{\frac{1-\lambda}{2}} f(x_{13},x_{12})\tau_{45}^{-\chi_{45}}\tau_{13}^{-\chi_{13}} f(\vartheta_\lambda^3,\tau_{13}\tau_{45})\\ &\cdot{\vartheta_\lambda}^\lambda\cdot f(\tau_{13}\tau_{45},\vartheta_\lambda^3)\tau_{45}^{\chi_{45}}\tau_{13}^{\chi_{13}} f(x_{12},x_{13})x_{12}^{\frac{\lambda-1}{2}},\end{aligned} \tag{6.5.3}$$

where $\eta = \tau_1\tau_2\tau_1$, $\xi = \tau_1\tau_2 \in B_3$ and $\vartheta = \tau_1\tau_2\tau_3\tau_4 \in \Gamma_0^{[5]}$. See §1, Lemma (1.1)(3) for recalling ϑ_λ.

Proof. Since (6.5.1-2) are easier, we only prove (6.5.3) here. First note that a lift of ϑ^3 on $M_{0,5}$ gives the standard path from $\mathcal{A}$ to $\boldsymbol{\rho}(\mathcal{A})$. Therefore, one can comupute $\sigma(\vartheta^3)$ after Theorem 1 (III$'$) and Lemma (1.1)(1) as follows:

$$\begin{aligned}\sigma(\vartheta^3) &= f(x_{12}, x_{23})^{-1}\vartheta^3\\ &= \begin{cases} k^{-1}\vartheta^3 k, & (\lambda \equiv \pm 1 \bmod 5),\\ k^{-1}V_\sharp \vartheta^4 V_\sharp^{-1} k & (\lambda \equiv \pm 2 \bmod 5),\end{cases}\\ &= \begin{cases} k^{-1}\vartheta^{3\lambda} k & (\lambda \equiv 1 \bmod 5),\\ k^{-1}\varepsilon\vartheta^{3\lambda}\varepsilon k & (\lambda \equiv -1 \bmod 5),\\ k^{-1}V_\sharp \vartheta^{3\lambda} V_\sharp^{-1} k & (\lambda \equiv -2 \bmod 5),\\ k^{-1}V_\sharp\varepsilon \vartheta^{3\lambda}\varepsilon V_\sharp^{-1} k & (\lambda \equiv 2 \bmod 5).\end{cases}\end{aligned}$$

This together with Lemma (1.1)(4) and Theorem A computes the conjugating factor of $\vartheta_\lambda^{3\lambda}$ in $\sigma(\vartheta^3)$. As $\vartheta = (\vartheta^3)^2$, the conjugating factor of $\vartheta_\lambda^\lambda$ in $\sigma(\vartheta)$ is the same. □

These hold for the Galois action, but what about the $\widehat{GT}$-action? We have less information in this case; we only know that the following precise version of the above conjugacy result holds in genus 0 for all $\sigma \in \widehat{GT}$: $\sigma(\gamma)$ is a conjugate of $\gamma^{\chi(\sigma)}$ for all finite-order $\gamma \in \hat{\Gamma}_0^n$ (in genus 0, the field of definition k is cyclotomic). It turns out (see [S]) that in a sense we will not explain here, the information contained in the automorphisms in genus 0 is enough to recover the information at infinity in all genera. Note again that it is not known whether the relations in [NT] and the present paper connecting g, h and k with f (as well as those involving f only, such as in Theorems 3 and B) are satisfied in the original version of $\widehat{GT}$ (defined in [Dr], see also [F]).

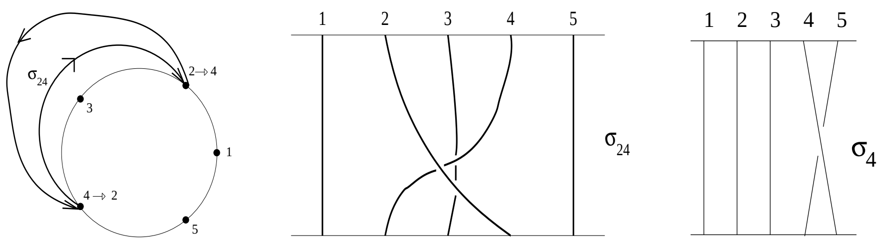
7. Appendix: Dictionary of conventions for paths and generators

In this Appendix, we introduce the "σ-convention" and the "τ-convention", each of which consists of a coherent set of rules on path composition, braid generator systems and associated definitions of $\widehat{GT}$-parameters. Both conventions have been used in recent papers on Grothendieck-Teichmüller theory, and the present paper is based on the τ-convention. We give a recipe for translating formulas into each other so that the reader will be equipped to easily read papers written in either convention.

[σ-convention]: Paths are composed from right to left. If γ_1 is a path from A to B and γ_2 is a path from B to C, then $\gamma_2\gamma_1$ denotes the composed path from A to C. Paths act on functions on the left. If γ is a path from A to B, and if f is a germ of functions defined near A, then $\gamma(f)$ means the germ of functions near B analytically continued along γ. We call this the monodromy action of γ. If c is a small counterclockwise loop around 0 of the affine t-line $\mathbf{A}^1(\mathbb{C})$, then $c(t^{1/n}) = \zeta_n t^{1/n}$. Given an etale cover $\phi : Y \to X$ and a path γ on X from x_0 to x_1, then γ induces a bijection $\gamma : \phi^{-1}(x_0) \to \phi^{-1}(x_1)$. In the monodromy action, we have $(\gamma\gamma')(*) = \gamma(\gamma'(*))$. The standard generators $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$ with $\mathbf{xyz} = 1$ are taken as follows: $\mathbf{x}$ is a small counterclockwise loop around 0; $\mathbf{y}$ is a loop running along the real segment [0,1], turning counterclockwise around 1 and running back along the real segment; $\mathbf{z}$ is a loop running mainly on the *lower* hemisphere and turning counterclockwise around ∞. Defining $\mathbf{p}$ as the path from $\overrightarrow{01}$ to $\overrightarrow{10}$, we introduce the non-commutative proword $\mathbf{f}_\sigma(*, *)$ by the equation in $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$: $\sigma(\mathbf{p}) = \mathbf{p}\mathbf{f}_\sigma(\mathbf{x}, \mathbf{y})$ for $\sigma \in G_\mathbb{Q}$. Pick n points $a_i = \zeta_n^i$ ($i \in \mathbb{Z}/n\mathbb{Z}$) on $\mathbf{P}^1(\mathbb{C}) \cong S^2$. The standard generator system $\{\sigma_{ij} | 1 \leq i \neq j \leq n\}$ of the sphere braid group on $a_1, \ldots, a_n$ is taken so that σ_{ij} interchanges a_i and a_j counterclockwise on the *lower* hemisphere. It is written by the minimal standard generators as

$$\sigma_{ij} = \sigma_i^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j \sigma_{j-1} \cdots \sigma_i \quad (1 \leq i < j \leq n).$$

Here, braids, like paths, are composed from right to left, and drawn on strands numbered from left to right, with σ_i denoting the braid in which the i-th strand crosses to the right *over* the $(i+1)$-st strand. We define the pure braid $\mathbf{x}_{ij} := \sigma_{ij}^2$; the elements σ_{24} and σ_4 are shown as a movement of points and a braid in the following figure.

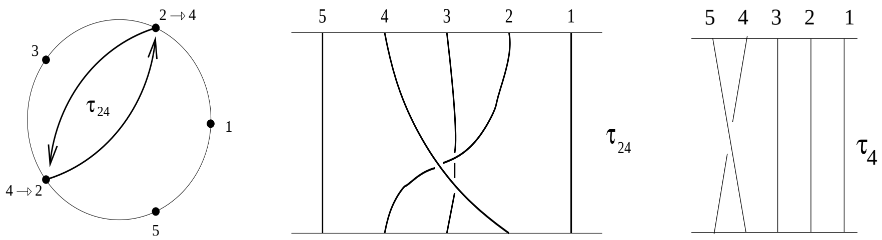


[τ-convention]: Paths are composed from left to right. If γ_1 is a path from A to B and γ_2 is a path from B to C, then $\gamma_1\gamma_2$ denotes the composed path from A to C. Paths act on functions on the left. If γ is a path from A to B, and if f is a germ of functions defined near B, then $\gamma(f)$ means the germ of functions near A analytically continued along γ. We call this the monodromy action of γ. If c is a small counterclockwise loop around 0 of the affine t-line $\mathbf{A}^1(\mathbb{C})$, then $c(t^{1/n}) = \zeta_n^{-1} t^{1/n}$. Given an etale cover $\phi : Y \to X$ and a path γ on X from x_0 to x_1, then γ induces a bijection $\gamma : \phi^{-1}(x_1) \to \phi^{-1}(x_0)$. In the monodromy action, we have $(\gamma\gamma')(*) = \gamma(\gamma'(*))$. The standard generators x, y, z of $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$ with $xyz = 1$ are taken as follows: x is a small counterclockwise loop around 0; y is a loop running along the real segment

[0,1] and turning counterclockwise around 1; z is a loop running mainly on the *upper* hemisphere and turning counterclockwise around ∞. Defining p as the path from $\overrightarrow{01}$ to $\overrightarrow{10}$, we introduce the non-commutative proword $f_\sigma(*,*)$ by the equation in $\pi_1(\mathbf{P}^1 - \{0,1,\infty\}, \overrightarrow{01})$: $\sigma(p) = f_\sigma(x,y)^{-1}p$ for $\sigma \in G_{\mathbb{Q}}$. The standard generator system $\{\tau_{ij} | 1 \leq i \neq j \leq n\}$ of the sphere braid group on the above $\{a_1, \ldots, a_n\} \subset \mathbf{P}^1(\mathbb{C})$ is taken so that τ_{ij} interchanges a_i and a_j counterclockwise on the *upper* hemisphere. It is written by the minimal standard generators as:

$$\tau_{ij} = \tau_i^{-1} \cdots \tau_{j-1}^{-1} \tau_j \tau_{j-1} \cdots \tau_i \quad (1 \leq i < j \leq n).$$

Note here that the braids are composed from left to right, and drawn on strands numbered from right to left, with the generator τ_i crossing the i-th strand to the left *under* the $(i+1)$-st strand. We define the pure braid $x_{ij} := \tau_{ij}^2$, and draw τ_{24} and τ_4 below.



Dictionary. We produce a dictionary between the above two conventions. Our principle is to identify paths as monodromy operators on germs of functions (i.e. we consider pictures of braids or loops only as "superficial expressions" within each convention, and do not base our convention translations on them). This helps us from extra-cares on reversing multiplication orders in translations, by virtue of the above definitions of monodromy in both conventions. So, $\mathbf{p} = p^{-1}$, $\mathbf{x} = x^{-1}$, $\mathbf{y} = y^{-1}$ while $\mathbf{z} = yz^{-1}y^{-1}$. Then, for $\sigma \in G_{\mathbb{Q}}$, $\sigma(\mathbf{p}) = \mathbf{p}\mathbf{f}_\sigma(\mathbf{x},\mathbf{y})$ is equal to $\sigma(p^{-1}) = p^{-1}f_\sigma(x,y) = \mathbf{p}f_\sigma(\mathbf{x}^{-1},\mathbf{y}^{-1})$. Therefore, as prowords of two non-commutative generators X, Y of $\hat{F}_2$, we have

$$f_\sigma(X,Y) = \mathbf{f}_\sigma(X^{-1}, Y^{-1}).$$

It is obvious that the $\widehat{GT}$-relation (I) for $\mathbf{f}_\sigma$ is equivalent to that for f_σ. On the other hand, the equivalence of the $\widehat{GT}$-relation (II) for $\mathbf{f}_\sigma$ and that for f_σ is a(n easy but) non-trivial exercise. We leave it for interested readers. The equivalence of the $\widehat{GT}$-relation (III) for both conventions can be assured from the translation of braids as monodromy operators: $\sigma_i = \tau_i^{-1}$. This gives, in general,

$$\sigma_{ij} = x_{i,i+1}\cdots x_{i,j-1}\tau_{ij}^{-1}x_{i,j-1}^{-1}\cdots x_{i,i+1}^{-1},$$
$$\tau_{ij} = \mathbf{x}_{i,i+1}^{-1}\cdots \mathbf{x}_{i,j-1}^{-1}\sigma_{ij}^{-1}\mathbf{x}_{i,j-1}\cdots \mathbf{x}_{i,i+1}$$

for $1 \leq i < j \leq n$. Under the above translation rule, we have, for example,

$$\mathbf{f}_\sigma(\mathbf{x}_{12}, \mathbf{x}_{23}) = f_\sigma(\mathbf{x}_{12}^{-1}, \mathbf{x}_{23}^{-1}) = f_\sigma(x_{12}, x_{23}).$$

Thus, $\widehat{GT}$-equations in the sphere braid groups look same in both conventions, *as long as* they involve only f-parameters and braid generators on two *consecutive* strings. The relation (III) is a case in point. But recently, more complicated equations have been studied in the theory of $\widehat{GT}$, which involve more general types of generators or local factors with Kummer type characters as exponents. To illustrate the situation, let us consider Theorem A of this paper, for the case $\lambda \equiv 1 \bmod 5$ for simplicity. Then, in the τ-convention, it reads:

$$k_\sigma(\{x_{ij}\}) = \vartheta^{-6\rho_2} f(\tau_{13}\tau_{45}, \vartheta^3)\tau_{13}^{\chi_{13}}\tau_{45}^{\chi_{45}} f(x_{12}, x_{13}) x_{12}^{\frac{\lambda-1}{2}}.$$

where $k_\sigma(\{x_{ij}\})$ is defined by $\sigma(v) = k_\sigma(\{x_{ij}\})^{-1}v$ for a certain path v from $\mathcal{A}$ to Q. The corresponding $\mathbf{k}_\sigma$ should be defined by $\sigma(\mathbf{v}) = \mathbf{v}\mathbf{k}_\sigma(\mathbf{x}_{ij})$ using $\mathbf{v} = v^{-1}$. Then, $\mathbf{k}_\sigma(\mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{34}, \mathbf{x}_{45}, \mathbf{x}_{51}) = k_\sigma(x_{12}, x_{23}, x_{34}, x_{45}, x_{51})$. The translation then gives us

$$\mathbf{k}_\sigma(\mathbf{x}_{ij}) = \boldsymbol{\vartheta}^{-6\rho_2}\mathbf{f}_\sigma(\mathbf{x}_{12}^{-1}\sigma_{13}\sigma_{45}\mathbf{x}_{12}, \boldsymbol{\vartheta}^2)\mathbf{x}_{12}^{-1}\sigma_{13}^{-\chi_{13}}\sigma_{45}^{-\chi_{45}}\mathbf{f}_\sigma(\mathbf{x}_{12}, \mathbf{x}_{13})\mathbf{x}_{12}^{\frac{3-\lambda}{2}}$$

where $\boldsymbol{\vartheta} = \sigma_1\sigma_2\sigma_3\sigma_4 = \tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\tau_4^{-1} = \tau_1\tau_2\tau_3\tau_4 = \vartheta$.

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(*Received January 28, 2004*)