Tangential base points and Eisenstein power series*

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In this note, we discuss a Galois theoretic topic where the two subjects of the title intersect. Three co-related sections will be arranged as follows. In Part I, we review basic notion of tangential base points for etale fundamental groups of schemes of characteristic zero. Then, in Part II, we introduce 'Eisenstein power series' as a main factor of the Galois representation "of Gassner-Magnus type" arising from an affine elliptic curve with 'Weierstrass tangential base point'. Part III is devoted to examining the Eisenstein power series in the case of the Tate elliptic curve over the formal power series ring $\mathbb{Q}[[q]]$ (introduced in Roquette [R], Deligne-Rapoport [DR]). We deduce then a certain explicit relation (Th.3.5) between such Eisenstein power series and Ihara's Jacobi-sum power series [I1].

I

In [GR], A.Grothendieck invented Galois theory for general connected schemes. It is based on axiomatic characterization of a "Galois category" which models on the category $\mathrm{Rev}(X)$ of all finite etale covers of a scheme X. In this theory, the role of a base point of π_1 is played by a certain "Galois functor" $\mathrm{Rev}(X) \to \{\text{finite sets}\}$ which axiomatizes the functor of taking fibre sets over a "base point" for all covers in $\mathrm{Rev}(X)$. Then, a chain between two "base points" is by definition an invertible natural transformation between such Galois functors. In particular, the fundamental group based at a Galois functor Φ is the functorial automorphism group $\mathrm{Aut}(\Phi)$, or equivalently, the automorphism group of the coherent sequence of finite sets $\{\Phi(Y)\}_{Y\in\mathrm{Rev}(X)}$ (with maps induced from those in $\mathrm{Rev}(X)$) topologized naturally as a profinite group.

Recently, the notion of "base point at infinity" seems to be calling certain attentions of Galois-theorists, according as fascinating problems of Grothendieck [G] are known (cf. V.G.Drinfeld [Dr], L.Schneps&P.Lochak [SL].) This notion was founded rigorously by P.Deligne [De] as "tangential base point" for more general π_1-theory of motives (including Betti, de Rham, etale realizations etc.) Still in the original Galois context, G.Anderson and Y.Ihara [AI] initiated effective use of Puiseux power series to represent such a base point, which has led to a number of practical applications in Galois-Teichmüller

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theory ([IM], [IN], [Ma], [N2,3],...) Inspired from these works, in this paper, we shall employ the following simple definition of a tangential base point.

(1.1) Definition. Let X be a connected scheme, and $k((t))$ be the field of Laurent power series in t over a field k of characteristic zero. A k-rational tangential base point on X is, by definition, a morphism $\vec{v} : \operatorname{Spec} k((t)) \to X$. This amounts to giving a scheme-theoretic point x of X together with an embedding of the residue field of x into the field $k((t))$. (The point x may or maynot be a k-rational point of X; see examples below.)

A basic motivation to introduce the above definition is that it has been often the case that a certain role of a "base point at infinity" can be played by a generic point of a 1-dimensional subscheme with specified 1-parameter "t". Let us explain how such a tangential base point $\vec{v}$ could work in the study of Galois representations in fundamental groups. Following Anderson-Ihara [AI], we fix an algebraically closed overfield $\Omega = \bar{k}\{\{t\}\}$, the field of Puiseux power series in the symbols "$t^{1/n}$" with $(t^{1/mn})^m = t^{1/n}$ $(m, n \in \mathbb{N})$, which is the union of the Laurent power series fields $\bar{k}((t^{1/n}))$ for $n \in \mathbb{N}$. Given such a $\vec{v}$ (and $\Omega_{\vec{v}}$), for each cover $Y \in \operatorname{Rev}(X)$, we may associate the set of its $\Omega_{\vec{v}}$-valued points $Y(\Omega_{\vec{v}})$. This is a finite set as the fibre of the finite etale morphism $Y \to X$ over the geometric point $\vec{v}$ on X. Noticing also that every morphism $Y' \to Y$ in $\operatorname{Rev}(X)$ induces a natural map $Y'(\Omega_{\vec{v}}) \to Y(\Omega_{\vec{v}})$, we get a coherent sequence of finite sets $\{Y(\Omega_{\vec{v}})\}$ indexed by the objects $Y \in \operatorname{Rev}(X)$, or equivalently, a fibre functor $\Phi_{\vec{v}} : \operatorname{Rev}(X) \to \{\text{finite sets}\}$ $(Y \mapsto \Phi_{\vec{v}}(Y) = Y(\Omega_{\vec{v}}))$.

Now, the absolute Galois group $G_k = \operatorname{Gal}(\bar{k}/k)$ acts on $\Omega_{\vec{v}}$ by the coefficientwise transformation of power series $\sum_{\alpha\in\mathbb{Q}} a_\alpha t^\alpha \mapsto \sum_{\alpha\in\mathbb{Q}} \sigma(a_\alpha) t^\alpha$, hence induces an automorphism of the sequence $\{Y(\Omega_{\vec{v}})\}_{Y\in\operatorname{Rev}(X)}$ coherently. Thus, we obtain a natural homomorphism

$$s_{\vec{v}} : G_k \to \pi_1(X, \vec{v}) := \operatorname{Aut}(\Phi_{\vec{v}}).$$

When X is defined to be geometrically connected over k, then $s_{\vec{v}}$ gives a splitting of the canonical exact sequence

$$1 \longrightarrow \pi_1(X_{\bar{k}}, \vec{v}) \longrightarrow \pi_1(X, \vec{v}) \xrightarrow{p_{X/k}} G_k \longrightarrow 1. \tag{1.2}$$

By conjugation, $s_{\vec{v}}$ defines a Galois representation

$$\varphi_{\vec{v}} : G_k \to \operatorname{Aut}(\pi_1(X_{\bar{k}}, \vec{v}))$$

which lifts the exterior Galois representation

$$\varphi : G_k \to \operatorname{Out}(\pi_1(X_{\bar{k}}, \vec{v}))$$

induced from the exact sequence (1.2). Generally speaking, the group-theoretic character of φ is independent of the choice of base points, while that of $\varphi_{\vec{v}}$ is dependent on $\vec{v}$.

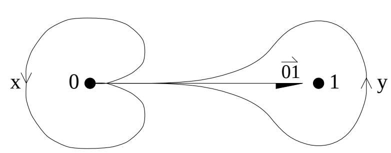
Example 0. Any k-rational point $x \in X(k)$ gives automatically a k-rational tangential base point via $k \hookrightarrow k((t))$.

Example 1. Let $X = \mathbf{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ be the projective t-line over $\mathbb{Q}$ minus the three points $t = 0, 1, \infty$. Then, the residue field of the generic point x of X can be identified with the rational function field $\mathbb{Q}(t)$. The obvious embedding $\mathbb{Q}(t) \hookrightarrow \mathbb{Q}((t))$ determines a morphism

$$\operatorname{Spec} \mathbb{Q}((t)) \to X = \mathbf{P}^1_t - \{0, 1, \infty\},$$

whose target lies on the generic point x of X. We call the tangential base point given by this morphism the standard tangential base point on X, and denote it by $\overrightarrow{01}$. Note that the notion of $\overrightarrow{01}$ depends on the choice of the normalized coordinate t of $\mathbf{P}^1$ setting the 3 punctures to be $t = 0, 1, \infty$.

Now, we have a natural compactification $\mathbf{P}^1$ of X, with respect to which the above $\overrightarrow{01}$ can be extended to the morphism $\operatorname{Spec} \mathbb{Q}[[t]] \to \mathbf{P}^1_t$. This means that the "point" determined by $\overrightarrow{01}$ is not only the generic point x of X but also a uniformizer of the (completed) local ring at $t = 0$ on $\mathbf{P}^1_t$, i.e., a 1-dimensional tangent vector (consisting of 'direction' + 'speed') starting from $t = 0$. (If we change normalization (e.g., scale) of $t \in \mathbb{Q}((t))$ relative to the standard coordinate t of $\mathbf{P}^1$, the represented vector will differ from $\overrightarrow{01}$. In a few contexts where X is regarded as the elliptic modular curve of level 2, another tangential base point "$\frac{1}{16}\overrightarrow{01}$" plays a crucial role, as pointed out in [N2-3].) According to this realization, we usually picture $\overrightarrow{01}$ as a unit tangent vector rooted at 0 towards 1. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ so that the geometric fundamental group $\pi_1(X_{\overline{\mathbb{Q}}}, \overrightarrow{01})$ may be identified with the profinite completion of its natural Betti correspondent. Then, $\pi_1(X_{\overline{\mathbb{Q}}}, \overrightarrow{01})$ is a free profinite group $\hat{F}_2$ freely generated by the standard loops x, y running around the punctures 0,1 respectively.



It is known that the Galois representation $\varphi_{\overrightarrow{01}}$ embeds $G_{\mathbb{Q}}$ into $\operatorname{Aut}\hat{F}_2$ in such a way that

$$\sigma(x) = x^{\chi(\sigma)}, \quad \sigma(y) = f_\sigma(x, y)^{-1} y^{\chi(\sigma)} f_\sigma(x, y) \quad (\sigma \in G_{\mathbb{Q}}) \tag{1.3}$$

with $f_\sigma(x, y) \in [\hat{F}_2, \hat{F}_2]$ (G.V.Belyi), where $\chi : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times$ denotes the cyclotomic character. The pro-word f_σ is uniquely determined by the above formula, and plays a central role in the Grothendieck-Teichmüller theory.

Example 3. (Ihara-Matsumoto [IM]) Let X_n be the affine n-space $\mathbf{A}^n_{\mathbb{Q}}$ minus the discriminant locus D_n whose geometric fundamental group is isomorphic to the profinite Artin braid group $\hat{B}_n$ generated by $\tau_1, \ldots, \tau_{n-1}$ with relations $\tau_i\tau_j = \tau_j\tau_i$ ($|i-j|>1$), $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ ($1 \le i < n-1$). The covering space Y_n corresponding to the pure braid group $\hat{P}_n \subset \hat{B}_n$ can be naturally regarded as $\{(t_1, \ldots, t_n) \in \mathbf{A}^n_{\mathbb{Q}} \mid t_i \neq t_j\,(i \neq j)\}$. Define a tangential base point $\vec{v}' : \mathrm{Spec}\,\mathbb{Q}((t)) \to Y_n$ via $t \mapsto (t, t^2, \ldots, t^n)$, and let $\vec{v}$ be the projection image of $\vec{v}'$ on X_n. Then, it turns out that $\varphi_{\vec{v}} : G_{\mathbb{Q}} \to \mathrm{Aut}\hat{B}_n$ provides the Galois representation of the form:

$$(1.4)\qquad \begin{cases} \sigma(\tau_1) &= \tau_1^{\chi(\sigma)}, \\ \sigma(\tau_i) &= f_\sigma(y_i, \tau_i^2)^{-1}\tau_i^{\chi(\sigma)} f_\sigma(y_i, \tau_i^2) \qquad (1 \le i \le n-1). \end{cases}$$

where $y_i = \tau_{i-1}\cdots\tau_1 \cdot \tau_1 \cdots \tau_{i-1}$. This Galois action is compatible with Drinfeld's formula discovered in the context of quasi-triangular, quasi-Hopf algebras ([Dr]).

Example 4. If a space X is given as a modular variety parametrizing certain types of objects, then to construct a tangential base point $\vec{v} : \mathrm{Spec}\,\mathbb{Q}((t)) \to X$ is equivalent to constructing such an object defined over $\mathbb{Q}((t))$. To do this, sometimes, formal patching method turns out to be useful in smoothing a specially degenerate object $Y_0/\mathbb{Q}$ over $\mathbb{Q}[[t]]$ whose generic fibre $Y_\eta/\mathbb{Q}((t))$ defines $\vec{v}$ with desired properties. We refer to [IN], [N2-3] for some of such examples of tangential base points constructed in the moduli spaces $M_{g,n}$ of the marked smooth curves. The method will also produce a "coalescing tangential base point" $\vec{v}(g)$ on the Hurwitz moduli space $\mathcal{H}(G; C_1, \ldots, C_r)$ associated to a Nielsen class $g \in Ni(G, C)$. Indeed, for any given transitive permutation group $G \subset S_n$ and for any generator system $g = (g_1, \ldots, g_r)$ with $g_1 \cdots g_r = 1$ (g_i lying in a conjugacy class $C_i \subset G$), define $r-2$ triples $\{(x_i, y_i, z_i) \mid i = 1, \ldots, r-2\}$ by setting $x_i = g_1 \cdots g_i$, $y_i = g_{i+1}$, $z_i = g_{i+2} \cdots g_r$. Then, since $x_i y_i z_i = 1$, each (x_i, y_i, z_i) (regarded as a branch cycle datum) defines a (not necessarily connected) branched cover $Y_i \to \mathbf{P}^1$ ramified only over $\{0, 1, \infty\}$. One obtains then an admissible cover $Y_s = \cup_i Y_i/\sim$ over a linear chain P of $\mathbf{P}^1_{01\infty}$ such that $\sim$ identifies the branch points on Y_i and Y_{i+1} according as the cycle orbits by $\langle z_i = x_{i+1}^{-1} \rangle$ in $\{1, \ldots, n\}$. One expects that suitable techniques for smoothing $Y \to P$ (cf. [HS], [W]) should yield a good $\vec{v}(g)$ on the Hurwitz moduli space, generalizing a prototype example given in [N2]. We hope to investigate some aspects of this construction in a circle of ideas of inverse Galois problems ([Fr]).

II

In [I1], Y.Ihara discovered deep aspects of arithmetic fundamental groups by interpolating complex multiplications of Fermat jacobians encoded in $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$. Among other things, he introduced new l-adic power series $\mathcal{F}_\sigma \in \mathbb{Z}_l[[T_1, T_2]]$ ($\sigma \in G_{\mathbb{Q}}$) whose special values at roots of unity recover Jacobi sum grössencharacters. He also conjectured the explicit forms of Galois characters appearing in the coefficients of $\mathcal{F}_\sigma$ in terms of Soule's l-adic cyclotomic elements, which was settled by Anderson [A], Coleman [C], Ihara-Kaneko-Yukinari [IKY] (see 3.6 below). Ihara [I2] also developed an l-adic theory of Fox's free differential calculus to control this power series $\mathcal{F}_\sigma$ in the framework of combinatorial group theory. This enables one to relate $\mathcal{F}_\sigma$ and f_σ of the previous section in a very simple way. We shall employ this treatment also here in a slightly more general setting applicable to higher genus curves.

Let X be a smooth projective curve of genus g over a field k of characteristic 0, S a non-empty closed subset of X with geometric cardinality n, and let $C = X - S$ be the affine complement curve. Practically, we shall be concerned with the "pure affine hyperbolic cases" of $(g, n) = (g, 1)$ or $(0, n)$ ($g \geq 1$, $n \geq 3$), where the geometric fundamental group is a nonabelian free profinite group with its 1-st homology group being pure of weight -1 or -2 respectively.

Fix a rational prime p, and pick a k-rational tangential base point $\vec{v}$ on C. The Galois group G_k acts on $\pi_1(C_{\bar{k}}, \vec{v})$ and hence on its maximal pro-p quotient π. We shall write this action as $\varphi_{\vec{v}}^{(p)} : G_k \to \mathrm{Aut}(\pi)$. Note that π is a free pro-p group of rank $r := 2g + n - 1$ and that its abelianization H is canonically identified with the p-adic etale homology group $H_1(C_{\bar{k}}, \mathbb{Z}_p)$ which is a free $\mathbb{Z}_p$-module of rank r. Our interest will be concentrated on the kernel part of the composition map:

$$\rho^{(p)} : G_k \xrightarrow{\varphi_{\vec{v}}^{(p)}} \mathrm{Aut}(\pi) \longrightarrow \mathrm{GL}(H). \tag{2.1}$$

The fixed field of the kernel of $\rho^{(p)}$ (resp. the kernel subgroup $\ker(\mathrm{Aut}(\pi) \to \mathrm{GL}(H))$ of $\mathrm{Aut}(\pi)$) will be denoted by $k(1)$ (resp. $\mathrm{Aut}_1\pi$).

In order to analyze the restriction $\varphi_{\vec{v}}^{(p)}|_{G_{k(1)}}$ closely, we construct a certain combinatorial (anti-)representation

$$\bar{\mathfrak{A}} : \mathrm{Aut}_1\pi \to \mathrm{GL}_r(\mathbb{Z}_p[[H]]) \tag{2.2}$$

in analogy with the Gassner-Magnus representation in combinatorial group theory (cf. Ihara [I2], see also [Bi], [Mo] for topological aspects). Here, $\mathbb{Z}_p[[H]]$ is the abelianization of the complete group algebra $\mathbb{Z}_p[[\pi]]$ which is by definition the projective limit of the finite group rings $(\mathbb{Z}/p^n\mathbb{Z})[\pi/N]$ over the open normal subgroups $N \subset \pi$ and $n \in \mathbb{N}$. The Gassner-Magnus representation is a basic device to look at operations on the maximal meta-abelian quotient of π (cf. 2.7 below). To give its precise definition, we shall first introduce some terminology of free differential calculus.

If $\{x_1, \ldots, x_r\}$ is a free generator system of π, then, as shown by Lazard [La], $\mathbb{Z}_p[[\pi]]$ can be regarded as the ring of formal power series in non-commutative variables $t_i := x_i - 1$ ($i = 1, \ldots, r$) over $\mathbb{Z}_p$. Each element

$\lambda \in \mathbb{Z}_p[[\pi]]$ is then written in the form

$$\lambda = \sum_w a_w \cdot w \qquad (a_w \in \mathbb{Z}_p),$$

where w runs over all finite words in $\{t_1, \ldots, t_r\}$ with non-negative exponents including the unity 1. We call a_1 the constant term of λ and denote it by $\varepsilon(\lambda)$. Classifying the other terms $a_w \cdot w$ $(w \neq 1)$ of λ according to the right most letters, we may write uniquely $\lambda = \varepsilon(\lambda) + \sum_{i=1}^r \lambda_i t_i$ $(\lambda_i \in \mathbb{Z}_p[[\pi]])$. This λ_i is by definition the i-th free differential of λ, and will be denoted by $\partial\lambda/\partial x_i$:

$$\lambda = \varepsilon(\lambda) + \sum_{i=1}^{r} \frac{\partial \lambda}{\partial x_i}(x_i - 1). \tag{2.3}$$

In the following, we use capital letters X_i, T_i to designate the images of $x_i, t_i \in \mathbb{Z}_p[[\pi]]$ in the abelianization $\mathbb{Z}_p[[H]]$ $(i = 1, 2)$. Obviously, it follows that

$$\mathbb{Z}_p[[H]] = \mathbb{Z}_p[[T_1, \ldots, T_r]] \qquad (T_i = X_i - 1).$$

For a general element $\lambda \in \mathbb{Z}_p[[\pi]]$, we write $\lambda^{\mathfrak{ab}}$ for its image in $\mathbb{Z}_p[[H]]$.

(2.4) Definition. For $\alpha \in \mathrm{Aut}_1\pi$, define its Gassner-Magnus matrix by

$$\bar{\mathfrak{A}}_\alpha := \left((\frac{\partial \alpha(x_i)}{\partial x_j})^{\mathfrak{ab}} \right)_{1 \leq i,j \leq r}.$$

(2.5) Proposition. *The mapping* $\bar{\mathfrak{A}} : \mathrm{Aut}_1\pi \to \mathrm{GL}_r(\mathbb{Z}_p[[H]])$ $(\alpha \mapsto \bar{\mathfrak{A}}_\alpha)$ *is an anti-representation, i.e.,* $\bar{\mathfrak{A}}_{\alpha\alpha'} = \bar{\mathfrak{A}}_{\alpha'}\bar{\mathfrak{A}}_\alpha$ $(\alpha, \alpha' \in \mathrm{Aut}_1\pi)$.

Proof. From direct computation, we have

$$\alpha\alpha'(x_i) = 1 + \sum_k \alpha\left(\frac{\partial \alpha'(x_i)}{\partial x_k}\right)\left(\sum_j \frac{\partial \alpha(x_k)}{\partial x_j}(x_j - 1)\right).$$

The result follows at once from the fact that α acts trivially on $\mathbb{Z}_p[[H]]$. □

(2.6) We note that the above construction of $\bar{\mathfrak{A}}$ depends on the choice of the free basis $(x_1, \ldots, x_r)$ of π. In order to see its dependency on the choice, it would be convenient to introduce, more primitively, the Magnus matrices $\mathfrak{A}_\alpha$ for $\alpha \in \mathrm{Aut}(\pi)$ by

$$\mathfrak{A}_\alpha := \left((\frac{\partial \alpha(x_i)}{\partial x_j}) \right)_{1 \leq i,j \leq r}$$

with entries in the non-commutative algebra $\mathbb{Z}_p[[\pi]]$ satisfying the anti-1-cocycle property: $\mathfrak{A}_{\alpha\beta} = \alpha(\mathfrak{A}_\beta) \cdot \mathfrak{A}_\alpha$ $(\alpha, \beta \in \mathrm{Aut}(\pi))$. Then, one can show that if the free basis $(x_1, \ldots, x_r)$ is replaced by another basis $(x'_1, \ldots, x'_r)$, then the respective Magnus matrices $\mathfrak{A}_\alpha$, $\mathfrak{A}'_\alpha$ are related by "Jacobian matrices" as follows:

$$\mathfrak{A}_\alpha \cdot \frac{\partial(x_1, \ldots, x_r)}{\partial(x'_1, \ldots, x'_r)} = \alpha\left(\frac{\partial(x_1, \ldots, x_r)}{\partial(x'_1, \ldots, x'_r)}\right) \cdot \mathfrak{A}'_\alpha \qquad (\alpha \in \mathrm{Aut}(\pi)).$$

Since $\alpha \in \mathrm{Aut}_1\pi$ acts trivially on $\mathbb{Z}_p[[H]]$, the above implies that the Gassner-Magnus matrix $\bar{\mathfrak{A}}'_\alpha$ w.r.t. $(x'_1, \ldots, x'_r)$ is just the conjugation of $\bar{\mathfrak{A}}_\alpha$ by the Jacobian matrix $(\frac{\partial(x'_1,\ldots,x'_r)}{\partial(x_1,\ldots,x_r)})^{\mathfrak{ab}}$

The usefulness of the anti-1-cocycle representation of $\mathrm{Aut}(\pi)$ through Magnus matrices was shown by Anderson-Ihara [AI] Part 2 in their close study of Galois representations in $\pi_1(\mathbf{P}^1 - \{n \text{ points}\})$ (where a more 'geometric' variant was employed). Independently, Morita [Mo] presented effective applications of Magnus matrices in his theory of "traces" of topological surface mapping classes.

(2.7) We shall now briefly explain how the Gassner-Magnus representation looks at the meta-abelian quotient of π. Let π'' be the double commutator subgroup of π, i.e., $\pi'' = [\pi', \pi']$ where $\pi' = [\pi, \pi]$. Then, the pro-p version of Blanchfield-Lyndon theorem (cf. Brumer [Br] (5.2.2), Ihara [I2] Th.2.2) tells us an exact sequence of $\mathbb{Z}_p[[H]]$-modules:

$$(BL_p) \qquad 0 \to \pi'/\pi'' \xrightarrow{\partial} \mathbb{Z}_p[[H]] \otimes_{\mathbb{Z}_p[[\pi]]} I(\pi) \xrightarrow{\delta} I(H) \to 0,$$

where $I(*)$ denotes the augmentation ideal of $\mathbb{Z}_p[[*]]$, and the maps ∂, δ are defined by $\partial(a \bmod \pi'') = 1 \otimes (a-1)$, $\delta(b \otimes c) = b \cdot c^{\mathfrak{ab}}$. Since $I(\pi)$ is known to be the free $\mathbb{Z}_p[[\pi]]$-module of rank r with basis $x_i - 1$ $(i = 1, \ldots, r)$, one can identify the middle module with $\bigoplus_{i=1}^r \mathbb{Z}_p[[H]] \otimes (x_i - 1) \cong \mathbb{Z}_p[[H]]^{\oplus r}$ so that

$$\partial(a) = \left((\frac{\partial a}{\partial x_1})^{\mathfrak{ab}}, \ldots, (\frac{\partial a}{\partial x_r})^{\mathfrak{ab}}\right) \in \mathbb{Z}_p[[H]]^{\oplus r}.$$

Each automorphism $\alpha \in \mathrm{Aut}(\pi)$ acts on the modules of (BL_p) compatibly, especially on the middle one by $\alpha(b \otimes c) = \alpha(b) \otimes \alpha(c)$. In particular, $\alpha \in \mathrm{Aut}_1(\pi)$ acts on it $\mathbb{Z}_p[[H]]$-linearly with matrix representation given by the (transpose of) Gassner-Magnus matrix $\bar{\mathfrak{A}}_\alpha$. Noticing that π'/π'' is embedded there by ∂, one sees at least that the representation of $\mathrm{Aut}_1(\pi)$ in π'/π'' should be analyzed well by the Gassner-Magnus matrices.

Returning to the situation of Galois representation $\varphi_{\vec{v}}^{(p)} : G_k \to \mathrm{Aut}(\pi)$, our main concern thus turns to look at the composition with the Gassner-Magnus representation:

$$\bar{\mathfrak{A}}_{\vec{v}} = \bar{\mathfrak{A}} \circ \varphi_{\vec{v}}^{(p)}|_{G_{k(1)}} : G_{k(1)} \to \mathrm{GL}_r(\mathbb{Z}_p[[H]]).$$

In the remainder of this section, we review known results on the most basic two cases of $(g,n)=(0,3)$, $(1,1)$. The former is Ihara's original case $C=\mathbf{P}^1-\{0,1,\infty\}$ ([I1,I2]), and the latter case is for $C=$ an elliptic curve minus one point, which was introduced/studied by Bloch [Bl], Tsunogai [T] and the author [N1].

Case 1: $C=\mathbf{P}^1-\{0,1,\infty\}$, $\vec{v}=\overrightarrow{01}$, $x_1=x$, $x_2=y$.

In this case, it follows from computations that

$$\bar{\mathfrak{A}}_{\overrightarrow{01}}(\sigma)=\begin{pmatrix} 1 & 0 \\ \left(\frac{\partial f_\sigma}{\partial x_1}(x_2-1)\right)^{\mathfrak{ab}} & \left(1+\frac{\partial f_\sigma}{\partial x_2}(x_2-1)\right)^{\mathfrak{ab}} \end{pmatrix}$$

for $\sigma\in G_{\mathbb{Q}(\mu_{p^\infty})}$. (Note that $\mathbb{Q}(1)=\mathbb{Q}(\mu_{p^\infty})$ now). The power series

$$\mathcal{F}_\sigma(T_1,T_2):=\det\bar{\mathfrak{A}}_{\overrightarrow{01}}(\sigma)=\left(1+\frac{\partial f_\sigma}{\partial x_2}(x_2-1)\right)^{\mathfrak{ab}}$$

is called the universal power series for Jacobi sums, or Ihara's power series ([I1,2], [Ic], [Mi]). In fact, Ihara showed that the mappings $\sigma\mapsto\mathcal{F}_\sigma(\zeta_{p^n}^a-1,\zeta_{p^n}^b-1)$ $(1\le a,b<p^n)$ represent Jacobi sum grössencharacters over $\mathbb{Q}(\mu_{p^n})$, and also investigated the p-adic local behaviors of the coefficient characters. As in [I1], $\mathcal{F}_\sigma$ can be defined for all $\sigma\in G_{\mathbb{Q}}$, but in the present paper, we content ourselves with treating it only over $\mathbb{Q}(\mu_{p^\infty})$. By the above definition and (2.6), we see that $\mathcal{F}_\sigma$ in this range is determined only by the abelianization of the free basis (x_1,x_2) of π. In view of the above $(BL)_p$ specialized to this case, the module π'/π'' turns out to be a free $\mathbb{Z}_p[[H]]$-module of rank 1 generated by the class of $[x_1,x_2]=x_1x_2x_1^{-1}x_2^{-1}$ mod π''. The image of ∂ is generated by $\partial([x_1,x_2])=(-T_2,T_1)\in\mathbb{Z}_p[[H]]^{\oplus 2}$, and from this follows that $\bar{\mathfrak{A}}_{\overrightarrow{01}}(\sigma)$ (hence $\varphi_{\overrightarrow{01}}(\sigma)$) acts on $\mathrm{Im}(\partial)\cong\pi'/\pi''$ by multiplication by $\mathcal{F}_\sigma(T_1,T_2)$. This was in fact Ihara's original definition of $\mathcal{F}_\sigma$ in [I1].

Case 2: $C: y^2=4x^3-g_2x-g_3$ $(g_2,g_3\in k)$, $\Delta=g_2^3-27g_3^2\neq 0$.

In this case, we will take suitable generators x_1,x_2,z of π with $[x_1,x_2]z=1$ so that z generates an inertia subgroup over the missed infinity point $O\in X$. For a tangential base point, we take $\vec{\mathfrak{w}}:\mathrm{Spec}\,k((t))\to C$ defined by $t:=-2x/y$ and call it the Weierstrass base point. The fixed field $k(1)$ of the kernel of G_k-action on $H=\pi/[\pi,\pi]$ is the field generated by all coordinates of the p-power division points of E over k. As shown in [N1] §6, there exists a unique power series $\mathcal{E}_\sigma(T_1,T_2)\in\mathbb{Z}_p[[T_1,T_2]]$ such that

$$\bar{\mathfrak{A}}_{\vec{\mathfrak{w}}}(\sigma)=\mathbf{1}_2+\mathcal{E}_\sigma\cdot\begin{pmatrix} T_1T_2 & -T_1^2 \\ T_2^2 & -T_1T_2\end{pmatrix}$$

for all $\sigma\in G_{k(1)}$. It is easy to see that $\mathcal{E}_\sigma$ depends only on the abelianization image $(\bar{x}_1,\bar{x}_2)$ of the free basis (x_1,x_2) of π. We shall call $\mathcal{E}_\sigma$ the Eisenstein power series associated to the Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ and the basis $(\bar{x}_1, \bar{x}_2)$ of H. As in Case 1, again, the image of π'/π'' in $\mathbb{Z}_p[[H]]^{\oplus 2}$ via ∂ of (BL_p) is the free $\mathbb{Z}_p[[H]]$-module of rank 1 generated by $\partial(z) = (T_2, -T_1)$, but this time the action of $\bar{\mathfrak{A}}_\alpha$ ($\alpha \in \mathrm{Aut}_1\pi$) on this image is trivial. This means that $\mathcal{E}_\sigma$ ($\sigma \in G_{k(1)}$) should be understood as an invariant of the 'unipotent' action of $\varphi_{\vec{\mathfrak{w}}}(\sigma)$ on the extension of (BL_p). In fact, using Bloch's construction described in [T],[N1], one can show more explicitly that

$$\varphi_{\vec{\mathfrak{w}}}(\sigma)(1 \otimes (x_i - 1)) = 1 \otimes (x_i - 1) + \mathcal{E}_\sigma(T_1, T_2)\, T_i \cdot \partial(z) \qquad (i = 1, 2)$$

holds for $\sigma \in G_{k(1)}$ in $\mathbb{Z}_p[[H]] \otimes_{\mathbb{Z}_p[[\pi]]} I(\pi)$.

(2.8) Remark. In [N1], we employed a special section $s : G_{k(1)} \to \pi_1(C)$ characterized by a certain group theoretical property instead of that induced from the above $\vec{\mathfrak{w}}$. The power series α_σ given in loc.cit. is the same as $\mathcal{E}_\sigma$ except that it misses constant term. If $p \geq 5$, the constant term of $\mathcal{E}_\sigma$ is $\frac{1}{12}\rho_\Delta(\sigma)$ where $\rho_\Delta : G_{k(1)} \to \mathbb{Z}_p$ is the Kummer character defined by the p-power roots of Δ.

In the next part III, we will show that $\mathcal{E}_\sigma$ for the Tate elliptic curve over $\mathbb{Q}((q))$ degenerates to a "logarithmic partial derivative" of Ihara's power series $\mathcal{F}_\sigma$.

III

In this section, we shall examine the Eisenstein power series arising from the Tate curve $T =$"$\mathbf{G}_m/q^{\mathbb{Z}}$" over the rational power series ring $\mathbb{Q}[[q]]$ in one variable q. The affine equation defining T (minus the origin O) is

$$y^2 + xy = x^3 + a_4(q)x + a_6(q), \tag{3.1}$$

where $a_4(q)$, $a_6(q) \in \mathbb{Q}[[q]]$ are given by

$$a_4(q) = -5s_3(q), \quad a_6(q) = -\frac{1}{12}(5s_3(q) + 7s_5(q)),$$
$$s_k(q) := \sum_{n \geq 1} \sigma_k(n)q^n = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n} \quad (k \geq 1).$$

The equation modulo q is $y^2 + xy = x^3$, hence T has a split multiplicative reduction. Indeed, the $\overline{\mathbb{Q}((q))}$-rational points of $T - \{O\}$ are uniformized by $u \in \overline{\mathbb{Q}((q))}^\times \setminus q^{\mathbb{Z}}$ through the formulae:

$$x(u, q) = \sum_{n \in \mathbb{Z}} \frac{q^n u}{(1 - q^n u)^2} - 2s_1(q), \quad y(u, q) = \sum_{n \in \mathbb{Z}} \frac{(q^n u)^2}{(1 - q^n u)^3} + s_1(q),$$

and the special fibre T_s at $q = 0$ may be regarded as the nodal projective u-line with two points $u = 0, \infty$ identified (cf. [Si] V §3.) The completed local neighborhood at $u = 1$ in T can be identified as $\mathrm{Spec}\,\mathbb{Q}[[q, u-1]]$ whose generic fibre is the spectrum of $\mathbb{Q}[[q, u-1]] \otimes_{\mathbb{Q}[[q]]} \mathbb{Q}((q))$, the ring of formal power series in $u-1$ with bounded coefficients from $\mathbb{Q}((q))$. In the latter ring, we may arrange the mapping $q \mapsto t'$, $u-1 \mapsto t'$ to define a tangential base point valued in $\mathbb{Q}((t'))$ on the generic fibre $T_\eta/\mathbb{Q}((q))$ of T minus the origin O_η. We write this tangential base point as $\vec{\mathfrak{t}} : \mathrm{Spec}\,\mathbb{Q}((t')) \to T_\eta - \{O_\eta\}$ and call it the Tate base point. In the following, we shall look at the Galois representation $\varphi_{\vec{\mathfrak{t}}} : G_{\mathbb{Q}} \to \mathrm{Aut}\,\pi_1(T_{\bar\eta} \setminus O)$, where $T_{\bar\eta} \setminus O$ denotes the generic geometric fibre of $T - \{O\}$.

First, let us connect the above $\mathbb{Q}$-rational base point $\vec{\mathfrak{t}}$ with the $\mathbb{Q}((q))$-rational Weiserstrass base point $\vec{\mathfrak{w}}$ on the generic elliptic curve T_η (introduced in the previous section). Indeed, we see that these two base points give essentially the same Galois action on $\pi_1(T_{\bar\eta} \setminus O)$ as follows. First, let us apply the change of variables "$X = x + \frac{1}{12}$, $Y = x + 2y$" to (3.1) to get the equation of Weierstrass form

$$Y^2 = 4X^3 - g_2(q)X - g_3(q), \tag{3.2}$$

where

$$g_2(q) = 20\left(-\frac{B_4}{8} + \sum_{n\geq 1} \sigma_3(n)q^n\right),$$
$$g_3(q) = \frac{7}{3}\left(-\frac{B_6}{12} + \sum_{n\geq 1} \sigma_5(n)q^n\right).$$

($B_4 = -1/30$, $B_6 = 1/42$ are the Bernoulli numbers.) Then, as explained in the previous section, the Weierstrass base point $\vec{\mathfrak{w}}$ on $T_\eta - O_\eta$ is defined as a tangential basepoint valued in $\mathbb{Q}((q))((t))$ by putting $t = -2X/Y$. Our claim here is that the Galois representation $\varphi_{\vec{\mathfrak{t}}} : G_{\mathbb{Q}} \to \mathrm{Aut}\,\pi_1(T_{\bar\eta} \setminus O)$ is essentially the same as the composite of $\varphi_{\vec{\mathfrak{w}}} : G_{\mathbb{Q}((q))} \to \mathrm{Aut}\,\pi_1(T_{\bar\eta} \setminus O)$ with the map $G_{\mathbb{Q}} \to G_{\mathbb{Q}((q))}$, where the last map is the one obtained from the coefficientwise $G_{\mathbb{Q}}$-action on the Puiseux power series in $\overline{\mathbb{Q}((q))} \hookrightarrow \overline{\mathbb{Q}}\{\{q\}\}$. Indeed, since x and y can be written respectively in the forms $(u-1)^{-2}(1 + \sum_m \alpha_m(u-1)^m)$, $-(u-1)^{-3}(1 + \sum_m \beta_m(u-1)^m)$ with $\alpha_m, \beta_m \in \mathbb{Q}[[q]]$ (cf. [Si] V §4), the coefficientwise $G_{\mathbb{Q}}$-actions on the two rings $\overline{\mathbb{Q}}[[q^{1/N}, t^{1/N}]]$, $\overline{\mathbb{Q}}[[q^{1/N}, (u-1)^{1/N}]]$ are compatible with their natural identification via $t = -2X/Y \equiv u-1 \bmod^{\times} (1 + (u-1)\mathbb{Q}[[q, u-1]])$. From this, the above relation of $\varphi_{\vec{\mathfrak{w}}}$ with $\varphi_{\vec{\mathfrak{t}}}$ follows.

Now, fix a rational prime p, and consider the p-adic Tate module $H = \varprojlim_m T_{\bar\eta}[p^m]$. As is well-known, in the Tate curve case, H is an extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p(1)$. But in our case, we may split the extension in a natural way as follows. In fact, by the Tate uniformization by "$\mathbf{G}_m$" of $T_{\bar\eta}$, the p^m-division points $T_{\bar\eta}[p^m]$ may be identified with the subset $\{\zeta_{p^m}^a q^{b/p^m} \mid 0 \le a, b < p^m\}$ of $\overline{\mathbb{Q}}\{\{q\}\}^\times$ with natural $G_{\mathbb{Q}((q))}$-action. (Here ζ_{p^m} is a primitive p^m-th roots of unity; we select those so that $\zeta_{p^n}^{p^m} = \zeta_{p^{n-m}}$ $(0 < m < n)$ once and for all.) Thus, we can take generators $\bar x_1, \bar x_2$ of H as projective sequences $\{q^{1/p^n}\}$, $\{\zeta_{p^m}\}$ respectively to obtain a splitting $H = \mathbb{Z}_p \bar x_1 \oplus \mathbb{Z}_p(1)\bar x_2$.

Let us then consider the maximal pro-p quotient π of $\pi_1(T_{\bar\eta} \setminus O, \vec{\mathfrak{t}})$ and the associated Galois representation $\varphi_{\vec{\mathfrak{t}}}^{(p)} : G_{\mathbb{Q}} \to \mathrm{Aut}(\pi)$. Then, with respect to the above basis $(\bar x_1, \bar x_2)$ of H, we have the Gassner-Magnus representation $\bar{\mathfrak{A}}_{\vec{\mathfrak{t}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p[[H]])$, which yields the (Tate-)Eisenstein power series $\mathcal{E}_\sigma^{\vec{\mathfrak{t}}}(T_1, T_2) \in \mathbb{Z}_p[[H]]$ $(\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})})$ defined by

$$\bar{\mathfrak{A}}_{\vec{\mathfrak{t}}}(\sigma) = \mathbf{1}_2 + \mathcal{E}_\sigma^{\vec{\mathfrak{t}}} \cdot \begin{pmatrix} T_1T_2 & -T_1^2 \\ T_2^2 & -T_1T_2 \end{pmatrix}.$$

(3.3) Theorem. *Let $U_i = \log(1+T_i)$ $(i = 1, 2)$. Then, in $\mathbb{Q}_p[[U_1, U_2]]$, we have*

$$\mathcal{E}_\sigma^{\vec{\mathfrak{t}}}(T_1, T_2) = \sum_{\substack{m \ge 2 \\ even}} \frac{\chi_{m+1}(\sigma)}{1 - p^m} \frac{U_2^m}{m!} \qquad (\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}).$$

Here $\chi_m : G_{\mathbb{Q}(\mu_{p^\infty})} \to \mathbb{Z}_p(m)$ is the m-th Soule character defined by the properties:

$$\left(\prod_{\substack{1 \le a < p^n \\ p \nmid a}} (1 - \zeta_{p^n}^a)^{a^{m-1}} \right)^{\frac{1}{p^n}(\sigma - 1)} = \zeta_{p^n}^{\chi_m(\sigma)} \quad (\forall n \ge 1).$$

Proof. The statement follows from a more general formula given in [N1] which states that the coefficient $\kappa_{ij}(\sigma)$ of $U_1^i U_2^j/(1 - l^{i+j}) i! j!$ $((i,j) \ne (0,0))$ is determined by the following Kummer properties:

$$\left(\prod_{\substack{0 \le a, b < p^n \\ p \nmid (a,b)}} (\theta_{ab}^{(p^n)})^{a^i b^j} \right)^{\frac{1}{p^n}(\sigma - 1)} = \zeta_{p^n}^{12\kappa_{ij}(\sigma)} \quad (\forall n \ge 1),$$

where, for $0 \le a, b < N = p^n$,

$$\theta_{ab}^{(N)} = q^{6B_2(\frac{a}{N})} \zeta_N^{6b(\frac{a}{N} - 1)} \left[(1 - q^{\frac{a}{N}} \zeta_N^b) \prod_{n \ge 1} (1 - q^{n + \frac{a}{N}} \zeta_N^b)(1 - q^{n - \frac{a}{N}} \zeta_N^{-b}) \right]^{12}.$$

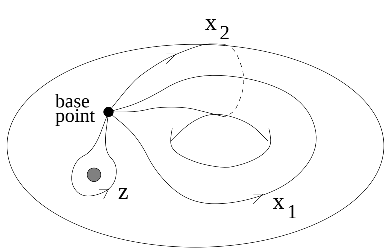
Here $B_2(T) = T^2 - T + \frac{1}{6}$ is the second Bernoulli polynomial. Observing that the coefficientwise $G_{\mathbb{Q}(\mu_{p^\infty})}$-action on the p-power roots of $\theta_{ab}^{(p^n)}$ is nontrivial only when $a = 0$, and noticing that $0^i = 1$ only when $i = 0$, we see that κ_{ij} occurs nontrivially only when $i = 0$, in which case it is equal to χ_{j+1}. The constant term turns out to vanish according to Remark(2.8) applied to $\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}$. □

The above result (3.3) may also be deduced by an alternative method relating $\mathcal{E}_\sigma^{\vec{\mathfrak{t}}}$ explicitly with Ihara's power series $\mathcal{F}_\sigma$. We begin with

(3.4) Theorem. *One can take suitable generators x_1, x_2, z of $\pi_1(T_{\bar\eta} \setminus O, \vec{\mathfrak{t}})$ with $[x_1, x_2]z = 1$ such that (x_1, x_2) lifts $(\bar x_1, \bar x_2)$ above and that the Galois representation $\varphi_{\vec{\mathfrak{t}}} : G_\mathbb{Q} \to \mathrm{Aut}\,\pi_1(T_{\bar\eta} \setminus O, \vec{\mathfrak{t}})$ is expressed by the following formulae in terms of $(\chi(\sigma), f_\sigma)$ of §1 Example 1:*

$$\begin{cases} x_1 & \mapsto z^{\frac{1-\chi(\sigma)}{2}} f_\sigma(x_1x_2x_1^{-1}, z)x_1 f_\sigma(x_2^{-1}, z)^{-1}, \\ x_2 & \mapsto f_\sigma(x_2^{-1}, z)x_2^{\chi(\sigma)} f_\sigma(x_2^{-1}, z)^{-1} \\ z & \mapsto z^{\chi(\sigma)} \end{cases}$$

Proof. This assertion is essentially [N3] Cor.(4.5), except that the choice of generators differs from loc.cit. We first consider '$\mathbf{G}_m/q^{n\mathbb{Z}}$' $(n \geq 2)$ over $\mathbb{Q}[[q]]$ and realize the fundamental group of generic geometric fibre minus sections (one for each component) as a Van-Kampen composite of copies $\pi(i)$ $(i \in \mathbb{Z}/n\mathbb{Z})$ of $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$. Identifying $\pi(i) = \langle 0_i, 1_i, \infty_i \mid 0_i 1_i \infty_i = 1\rangle$, we compute the composite as the amalgamated product of the $\pi(i)$'s and $\langle e\rangle$ over the relations $\infty_i^{-1} = 0_{i+1}$ $(0 \leq i < n-1)$, $\infty_{n-1}^{-1} = e0_0e^{-1}$. Setting then standard generators $x_1 = e$, $x_2 = 0_0^{-1}$, $z_i = 1_{i-1}$ $(1 \leq i \leq n)$, we get the relation $[x_1, x_2]z_1 \cdots z_n = 1$. Then, [N3] Th.(3.15) computes the limit Galois representation $\varphi_{\vec{\mathfrak{t}}}$ on these generators in terms of the parameters $\chi(\sigma), f_\sigma$. The desired Galois representation follows from this computation after reducing $z_1 = z$, $z_2 = \cdots = z_n = 1$ (and checking its subtle independence of n). □



Using the above, we shall compute $\mathcal{E}_\sigma^{\vec{\mathfrak{t}}}$ directly from the definition. Note that it suffices to look at $\frac{\partial\varphi_{\vec{\mathfrak{t}}}^{(p)}(\sigma)(x_2)}{\partial x_2} - 1$ divided by $-T_1T_2$ for $\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}$.

First, we compute

$$\frac{\partial \varphi^{(p)}_{\vec{\mathfrak{t}}}(\sigma)(x_2)}{\partial x_2} - 1 = \frac{\partial f_\sigma x_2 f_\sigma^{-1}}{\partial x_2} - 1 = (1 - f_\sigma x_2 f_\sigma^{-1}) \frac{\partial f_\sigma}{\partial x_2} + f_\sigma - 1,$$

for $\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}$, which maps to $-T_2(\frac{\partial f_\sigma}{\partial x_2})^{\mathfrak{ab}} + 0$ in $\mathbb{Z}_p[[H]]$. But recalling $f_\sigma = f_\sigma(x_2^{-1}, z)$ here, we have

$$\frac{\partial f_\sigma}{\partial x_2} = \frac{\partial f_\sigma(x_2^{-1}, z)}{\partial x_2^{-1}} \frac{\partial x_2^{-1}}{\partial x_2} + \frac{\partial f_\sigma(x_2^{-1}, z)}{\partial z} \frac{\partial z}{\partial x_2},$$

where its first term must vanish in $\mathbb{Z}_p[[H]]$ because $\frac{\partial f_\sigma(x_2^{-1},z)}{\partial x_2^{-1}}(x_2^{-1} - 1)$ is equal to $f_\sigma - 1 - \frac{\partial f_\sigma(x_2^{-1},z)}{\partial z}(z-1)$ which vanishes in $\mathbb{Z}_p[[H]]$. Then since $(\frac{\partial z}{\partial x_2})^{\mathfrak{ab}} = -T_1$, it follows that

$$\mathcal{E}^{\vec{\mathfrak{t}}}_\sigma(T_1, T_2) = -(\lim_{\substack{w_1 \to x_2^{-1} \\ w_2 \to z}} \frac{\partial f_\sigma(w_1, w_2)}{\partial w_2})^{\mathfrak{ab}} = -\lim_{\substack{W_1 \to X_2^{-1} \\ W_2 \to 1}} \frac{\mathcal{F}_\sigma(W_1 - 1, W_2 - 1) - 1}{W_2 - 1}.$$

In terms of the variables $U_i = \log X_i$ $(i = 1, 2)$, we conclude (after de l'Hospital's limit rule) the following relation between the Tate-Eisenstein power series and Jacobi sum power series.

(3.5) Theorem.

$$\mathcal{E}^{\vec{\mathfrak{t}}}_\sigma(T_1, T_2) = -\frac{\partial}{\partial T} \log \mathcal{F}_\sigma(S, T)\Big|_{\substack{S = \exp(-U_2) - 1, \\ T = 0}}$$

for $\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}$. □

This sort of relation between genus 1 and 0 was first expected by Takayuki Oda in his comments on a seminar talk by the author at RIMS, Kyoto University in 1993. The above formula was then obtained in the course of studies along [IN, N3] with Y.Ihara. Theorem (3.3) can then be deduced also by combining Theorem (3.5) with the following formula:

(3.6) Theorem. (Anderson [A], Coleman [C], Ihara-Kaneko-Yukinari [IKY])

$$\mathcal{F}_\sigma(T_1, T_2) = \exp\left(\sum_{\substack{m \geq 3 \\ odd}} \frac{\chi_m(\sigma)}{l^{m-1} - 1} \sum_{\substack{i+j=m \\ i,j \geq 1}} \frac{U_1^i U_2^j}{i! j!} \right) \qquad (\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}). \quad \square$$

The explicit formula (3.6) was proved by Anderson [A], Coleman [C] and Ihara-Kaneko-Yukinari [IKY] independently around 1985. Later Ichimura

[Ic], Miki [Mi] gave simplifications of the proof. All these proofs so far depended on the interpolation properties of the values $\mathcal{F}_\sigma(\zeta_{p^n}^a - 1, \zeta_{p^n}^b - 1)$ by Jacobi sums.

* * *

Recently, (in a more general profinite context) Ihara [I3], using the 5-cyclic relation of the Grothendieck-Teichmüller group, gave a purely algebraic proof of the factorization

$$\mathcal{F}_\sigma(T_1, T_2) = \Gamma_\sigma(T_1)\Gamma_\sigma(T_2)/\Gamma_\sigma((1+T_1)(1+T_2)-1),$$

where $\Gamma_\sigma(T) \in \mathbb{W}_p[[T]]$ is Anderson's Gamma series [A] ($\mathbb{W}_p$: the ring of Witt vectors of $\bar{\mathbb{F}}_p$). In particular, $\mathcal{F}_\sigma$ has to be of the form

$$\mathcal{F}_\sigma = \exp(\sum_{\substack{m\geq 3 \\ odd}} c_m \sum_{i+j=m} \frac{U_1^i U_2^j}{i!j!})$$

with some constants c_m. Then, he derived $c_m = \chi_m(\sigma)/(l^{m-1}-1)$ by a direct method observing meta-cyclic covers of $\mathbf{P}^1 - \{0, 1, \infty\}$ (cf. also [De] §16 for the last technique). Thus, we now have a purely geometric proof of Theorem (3.6) without use of Jacobi sums.

Returning to our elliptic context, we see that combination of Theorems (3.3), (3.5) may also reconfirm the same values of c_m's independently, leading us to an elliptic interpretation of the logarithmic derivative of Anderson's Gamma series (with constant term dropped):

$$D\log\Gamma_\sigma(T_2) - D\log\Gamma_\sigma(0) = \mathcal{E}_\sigma^{\vec{\mathbf{t}}}(T_1, T_2).$$

If p-adic Tate curves "$\mathbb{Q}_p^\times/q^{p^n\mathbb{Z}}$" ($n \in \mathbb{N}$) are employed instead of a single Tate curve over $\mathbb{Q}[[q]]$, then it can be shown that those Eisenstein power series $\{\mathcal{E}_\sigma^{(p^n)}(0, T_2)\}_{n\in\mathbb{N}}$ produce a $\mathbb{Q}_p$-valued distribution whose 'asymptotic expansion' gives a power series

$$2\sum_{\substack{m\geq 2 \\ \text{even}}} E_{-m}^{(p)}(q)\varphi_{m+1}(\sigma)\frac{U_2^m}{m!}$$

for σ in the ramification subgroup $\mathcal{R}$ of $G_{\mathbb{Q}_p(\mu_{p^\infty})}$, where $\varphi_m : \mathcal{R} \to \mathbb{Z}_p$ represents the m-th Coates-Wiles homomorphism, and $E_s^{(p)}(q)$ is the p-adic Eisenstein series $\frac{1}{2}\zeta_p(1-s) + \sum_n \sigma_{s-1}^{(p)}(n)q^n$ of weight s introduced by J.P.Serre [Se]. For this and other arithmetic aspects, we will have more discussions in subsequent works.

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[revisions after printed:]
Figures on p.204, p.213 inserted
p.211: sign of $g_3(q)$
p.212, line 15, $1-l^m$ should read $1-p^m$ (displayed formula)
p.213, line 6, $(1-q^n)^{24}$; line 22, ∞_{n-1}^{-1}

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