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Limits of Galois representations in fundamental groups along maximal degeneration of marked curves, II

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By virtue of Belyi's result [Be], there is a standard way that the elements of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = G_{\mathbb{Q}}$ are faithfully parametrized by a set of pairs $(\chi, \mathfrak{f}) \in \hat{\mathbb{Z}} \times \hat{F}_2$ in the product of the free profinite groups of rank 1 and 2 (cf. [Ih1,2], [Sc]). This paper continues our previous work [N99] of the same title (which will be referred to as Part I below), where we investigated local behaviors of Galois representations in the fundamental groups of marked curves in terms of $(\chi, \mathfrak{f})$. In fact, we showed an explicit method for calculating the elementwise action of $G_{\mathbb{Q}}$ on certain standard generator systems of the fundamental groups of smooth marked curves which are infinitesimally tangent to maximally degenerate stable marked curves. Since the parameter $(\chi, \mathfrak{f})$ is defined by using the fundamental group of $\mathbf{P}^1 - \{0, 1, \infty\}$, our main task was to combine copies of $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$ suitably together into the issued fundamental groups with the van Kampen method.

The purpose of this paper is to push forward this program to the Galois actions on the fundamental groups of *moduli spaces of curves*. We call these fundamental groups the profinite Teichmüller modular groups. In fact, already in previous

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works [IM], [Ma], [N97], this program was studied in special cases. To establish a firm understanding of the general case, in [LNS]-[NS], the following strategy was employed: First, we invented a group $\mathbb{I}\Gamma$ (as a subgroup of what is called the Grothendieck-Teichmüller group $\widehat{GT}$) consisting of pairs $(\chi, \mathfrak{f}) \in \hat{\mathbb{Z}} \times \hat{F}_2$ with a certain finite number of axioms satisfied by the image of $G_{\mathbb{Q}}$. Then, we defined actions of $(\chi, \mathfrak{f}) \in \mathbb{I}\Gamma$ on all the Dehn twist elements in the profinite completion of the surface mapping class group which is isomorphic to the profinite Teichmüller modular group of the corresponding type. We showed that our axioms of $\mathbb{I}\Gamma$ "resolve" the relations among the Dehn twists to insure the well definedness of our $\mathbb{I}\Gamma$-action. As a consequence of our construction, the $\mathbb{I}\Gamma$-actions turned out to be compatible with the cutting and pasting alterations of the underlying surfaces, and established was an algorithm for computing how each element of $\mathbb{I}\Gamma(\supset G_{\mathbb{Q}})$ acts on the Dehn twists in terms of pairs $(\chi, \mathfrak{f})$.

In the last part of [NS], we also presented an explicit formula which describes a standard $\mathbb{I}\Gamma$-action on a certain finite number of Dehn twist generators of any type of mapping class group. The main purpose of the present paper is to show that this formula is indeed properly compatible with the Galois representation arising in the profinite Teichmüller modular group of the corresponding type in algebraic geometry:

THEOREM 5.8 (ABRIDGED FORM). *There is a tangential base point $\vec{a}$ on the moduli space of curves $M_{g,r}$ at which the induced Galois actions on the Dehn twist generators of $\pi_1(M_{g,r} \otimes \overline{\mathbb{Q}}, \vec{a})$ coincide with the restriction to $G_{\mathbb{Q}}$ of the standard $\mathbb{I}\Gamma$-actions on them computed topologically in [NS] §11.*

We will basically use the notation system used in Part I. We write $M_{g,r}$ (resp. $\mathcal{M}_{g,r}$) for the moduli stack over $\mathbb{Q}$ of the proper smooth (resp. stable) curves of genus g with (ordered) r marked points. For a circle c on a (marked) topological surface Σ, D_c denotes the left Dehn twist element along c in the mapping class group $\Gamma(\Sigma)$.

The content of each section of the present Part II is summarized as follows. In §5, we start by constructing a complex analytic model corresponding to the deformation curve associated with the $\mathbf{P}^1_{01\infty}$-diagram of [IN]. This allows us to talk about "Dehn twists in $\pi_1(M_{g,r})$" by the natural comparison isomorphism $\hat{\Gamma}(\Sigma) \cong \pi_1(M_{g,r} \times \overline{\mathbb{Q}})$. Then, we present our main theorem (Theorem 5.8) stating that the Galois representation arising from a certain tangential base point is given by the same explicit formulae of the actions on the Dehn twist generators as those calculated in [NS] §11. In §6, we present the formula of I,§3 on the limit Galois action in the general case of genus g and r marked points. In §7, partially reviewing the result of [N97], we show Theorem 5.8 in the special cases of $M_{g,1}$ and $M_{g,2}$. Here, the new deformation technique of Harbater-Stevenson [HS] enables us to examine a certain subtle factor arising in comparing Galois representations. In §8, we utilize the coupling technique used in [N96] to combine Galois actions on $\pi_1(M_{g,1})$ and $\pi_1(M_{0,r+1})$ inside $\pi_1(M_{g,r})$. This together with the limit Galois information of §6 on the kernel part of the forgetful homomorphism $\pi_1(M_{g,r}) \to \pi_1(M_{g,r-1})$ covers the total of $\pi_1(M_{g,r})$. In §9, we conclude the proof of Theorem 5.8 with reviewing necessary part of the algorithm established in [NS]. In §10, we discuss a generalization of our construction of a tangential base point to the case of Hurwitz moduli spaces. Finally in Appendix we supply a concise proof of the (classical) fact that the finite set of Dehn twists considered in Theorem 5.8 give a generator system of the mapping class group of type (g, r).

Before closing Introduction, we will mention briefly some related works which motivated our present paper but will not be mentioned in the text below. The problem of describing the Galois representations in $\pi_1(M_{g,r})$ by combining those in low levels was initially posed by A.Grothendieck [G] in his philosophy "Lego game of Galois-Teichmüller". (A rigorous definition of $\pi_1(M_{g,r})$ through resolution of the moduli stack by simplicial schemes is given in T.Oda [Od].) The use of graph of groups in the study of the universal monodromy representation of $\pi_1(M_{g,r})$ was initiated in the paper by Asada-Matsumoto-Oda [AMO]. As mentioned in Introduction of Part I, one can think of our present work as standing at a crossroad of these two research streams. Recently, T.Ichikawa [Ic] brought into the field another interesting viewpoint of Mumford's uniformization of curves, which supports our topological 'A-move' algorithm of [NS] in his algebro-geometric context. We would like to expect future work illustrating another 'S-move' algorithm in the context of algebraic geometry.

§5. Complex analytic model: prelude to Part II.

5.1. Suppose we are given a maximally degenerate marked stable curve (viz. $\mathbf{P}^1_{01\infty}$-diagram) over $\mathbb{Q}$ by the collection of data:

$$\left(X^0 = \bigcup_{\lambda\in\Lambda} X^0_\lambda, \{P^0_\mu\}_{\mu\in M}, \{Q^0_\nu\}_{\nu\in N}\right),$$

where X^0 consists of the rational irreducible components $X^0_\lambda(\cong \mathbf{P}^1_{\mathbb{Q}})$ $(\lambda \in \Lambda)$ connected by the ordinary double $\mathbb{Q}$-points $\{P^0_\mu\}_{\mu\in M}$ in such a way that each component X^0_λ has exactly three *distinguished* ($\mathbb{Q}$-)*points*, i.e., from $\{P^0_\mu\}_{\mu\in M} \cup \{Q^0_\nu\}_{\nu\in N}$ (cf. I,§3). For simplicity, we often write X^0 to designate all the above data. We call $\{Q^0_\nu\}_\nu$ the set of marked points on X^0, and write μ/λ (resp. ν/λ) when the point P^0_μ (resp. Q^0_ν) lies on the component X^0_λ. With the above collection of data, naturally associated is the dual graph Δ^0 having the vertex set Λ, the edge set M and the leg (= half edge) set N so that each vertex has valency 3 (counting adjacent edges and legs together). The rank g of the 1-st homology of Δ^0 is by definition the genus of X^0. We call X^0 a $\mathbf{P}^1_{01\infty}$-diagram of type $(g, |N|)$. Assume $2 - 2g - |N| < 0$ and $(g, |N|) \neq (0, 3)$.

A tangential structure $\mathcal{T}$ on X^0 means a collection $\{t_{\mu/\lambda}\}$ of coordinates $t_{\mu/\lambda}$ of X^0_λ $(\lambda \in \Lambda)$ which are chosen for all incidence pairs μ/λ under the rule that $t_{\mu/\lambda}$ should have value 0 (resp. 1 or ∞) at the point P^0_μ (resp. at any distinguished point on X^0_λ other than P^0_μ).

Let q be a formal variable. In [IN, Th.1′], applying the Grothendieck formal patching method with the equations $t_{\mu/\lambda}t_{\mu/\lambda'} = q$ $(\mu/\lambda, \lambda', \lambda \neq \lambda')$, we constructed a deformation scheme $X/\mathbb{Q}[\![q]\!]$ of $X^0/\mathbb{Q}$ equipped with marked sections $\{Q_\nu\colon \mathrm{Spec}\,\mathbb{Q}[\![q]\!] \to X\}_{\nu\in N}$ extending the set $\{Q^0_\nu\}_{\nu\in N}$. We call $X/\mathbb{Q}[\![q]\!]$ the *standard deformation of* $(X^0, \mathcal{T})$. In this section, we shall consider the complex analytic correspondent to the deformation space X.

5.2. We once and for all fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let us construct an analytic stable family $\{\mathcal{X}^q, \{Q^q_\nu\}_\nu\}$ of marked Riemann surfaces over the disk $\mathbb{U} = \{q \in$

$\mathbb{C}; |q| < \frac{1}{4}\}$ so that it extends the degenerate fibre $\mathcal{X}^0 = X^0(\mathbb{C})$. For $q \in \mathbb{U} \setminus \{0\}$, let us construct the smooth Riemann surface $\mathcal{X}^q$ as follows. First define, for each $\lambda \in \Lambda$, the local piece $\mathcal{X}^q_\lambda$ to be the (marked) bounded domain of $X^0_\lambda(\mathbb{C}) \cong \mathbf{P}^1(\mathbb{C})$ complementary to the open disks $\{|t_{\mu/\lambda}| < \sqrt{|q|}\}$ for all μ/λ ($\mu \in M$) with leaving marked points Q^0_ν for all ν/λ ($\nu \in N$) as Q^q_ν. Since $|q| < min\{\frac{1}{2^2}, (\frac{\sqrt{5}-1}{2})^2\}$, one finds easily that each of these $\mathcal{X}^q_\lambda$ is a Riemann sphere either with three holes, with two holes and one marked point, or with one hole and two marked points as in Figure 5.1. (In [IN] Remark 2.3.11, ε should have read $\frac{1}{2}$). We shall call this type of bounded Riemann sphere (with marked points) *a pair of pants.*

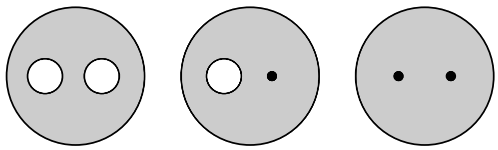

FIGURE 5.1

Next, we shall analytically connect $\mathcal{X}^q_\lambda$'s ($\lambda \in \Lambda$) together as follows. Whenever given incidence relations $\mu/\lambda, \lambda', \lambda \neq \lambda'$ ($\lambda, \lambda' \in \Lambda$, $\mu \in M$), identify $\mathcal{X}^q_\lambda$ (resp. $\mathcal{X}^q_{\lambda'}$) with a piece of the affine line $L = \{(t,t') \in \mathbb{C}^2 \mid tt' = q\}$ by $t_{\mu/\lambda} \mapsto t$ (resp. by $t_{\mu/\lambda'} \mapsto t'$). Then, it is easy to see that $\mathcal{X}^q_\lambda$ and $\mathcal{X}^q_{\lambda'}$ are fit in each other along one of their boundaries in the line L.

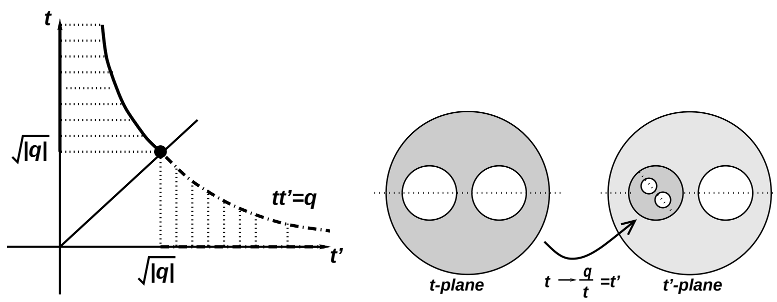

FIGURE 5.2

Continuing this process entirely, we finally obtain a closed Riemann surface $\mathcal{X}^q$ of genus g with $|N|$-marked points $\{Q^q_\nu\}_{\nu \in N}$. It is also obvious that the above construction varies analytically with respect to $q \in \mathbb{U}$ so that $\lim_{q \to 0} \mathcal{X}^q = \mathcal{X}^0$; hence gives our desired family $\{\mathcal{X}^q, \{Q^q_\nu\}_{\nu \in N}\}_{q \in \mathbb{U}}$.

5.3. In the above construction, each $\mathcal{X}^q$ has a pants decomposition $\mathcal{X}^q = \bigcup_\lambda \mathcal{X}^q_\lambda$. Now, suppose moreover that q is taken to be a positive real $\varepsilon \in \mathbb{U}$, i.e., $0 < \varepsilon < \frac{1}{4}$. In this case, the real loci of each $\mathcal{X}^\varepsilon_\lambda$ are continued to those of adjacent $\mathcal{X}^\varepsilon_{\lambda'}$ across their boundary. (See Figure 5.3 for the typical situation.)

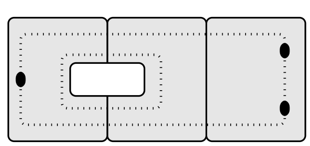

FIGURE 5.3

On $\mathcal{X}^\varepsilon$, let us draw the boundaries of $\mathcal{X}^\varepsilon_\lambda$'s by normal lines (call them the circles of the pants decomposition), and the collection of the real loci by dotted lines. Then, by the above remark, the marked Riemann surface $\mathcal{X}^\varepsilon$ acquires, in addition to the pants decomposition structure $\cup_\lambda \mathcal{X}^\varepsilon_\lambda$, a *quilt decomposition structure (over the pants decomposition)* which is, by definition, a way of decomposing each pair of pants into two "hexagonal rags" so that "seams" (illustrated by dotted lines) meet each other among themselves at exactly two points on each circle of the pants decomposition. (See [NS] for the more elaborated description concerning definitions of quilt, seam etc.) When we make ε approach 0, the $\mathcal{X}^\varepsilon$ is to be fastened at each boundary circle of the pants decomposition, but the combinatorial structure of the quilt/pants decomposition is invariant. In this way, given any $\mathbf{P}^1_{01\infty}$-diagram with a tangential structure, we say that there is a standard quilt decomposition on the marked Riemann surface $\mathcal{X}^\varepsilon$ of type $(g, |N|)$.

5.4. In the above construction, we could have used independent variables q_μ for respective edges $\mu \in M$ to produce an analytic family $\mathcal{X}^{univ}$ over the polydisc $\mathbb{U}^M := \{(q_\mu)_{\mu\in M}; |q_\mu| < \frac{1}{4}\}$. The cardinality of M is exactly $3g-3+|N|$, and the above family gives a local neighborhood of $\mathcal{X}^0$ in the universal family of stable marked curves over the moduli space. The diagonal 1-parameter family over $\{(q,\ldots,q) \in \mathbb{U}^M; |q| < \frac{1}{4}\}$ is nothing but $\mathcal{X}^q$ of §5.2-3. Letting $\varepsilon > 0$ approach 0, we may regard the locus $(\varepsilon,\ldots,\varepsilon)$ of $\mathcal{X}^\varepsilon$ as defining a tangential base point $\vec{b}$ on the moduli stack $M_{g,|N|}(\mathbb{C})$. The resulting (orbifold) fundamental group is in the well known manner identified with $\Gamma(\mathcal{X}^\varepsilon; \{Q^\varepsilon_\nu\}_{\nu\in N})$, the mapping class group of the marked Riemann surface $(\mathcal{X}^\varepsilon; \{Q^\varepsilon_\nu\}_\nu)$ which is by definition the connected component group of the orientation preserving diffeomorphisms of the marked Riemann surface $(\mathcal{X}^\varepsilon, \{Q^\varepsilon_\nu\})$ (marks are pointwise preserved):

$$\pi_1^{orb}(M_{g,|N|}(\mathbb{C}), \vec{b}) = \Gamma(\mathcal{X}^\varepsilon; \{Q^\varepsilon_\nu\}_{\nu\in N}).$$

For example, regard the loop based at $\vec{b}$ with the locus of $(q_\mu(t))_\mu$ $(0 \le t \le 1)$ by

$$q_\mu(t) = \begin{cases} \varepsilon e^{-2\pi i t} & (\mu = \mu_0), \\ \varepsilon & (\mu \neq \mu_0) \end{cases}$$

as an element of $\pi_1(M_{g,|N|}, \vec{b})$. Then, the corresponding element in the mapping class group $\Gamma(\mathcal{X}^\varepsilon; \{Q^\varepsilon_\nu\}_{\nu\in N})$ is the left Dehn twist along the circle of the pants decomposition appearing around the edge μ_0. Since the quilt structure introduced in §5.3 gives a cell decomposition of the marked surface $\mathcal{X}^\varepsilon$, one can detect any other Dehn twists of $\pi_1(M_{g,|N|}, \vec{b})$ also through circles drawn with the quilt on $\mathcal{X}^\varepsilon$.

5.5. Now, let $g, r \in \mathbb{N}$ be positive integers with $2-2g-r<0$, and consider the $\mathbf{P}^1_{01\infty}$-diagram $Y^0 = Y^0_{g,r+1}$ of type $(g, r+1)$ whose dual graph has

$$\begin{cases} \text{vertex set } \Lambda = \{\lambda_1,\ldots,\lambda_{2g},\epsilon_1,\ldots,\epsilon_{r-1}\}, \\ \text{edge set } M = \{\kappa_{\pm(2j-1)}, \kappa_{2j}, \mu_i \mid 1 \le i \le r-2, 1 \le j \le g\}, \\ \text{leg set } N = \{\kappa_0, \nu_1, \ldots, \nu_r\}, \end{cases}$$

connected as in Figure 5.4.

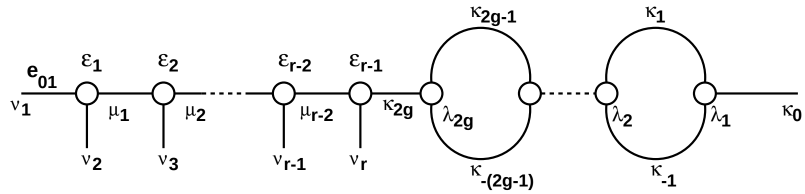

FIGURE 5.4

We introduce a collection of *distinguished coordinates* $\{t_{*/\lambda}\}$ for all incidence pairs $*/\lambda$ ($* \in M \cup N$, $\lambda \in \Lambda$). The choice has $t_{*/\lambda}$ a coordinate of $Y_\lambda^0 \cong \mathbf{P}^1_{\mathbb{Q}}$ valued in $\{0, 1, \infty\}$ at its three distinguished points so that $t_{*/\lambda}(P_*^0) = 0$ (resp. $t_{*/\lambda}(Q_*^0) = 0$) for $* \in M$ (resp. $* \in N$) (cf. §5.1). Further:

(5.5.1)
$$t_{\kappa_{2j}/\lambda_{2j}}(P^0_{\kappa_{-(2j-1)}}) = t_{\kappa_{2j-2}/\lambda_{2j-1}}(P^0_{\kappa_{-(2j-1)}}) = 1, \quad (1 \le j \le g),$$
(5.5.2)
$$t_{\kappa_{\pm(2j-1)}/\lambda_{2j}}(P^0_{\kappa_{2j}}) = t_{\kappa_{\pm(2j-1)}/\lambda_{2j-1}}(P^0_{\kappa_{2j-2}}) = 1, \quad (1 \le j \le g),$$
(5.5.3)
$$t_{\kappa_{2g}/\epsilon_{r-1}}(Q^0_{\nu_r}) = 1,$$
(5.5.4)
$$t_{\mu_i/\epsilon_i}(Q^0_{\nu_{i+1}}) = t_{\mu_i/\epsilon_{i+1}}(Q^0_{\nu_{i+2}}) = 1, \quad (1 \le i \le r-2),$$
(5.5.5)
$$t_{\nu_1/\epsilon_1}(Q^0_{\nu_2}) = t_{\nu_r/\epsilon_{r-1}}(P^0_{\kappa_{2g}}) = t_{\nu_{i+1}/\epsilon_i}(P^0_{\mu_i}) = 1, \quad (1 \le i \le r-2).$$

Define the tangential structure $\mathcal{T}_{g,r+1}$ on $Y^0_{g,r+1}$ to be the subset
$$\{t_{\mu/\lambda} \mid \mu/\lambda; \mu \in M, \lambda \in \Lambda\}.$$
The generic fibre of the standard deformation $Y_{g,r+1}/\mathbb{Q}[\![q]\!]$ of $(Y^0_{g,r+1}, \mathcal{T}_{g,r+1})$ then defines a tangential base point $\vec{a}'$ on $M_{g,r+1}$. Our principal object in the present part II of this paper is the image $\vec{a}$ of $\vec{a}'$ by the morphism $M_{g,r+1} \to M_{g,r}$ obtained by forgetting the marked point Q_{κ_0}. At the tangential base point $\vec{a}$, we have a canonical splitting of the exact sequence
$$1 \longrightarrow \pi_1(M_{g,r} \otimes \overline{\mathbb{Q}}, \vec{a}) \longrightarrow \pi_1(M_{g,r}, \vec{a}) \overset{s_{\vec{a}}}{\curvearrowleft\!\!\longrightarrow} G_{\mathbb{Q}} \longrightarrow 1,$$
and the conjugate $G_{\mathbb{Q}}$-action through the section $s_{\vec{a}}$ on $\pi_1(M_{g,r} \otimes \overline{\mathbb{Q}}, \vec{a})$. If we consider the complex analytic model $\mathcal{Y}^\varepsilon$ for $\vec{a}'$ with marked points $Q^\varepsilon_{\nu_1}, \dots, Q^\varepsilon_{\nu_r}$ (forgetting $Q^\varepsilon_{\kappa_0}$), the geometric fundamental group can be identified with $\hat{\Gamma}(\mathcal{Y}^\varepsilon; \{Q^\varepsilon_{\nu_i}\}_{i=1}^r)$, the profinite completion of the mapping class group of the marked Riemann surface $(\mathcal{Y}^\varepsilon; Q^\varepsilon_{\nu_1}, \dots, Q^\varepsilon_{\nu_r})$. As explained in §5.4, we can detect Dehn twist elements of the latter mapping class group by drawing circles on the complex model. We will examine explicitly the Galois action in terms of certain Dehn twist generators and the parameters in the Grothendieck-Teichmüller group $\widehat{GT}$.

5.6. As presented in §5.3, there is associated the pants (and quilt) decomposition on $\mathcal{Y}^\varepsilon$, which we shall illustrate as in the following Figure 5.5. Here, we understand that the seams for the quilt are given as the "ridges" of the figure dividing each pair of pants into front and back hexagons.

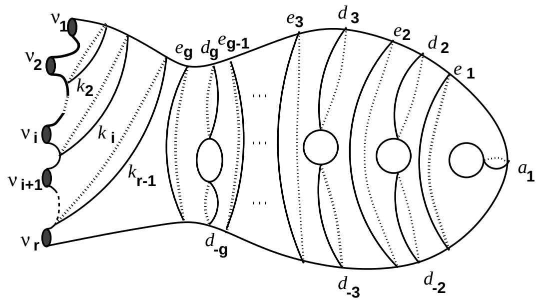

FIGURE 5.5

It is well known that the mapping class group is generated by a finite number of Dehn twists, shown by M.Dehn, W.B.R.Lickorish (cf. also Humphries [H], Mumford [M]). For example, as shown in Appendix, we can take as a generator system the set of Dehn twists along the circles $a_1, \ldots, a_{2g}$, $d_{\pm 2}, \ldots, d_{\pm g}$, $e_1, \ldots, e_g$, $k_2, \ldots, k_{r-1}$, $h_1, \ldots, h_r$ and u_{ij} $(1 \leq i < j \leq r)$ illustrated in Figure 5.5-6.

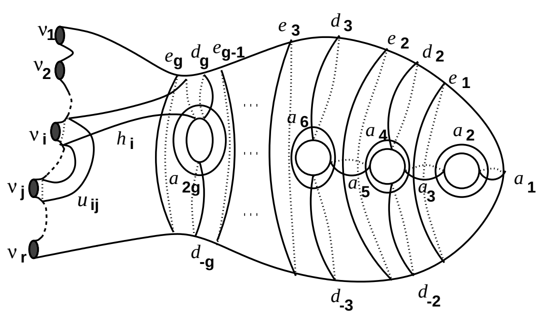

FIGURE 5.6

To describe the $G_{\mathbb{Q}}$-action on these generators, we need to introduce several other notations of circles lying in the left half part of Figure 5.5-6. Let us cut off the part by the circles d_g and d_{-g}, and develop it by regarding d_{-g} as a frame so that the issued part looks as in Figure 5.7. We would like to introduce other types of circles b_{st}, k_{st} $(1 \leq s, t \leq r; s < t)$ as illustrated in the figure.

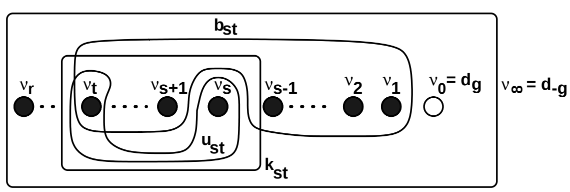

FIGURE 5.7

Note that $k_{1,s} = k_s$, $k_{s,s+1} = u_{s,s+1}$, $k_{2,s} = b_{1,s}$, $k_{1,r} = e_g$. Sometimes it is also convenient to regard $k_s =$'$b_{s+1,s+1}$', $d_g =$'Q_{ν_0}', $d_{-g} =$'Q_{ν_∞}', and set $h_s = u_{0,s} = b_{s,\infty}$, $h_1 = k_{01}$. One can write the Dehn twists along these new circles by the above generator system as follows.

LEMMA 5.7. *The following relations hold for $1 \le i < j \le r$.*

$$D_{k_{ij}} = (D_{u_{i,i+1}})(D_{u_{i,i+2}} D_{u_{i+1,i+2}}) \cdots (D_{u_{ij}} \cdots D_{u_{j-1,j}}), \tag{5.7.1}$$

$$D_{k_{1,i}} D_{k_{ij}} D_{b_{ij}} = D_{k_{1,i-1}} D_{k_{i+1,j}} D_{k_{1,j}}. \tag{5.7.2}$$

PROOF. (5.7.1) is proved in [NS] Lemma 11.1. (5.7.2) is so called the lantern relation among the Dehn twists along the circles drawn in Figure 5.8. □

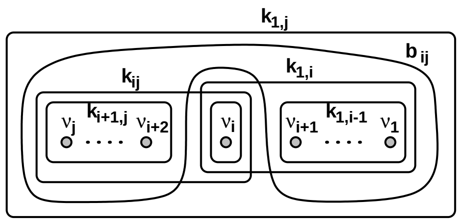

FIGURE 5.8

REMARK. We save notations here by denoting v_{ij} of [NS] by k_{ij}. It is also noted that in the statement of Lemma 11.1 [NS], the definitions of ϵ_0 and ϵ_∞ should have been reversely typeset.

In [NS], we constructed a subgroup $\mathbb{\Gamma}$ of $\widehat{GT}$ containing $G_{\mathbb{Q}}$, and defined actions of $\mathbb{\Gamma}$ on all types of the surface mapping class groups in a certain consistent topological way. The purpose of this paper is to show the following Theorem 5.8 stating that the $G_{\mathbb{Q}}$-action at the tangential base point $\vec{a}$ introduced in §5.5 coincides with that which is obtained from the $\mathbb{\Gamma}$-action of [NS] §11 by restriction to $G_{\mathbb{Q}}$. Let $\chi : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times$ be the cyclotomic character, and put $\rho_{-1}(\sigma) = \frac{\chi(\sigma)-1}{2}$ $(\sigma \in G_{\mathbb{Q}})$. Let $\rho_2 : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}$ be the Kummer 1-cocycle defined by $\sigma(\sqrt[n]{2}) = \sqrt[n]{2}\,\zeta_n^{\rho_2(\sigma)}$ $(n \ge 1, \sqrt[n]{2} > 0)$.

THEOREM 5.8. *Notations being as above, the action of $\sigma \in G_{\mathbb{Q}}$ on the Dehn twist generators of $\pi_1(M_{g,r} \otimes \overline{\mathbb{Q}}, \vec{a})$ can be written explicitly as follows:*

(1) $D_{d_i} \mapsto D_{d_i}^{\chi}$, $D_{d_{-i}} \mapsto D_{d_{-i}}^{\chi}$, $D_{e_i} \mapsto D_{e_i}^{\chi}$, $\quad (1 \le i \le g)$,
$D_{k_i} \mapsto D_{k_i}^{\chi}$, $\quad (2 \le j \le r-1)$

(2) $D_{a_{2i-1}} \mapsto w_{2i-1}^{4\rho_2} f(D_{a_{2i-1}}^2, w_{2i-1}) D_{a_{2i-1}}^{\chi} f(w_{2i-1}, D_{a_{2i-1}}^2) w_{2i-1}^{-4\rho_2}$,
$D_{a_{2i}} \mapsto w_{2i}^{-4\rho_2} f(D_{a_{2i}}^2, w_{2i}) D_{a_{2i}}^{\chi} f(w_{2i}, D_{a_{2i}}^2) w_{2i}^{4\rho_2}$. $\quad (1 \le i \le g)$

(3) $D_{h_s} \mapsto \mathcal{F}_s D_{h_s}^{\chi} \mathcal{F}_s^{-1}$ $(1 \le s \le r)$, *where $\mathcal{F}_s$ is given by*
$$\mathcal{F}_s = f(D_{k_{0,r-1}}, D_{k_{1,r}}) \cdots f(D_{k_{0,s}}, D_{k_{1,s+1}}) \cdot D_{k_{1,s}}^{\rho_{-1}} f(D_{h_s}, D_{k_{1,s}}).$$

(4) $D_{u_{st}} \mapsto \mathcal{F}_{st} D_{u_{st}}^{\chi} \mathcal{F}_{st}^{-1}$ $\quad (1 \le s < t \le r)$, *where $\mathcal{F}_{st}$ is given by*
$$\mathcal{F}_{st} = \left(\prod_{i=0}^{t-s-2} f(D_{k_{t-2-i,t-1}}, D_{k_{1,t-2-i}}) \right) f(D_{k_{st}}, D_{k_{1,t-1}}) D_{k_{s,t-1}}^{-\rho_{-1}} f(D_{u_{st}}, D_{k_{s,t-1}}).$$

Here, $\chi = \chi(\sigma)$, $\rho_{-1} = \rho_{-1}(\sigma)$, $\rho_2 = \rho_2(\sigma)$, $d_{\pm 1} = a_1$, and in the case $t = s+1$, the product in i of (4) is understood to be trivial.

5.9. Before closing this section, we introduce a notion of equivalence relation between tangential base points. Let X be a normal algebraic variety, k a field of characteristic zero, and suppose that we are given two k-rational tangential base points $\vec{v}_i : \operatorname{Spec} k((t)) \to X$ $(i=1,2)$. (See [N98] for an elementary exposition on the notion of tangential base points.) We shall say that $\vec{v}_1$ and $\vec{v}_2$ *share a common support of equivalence*, if there exist

1. a morphism $f : (U_t =)\operatorname{Spec} k[\![t_1,\dots,t_n]\!][\frac{1}{t_1},\dots,\frac{1}{t_n}] \to X$,
2. two k-morphisms $g_i : \operatorname{Spec} k((t)) \to U_t$ $(i=1,2)$ with $\vec{v}_i = f\circ g_i$,

such that both g_1 and g_2 are defined by power series in t with principal coefficients 1, in other words, $g_i^*(t_j)$ $(i=1,2;\ j=1,\dots,n)$ are of the form $t^{k_{ij}}(1+O(t))$ $(k_{ij}>0)$. The usefulness of this property is in the fact that when this is the case,

> *there is a natural isomorphism* $\pi_1(U,\vec{v}_1) \cong \pi_1(U,\vec{v}_2)$ *preserving their Galois sections* $s_{\vec{v}_i} : G_k \to \pi_1(U,\vec{v}_i)$ $(i=1,2)$.

Indeed, let $\operatorname{Rev}(X)$ be the Galois category of etale covers of X. By Abhyankar's lemma [GR, XIII], the components of the pullbacks of objects in $\operatorname{Rev}(X)$ over U_t are dominated by the spectrum of $\Omega := \bigcup_n \bar{k}[\![t_1^{1/n},\dots,t_n^{1/n}]\!][\frac{1}{t_1},\dots,\frac{1}{t_n}]$; hence one can define a base point functor $\operatorname{Rev}(X) \to \mathit{Sets}$ to be that which takes the set of Ω-valued points for each cover. This functor is naturally equivalent with that which takes the set of $\bigcup_n \bar{k}((t^{1/n}))$-valued points of each cover via $\vec{v}_i$ $(i=1,2)$ respectively. By virtue of the condition of the unit principal coefficients, two coefficientwise G_k-actions on those sets via $\vec{v}_1$, $\vec{v}_2$ are originated from the common G_k-action on the coefficients of power series in the ring Ω, hence it turns out that the Galois sections $s_{\vec{v}_i}$ $(i=1,2)$ should coincide with each other. This proves the above assertion.

We will call two tangential base points $\vec{v}$ and $\vec{v}'$ (Galois) *equivalent* (and write $\vec{v} \approx \vec{v}'$), if they are connected by a finite sequence $\vec{v} = \vec{v}_0, \vec{v}_1, \dots, \vec{v}_n = \vec{v}'$ such that $\vec{v}_{i-1}$ and $\vec{v}_i$ share a common support of equivalence for $i=1,\dots,n$. By repeating the above process consecutively, we may identify the Galois functors on $\operatorname{Rev}(X)$ (and the Galois sections $G_k \to \pi_1(X,*)$) induced from equivalent tangential base points. This naive formulation of equivalence, after the idea of "toroidal transformations of tangential base points" due to P.Deligne (cf. [De] §15), will be used essentially in several contexts of our work (including Part I as well as [N97]). To illustrate a typical example, suppose we are given a $\mathbf{P}^1_{01\infty}$-diagram $X^0 = \bigcup_\lambda X^0_\lambda$ of type (g,r) together with a tangential structure $\mathcal{T} = \{t_{\mu/\lambda}\}$ on it. Pick adjacent pairs $\mu/\lambda, \lambda'$ $(\lambda \neq \lambda')$ and consider another tangential structure $\mathcal{T}'$ which differs from $\mathcal{T}$ only in employing $\frac{t_{\mu/\lambda}}{t_{\mu/\lambda}-1}$, $\frac{t_{\mu/\lambda'}}{t_{\mu/\lambda'}-1}$ instead of $t_{\mu/\lambda}$, $t_{\mu/\lambda'}$. Then, by the similar argument to [N97] Lemma (4.3), one can show that the standard deformations of $(X^0,\mathcal{T})$ and of $(X^0,\mathcal{T}')$ determine mutually equivalent tangential base points on $M_{g,r}$. This justifies how one can associate an equivalence class of tangential base points with a topological quilt structure on a marked surface.

§6. Limit Galois representation in genus g case.

6.1. Following the general procedure of Part I §3, we shall calculate the limit Galois representation arising in deformation of a 2-point marked stable curve of genus g. Consider a standard $\mathbf{P}^1_{01\infty}$-diagram of genus g — a chain of 2-point marked Tate elliptic curves — consisting of the data:

$$\Big(Y^0 = \bigcup_{i=1}^{2g} Y^0_{\lambda_i}, \{P^0_{\kappa_{\pm(2i-1)}}, P^0_{\kappa_{2j}}\}_{\substack{1\le i\le g\\ 1\le j<g}}, \{Q^0_{\kappa_0}, Q^0_{\kappa_{2g}}\}\Big)$$

where each $Y^0_{\lambda_i}$ denotes a copy of $\mathbf{P}^1_{\mathbb{Q}}$ with three distinguished $\mathbb{Q}$-rational points lying in the set of double points $\{P^0_{\kappa_{\pm(2i-1)}}, P^0_{\kappa_{2j}}\}_{i,j}$ or in the set of marked points $\{Q^0_{\kappa_0}, Q^0_{\kappa_{2g}}\}$ of Y^0. The incidence relations appear on the following data: The dual graph Δ^0 on the vertex set $\Lambda = \{\lambda_i \mid i = 1, \dots, 2g\}$; the edge set $M = \{\kappa_{\pm(2i-1)}, \kappa_{2j} \mid 1 \le i \le g, 1 \le j < g\}$; and the leg (= half edge) set $N = \{\kappa_0, \kappa_{2g}\}$ connected as in Figure 6.1.

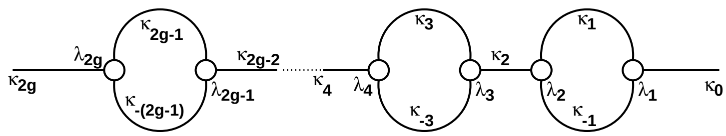

FIGURE 6.1

The above Y^0 (written $Y^0_{g,2}$ occasionally later) is a maximally degenerate marked stable curve of genus g with two marked points over $\mathbb{Q}$. We wish to deform it into a curve over the power series ring $\mathbb{Q}[\![q]\!]$. In order to do it according to the procedure of I§3, let us introduce a tangential structure $\mathcal{T}$ which is a collection of distinguished coordinates for all incidence pairs κ_*/λ_*:

$$\mathcal{T} := \left\{ t_{\kappa_{2j}/\lambda_{2j}}, t_{\kappa_{2j-2}/\lambda_{2j-1}}, t_{\kappa_{\pm(2j-1)}/\lambda_{2j}}, t_{\kappa_{\pm(2j-1)}/\lambda_{2j-1}} \;\middle|\; j = 1, \dots, g \right\}$$

whose members are determined by the following conditions we impose on them:

$$\begin{cases} t_{\kappa_{2j}/\lambda_{2j}}(P^0_{\kappa_{-(2j-1)}}) = t_{\kappa_{2j-2}/\lambda_{2j-1}}(P^0_{\kappa_{-(2j-1)}}) = 1, \\ t_{\kappa_{\pm(2j-1)}/\lambda_{2j}}(P^0_{\kappa_{2j}}) = t_{\kappa_{\pm(2j-1)}/\lambda_{2j-1}}(P^0_{\kappa_{2j-2}}) = 1 \quad (j = 1, \dots, g). \end{cases}$$

Here we also understand $P^0_{\kappa_{2g}} = Q^0_{\kappa_{2g}}$ and $P^0_{\kappa_0} = Q^0_{\kappa_0}$.

Let $Y/\mathbb{Q}[\![q]\!]$ be the standard deformation (§5.1) of $Y^0/\mathbb{Q}$ from the above data $(Y^0, \mathcal{T})$ together with sections $Q_{\kappa_0}, Q_{\kappa_{2g}} : \operatorname{Spec}\mathbb{Q}[\![q]\!] \to Y$ extending the marked points $Q^0_{\kappa_0}, Q^0_{\kappa_{2g}}$ respectively.

6.2. Now, let $\mathcal{D} \subset Y$ be the normal crossing divisor formed by the singular fibre Y^0 and the images of marking sections Q_{κ_0}, $Q_{\kappa_{2g}}$. Our first task is to compute $\pi_1^{\mathcal{D}}(Y, \widetilde{\kappa_{2g}})$, the fundamental group of Y admitting (tame) ramifications along $\mathcal{D}$ with the (standard) tangential base point $\widetilde{\kappa_{2g}}$ at $Q^0_{\kappa_{2g}}$ determined by the Puiseux ring $\overline{\mathbb{Q}}[\![q^{1/n}, t^{1/n}_{\kappa_{2g}/\lambda_{2g}}]\!]_{n\ge 1}$. According to the procedure in I§3, we shall fix a rooted maximal tree (T, e_0) of Δ^0: we set T to be the tree formed by the edges $\{\kappa_{-(2i-1)}, \kappa_{2j} \mid 1 \le i \le g, 1 \le j < g\}$, and set $e_0 = e_{\kappa_{2g}/\lambda_{2g}}$ (see Figure 6.2).

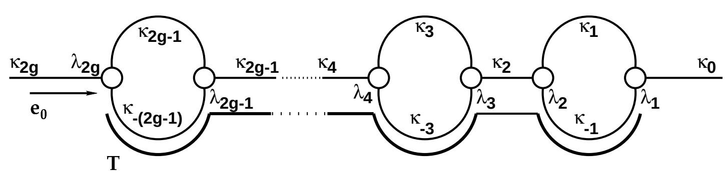

FIGURE 6.2

By Theorem 3.11, $\pi_1^{\mathcal{D}}(Y, \widetilde{\kappa_{2g}})$ is isomorphic to $\pi_1(\Pi/\Delta^0, T)$, the fundamental group of a graph of profinite groups Π/Δ^0 w.r.t. T. Let us now recall the construction of Π/Δ^0: It is a collection of the vertex groups Π_λ ($\lambda \in \Lambda$), the edge groups Π_μ ($\mu \in M$) and the connecting homomorphisms $\mathbf{j}_{\mu/\lambda} : \Pi_\mu \to \Pi_\lambda$ for all incidence pairs μ/λ. The local component groups are of the following forms:

$$\Pi_\lambda = i_\lambda(G_\mathbb{Q}) \ltimes (\langle 0_\lambda, 1_\lambda, \infty_\lambda \mid 0_\lambda 1_\lambda \infty_\lambda = 1\rangle \oplus \langle \tau_\lambda \rangle),$$
$$\Pi_\mu = i_\mu(G_\mathbb{Q}) \ltimes (\langle \tau_{\mu/\lambda}\rangle \oplus \langle \tau_{\mu/\lambda'}\rangle),$$

where 0_λ, 1_λ, ∞_λ, τ_λ, $\tau_{\mu/\lambda}$ and $\tau_{\mu/\lambda'}$ denote free symbols corresponding to certain generators of local fundamental groups. The isomorphic images of $i_\lambda(G_\mathbb{Q})$, $i_\mu(G_\mathbb{Q})$ of the Galois group $G_\mathbb{Q}$ are here acting on τ_*'s by the cyclotomic character $\chi : G_\mathbb{Q} \to \hat{\mathbb{Z}}^\times$. We also understand that $i_\lambda(G_\mathbb{Q})$ is acting on 0_λ, 1_λ, ∞_λ exactly as in $\pi_1(\mathbf{P}^1 - \{0,1,\infty\}, \overrightarrow{01})$, i.e.,

$$0_\lambda \mapsto 0_\lambda^{\chi(\sigma)}, \quad 1_\lambda \mapsto \mathfrak{f}_\sigma(0_\lambda, 1_\lambda)^{-1} 1_\lambda^{\chi(\sigma)} \mathfrak{f}_\sigma(0_\lambda, 1_\lambda),$$

where $\mathfrak{f}_\sigma(X,Y) \in [\hat{F}_2, \hat{F}_2]$ is the principal parameter of the image of $G_\mathbb{Q} \hookrightarrow \widehat{GT}$. In I§2, we introduced auxiliary parameters $\mathfrak{f}_\sigma^{\overrightarrow{ab}} \in \hat{F}_2$ for any pairs $a, b \in \{0,1,\infty\}$ ($a \neq b$) so that $\mathfrak{f}_\sigma^{\overrightarrow{01}} = 1$, $\mathfrak{f}_\sigma^{\overrightarrow{10}} = \mathfrak{f}_\sigma$ ($\sigma \in G_\mathbb{Q}$). Using these, the connecting homomorphism $\mathbf{j}_{\mu/\lambda}$ can be described as

$$\mathbf{j}_{\mu/\lambda} : \begin{cases} i_\mu(\sigma) & \mapsto \mathfrak{f}_\sigma^{\overrightarrow{ab}}(0_\lambda, 1_\lambda) \cdot i_\lambda(\sigma) \qquad (\sigma \in G_\mathbb{Q}), \\ \tau_{\mu/\lambda} & \mapsto \tau_\lambda, \\ \tau_{\mu/\lambda'} & \mapsto \tau_\lambda \cdot a_\lambda, \end{cases}$$

where a, b are respectively the $t_{\mu'/\lambda}$-coordinates of the points $t_{\mu/\lambda} = 0, 1$ for the unique μ'/λ lying on the reduced path between e_0 and λ in T. We denote this $\overrightarrow{ab}$ by $\vec{v}(\mu/\lambda)$.

6.3. Let Ω denote the algebraic closure of the local field $\mathbb{Q}((q))$ which is identified with $\overline{\mathbb{Q}}\{\{q\}\}$, the field of "Puiseux power series with bounded coefficients". Writing Y_Ω for the generic geometric fiber of $Y \to \operatorname{Spec} \mathbb{Q}[[q]]$, we have a canonical exact sequence

$$1 \longrightarrow \pi_1(Y_\Omega \smallsetminus \{Q_{\kappa_0}, Q_{\kappa_{2g}}\}, \widetilde{\kappa_{2g}}) \longrightarrow \pi_1^{\mathcal{D}}(Y, \widetilde{\kappa_{2g}}) \longrightarrow \operatorname{Gal}(\Omega/\mathbb{Q}((q))) \longrightarrow 1$$

together with a canonical splitting caused by the tangential base point $\widetilde{\kappa_{2g}}$. Since $\operatorname{Gal}(\Omega/\mathbb{Q}((q)))$ has also a canonical splitting $G_\mathbb{Q} \ltimes \hat{\mathbb{Z}}$ through the action of $G_\mathbb{Q}$ on the coefficients of each Puiseux series in Ω, the conjugate action

$$G_\mathbb{Q} \to \operatorname{Aut}\left(\pi_1(Y_\Omega \smallsetminus \{Q_{\kappa_0}, Q_{\kappa_{2g}}\}, \widetilde{\kappa_{2g}})\right)$$

arises. This is the limit Galois representation we are seeking now.

6.4. We shall standardize a generator system of the kernel part

$$\Pi_{g,2} := \pi_1(Y_\Omega \smallsetminus \{Q_{\kappa_0}, Q_{\kappa_{2g}}\}, \widetilde{\kappa_{2g}})$$

by using symbols of §6.2. Let us consider first the situation near the points $P^0_{\kappa_{\pm(2j-1)}}$ ($1 \leq j \leq g$), $P^0_{\kappa_{2j-2}}$ ($2 \leq j \leq g$) for a fixed j. First, check that

$$\vec{v}(\kappa_{2j-1}/\lambda_{2j}) = \overrightarrow{\infty 0}, \qquad \vec{v}(\kappa_{2j-1}/\lambda_{2j-1}) = \overrightarrow{\infty 1},$$
$$\vec{v}(\kappa_{-(2j-1)}/\lambda_{2j}) = \overrightarrow{10}, \qquad \vec{v}(\kappa_{-(2j-1)}/\lambda_{2j-1}) = \overrightarrow{01},$$
$$\vec{v}(\kappa_{2j-2}/\lambda_{2j-1}) = \overrightarrow{10}, \qquad \vec{v}(\kappa_{2j-2}/\lambda_{2j-2}) = \overrightarrow{01}.$$

Then, we trace the connecting homomorphisms:

$$\mathbf{j}_{\kappa_{2j-1}/\lambda_{2j}}:\begin{cases}\tau_{\kappa_{2j-1}/\lambda_{2j}}\mapsto\tau_{\lambda_{2j}},\\ \tau_{\kappa_{2j-1}/\lambda_{2j-1}}\mapsto\tau_{\lambda_{2j}}\infty_{\lambda_{2j}};\end{cases}$$
$$\mathbf{j}_{\kappa_{2j-1}/\lambda_{2j-1}}:\begin{cases}\tau_{\kappa_{2j-1}/\lambda_{2j}}\mapsto\tau_{\lambda_{2j-1}}\infty_{\lambda_{2j-1}},\\ \tau_{\kappa_{2j-1}/\lambda_{2j-1}}\mapsto\tau_{\lambda_{2j-1}}.\end{cases}$$

Upon gluing $\Pi_{\lambda_{2j}}$ and $\Pi_{\lambda_{2j-1}}$ along $\Pi_{\kappa_{2j-1}}$, we have to introduce a new edge symbol $e_{\kappa_{2j-1}/\lambda_{2j-1}}$ $(=e^{-1}_{\kappa_{2j-1}/\lambda_{2j}})$ to amalgamate them by the relation

$$\mathbf{j}_{\kappa_{2j-1}/\lambda_{2j}}(g)=e_{\kappa_{2j-1}/\lambda_{2j-1}}\mathbf{j}_{\kappa_{2j-1}/\lambda_{2j-1}}(g)\,e^{-1}_{\kappa_{2j-1}/\lambda_{2j-1}}\qquad(g\in\Pi_{\kappa_{2j-1}}).$$

In particular, we obtain

$$\infty^{-1}_{\lambda_{2j}}=e_{\kappa_{2j-1}/\lambda_{2j-1}}\infty_{\lambda_{2j-1}}e^{-1}_{\kappa_{2j-1}/\lambda_{2j-1}}. \tag{6.4.1}$$

Processing similar computation on the adjacent pairs "$\kappa_{-(2j-1)}/\lambda_{2j},\lambda_{2j-1}$" and also on "$\kappa_{2j-2}/\lambda_{2j-1},\lambda_{2j-2}$" yields

$$1^{-1}_{\lambda_{2j}}=e_{\kappa_{-(2j-1)}/\lambda_{2j-1}}0_{\lambda_{2j-1}}e^{-1}_{\kappa_{-(2j-1)}/\lambda_{2j-1}}\qquad(1\le j\le g), \tag{6.4.2}$$
$$1^{-1}_{\lambda_{2j-1}}=e_{\kappa_{2j-2}/\lambda_{2j-1}}0_{\lambda_{2j-2}}e^{-1}_{\kappa_{2j-2}/\lambda_{2j-1}}\qquad(2\le j\le g). \tag{6.4.3}$$

But since the edges $\kappa_{-(2j-1)}$, κ_{2j-2} are included in the maximal tree T, the symbols e_*'s in (6.4.2-3) are killed in $\pi_1(\Pi/\Delta^0,T)$. Thus, combining (6.4.1-3) together with $0_\lambda 1_\lambda\infty_\lambda=1$ $(\lambda\in\Lambda)$, we obtain

$$\infty_{\lambda_{2j-1}}e_{\kappa_{2j-1}/\lambda_{2j-1}}\infty^{-1}_{\lambda_{2j-1}}e^{-1}_{\kappa_{2j-1}/\lambda_{2j-1}}0_{\lambda_{2j}}=1^{-1}_{\lambda_{2j-1}}\qquad(j\ge1)$$
$$\big(=0_{\lambda_{2j-2}}\quad\text{when } j\ge2\big).$$

Coming to this stage, one can then compose the above relations for $j=1,\dots,g$. Then, after new notations $x_j:=\infty_{\lambda_{2j-1}}$, $y_j:=e_{\kappa_{2j-1}/\lambda_{2j-1}}$ $(1\le j\le g)$, $z_1=0_{\lambda_{2g}}$, $z_0=1_{\lambda_1}$ are introduced, the resulting relation becomes

$$[x_1,y_1]\cdots[x_g,y_g]z_1z_0=1, \tag{6.4.4}$$

where $[\xi,\eta]=\xi\eta\xi^{-1}\eta^{-1}$. Thus, we have managed to identify a standard generator system of the profinite surface group $\Pi_{g,2}=\pi_1(Y_\Omega\smallsetminus\{Q_{\kappa_0},Q_{\kappa_{2g}}\},\widetilde{\kappa_{2g}})$ in the language of the graph of groups Π/Δ^0. (See also Figure 6.3 for a topological illustration of the relationships between the issued loops after the van Kampen gluing of our local fundamental groups.)

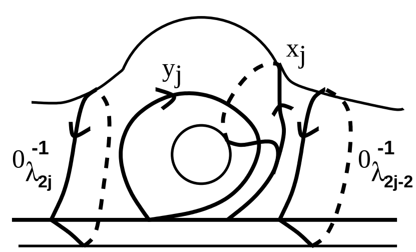

FIGURE 6.3

Moreover, this generator system may be used to express 0_{λ_i}, 1_{λ_i} as follows:

(6.4.5)
$$\begin{cases} 0_{\lambda_1} = x_1^{-1} z_0^{-1}, \\ 0_{\lambda_{2j}} = [y_j, x_j] \cdots [y_1, x_1] z_0^{-1}, \quad (1 \le j \le g), \\ 0_{\lambda_{2j+1}} = x_{j+1}^{-1} [y_j, x_j] \cdots [y_1, x_1] z_0^{-1}, \quad (1 \le j \le g-1); \end{cases}$$

(6.4.6)
$$\begin{cases} 1_{\lambda_1} = z_0, \quad 1_{\lambda_2} = z_0 x_1, \\ 1_{\lambda_{2j+1}} = z_0 [x_1, y_1] \cdots [x_j, y_j], \quad (1 \le j \le g-1), \\ 1_{\lambda_{2j}} = z_0 [x_1, y_1] \cdots [x_{j-1}, y_{j-1}] x_j, \quad (2 \le j \le g). \end{cases}$$

THEOREM 6.5. *Under the above presentation of*

$$\Pi_{g,2} = \pi_1(Y_\Omega \setminus \{Q_{\kappa_0}, Q_{\kappa_{2g}}\}, \widetilde{\kappa_{2g}})$$

via the generator system $\{x_1, y_1 \ldots, x_g, y_g, z_0, z_1\}$*, the limit Galois representation of §6.3 can be described by the following formula* ($\sigma \in G_{\mathbb{Q}}$)*:*

$$\begin{aligned} &\sigma(x_i) = \mathfrak{f}_{2i}(\sigma)^{-1} \mathfrak{f}_\sigma^{\overrightarrow{\infty 0}}(0_{\lambda_{2i-1}}, 1_{\lambda_{2i-1}})^{-1} x_i^{\chi(\sigma)} \mathfrak{f}_\sigma^{\overrightarrow{\infty 0}}(0_{\lambda_{2i-1}}, 1_{\lambda_{2i-1}}) \mathfrak{f}_{2i}(\sigma), \\ &\sigma(y_i) = \mathfrak{f}_{2i+1}(\sigma)^{-1} \mathfrak{f}_\sigma^{\overrightarrow{\infty 0}}(0_{\lambda_{2i}}, 1_{\lambda_{2i}})^{-1} \cdot y_i \cdot \mathfrak{f}_\sigma^{\overrightarrow{\infty 1}}(0_{\lambda_{2i-1}}, 1_{\lambda_{2i-1}}) \mathfrak{f}_{2i}(\sigma) \quad (i = 1, \ldots, g), \\ &\sigma(z_1) = z_1^{\chi(\sigma)}, \\ &\sigma(z_0) = \mathfrak{f}_1(\sigma)^{-1} z_0^{\chi(\sigma)} \mathfrak{f}_1(\sigma). \end{aligned}$$

Here, $\mathfrak{f}_i(\sigma) = \prod_{s=i}^{2g} \mathfrak{f}_\sigma(0_{\lambda_s}, 1_{\lambda_s})$ $(= 1$ *if* $i > 2g)$*. We also recall that*

$$\mathfrak{f}_\sigma^{\overrightarrow{\infty 1}}(x, y) = \mathfrak{f}_\sigma(y, z) y^{\frac{\chi(\sigma)-1}{2}} \mathfrak{f}_\sigma(x, y) \ (= z^{\frac{1-\chi(\sigma)}{2}} \mathfrak{f}_\sigma(x, z) x^{\frac{1-\chi(\sigma)}{2}})$$

and $\mathfrak{f}_\sigma^{\overrightarrow{\infty 0}}(x, y) = \mathfrak{f}_\sigma(x, z) x^{\frac{1-\chi(\sigma)}{2}}$ *for* x, y, z *with* $xyz = 1$.

PROOF. This is just a direct application of the formula given in Part I, Theorem 3.15 to our present setting of §§6.1-6.4. The last reminder is from I, Prop.2.11. □

6.6. Now, let us graft the standard tree of I, Example (3.18) to the above Y^0 by identifying ν_n with κ_{2g}. We shall denote by $Y^0_{g,r+1}$ the composed $\mathbf{P}^1_{01\infty}$-diagram whose dual graph is illustrated in Figure 5.4. Here, note that we set $r = n - 1 (\ge 2)$, and change notations of indices for components from λ_i of I,(3.18) to ϵ_i. The tangential structure $\mathcal{T}_{g,r+1}$ we put on $Y^0_{g,r+1}$ is, as the union of those introduced in §6.1 and I,(3.18), consisting of members of $\mathcal{T}$ together with the other set of the coordinates involved in the tree part: t_{μ_i/ϵ_i}, $t_{\mu_i/\epsilon_{i+1}}$ $(i = 1, \ldots, r-2)$, $t_{\kappa_{2g}/\lambda_{2g}}$ and $t_{\kappa_{2g}/\epsilon_{r-1}}$. The latter coordinates are chosen to take the value 1 at the involving distinguished marked points of $Q^0_{\nu_i}$ $(i = 2, \ldots, r)$ as in §5.5. We then deform $Y^0_{g,r+1}/\mathbb{Q}$ to $Y_{g,r+1}/\mathbb{Q}[\![q]\!]$ according to $\mathcal{T}_{g,r+1}$ in the standard way (§5.1), and equip it with the canonical sections Q_{ν_i} (resp. Q_{κ_0}) extending $Q^0_{\nu_i}$ $(i = 1, \ldots, r)$ (resp. $Q^0_{\kappa_0}$). Let t_{ν_1/ϵ_1} be the coordinate of $Y^0_{\epsilon_1}$ taking $0, 1, \infty$ at $Q^0_{\nu_1}$, $Q^0_{\nu_2}$, $P^0_{\mu_1}$ respectively (5.5.5), and let $\tilde{\nu_1}$ be the tangential base point on $Y_{g,r+1}$ defined by the Puiseux ring $\overline{\mathbb{Q}}[\![q^{1/n}, t^{1/n}_{\nu_1/\epsilon_1}]\!]_{n \ge 1}$.

6.7. Computation of the limit Galois action on the fundamental group

$$\Pi_{g,r+1} = \pi_1(Y_{g,r+1} \otimes \Omega \setminus \{Q_{\nu_1}, \ldots, Q_{\nu_r}, Q_{\kappa_0}\}, \tilde{\nu}_1)$$

is not a tedious task in this stage. Taking ν_1 as the initial segment e_{01} together with the obvious maximal tree T extending that in §6.2, it is now an almost automatic combination of our computations so far. In fact, the presentation of $\Pi_{g,r+1}$ is given by the amalgamation of $\Pi_{g,2}$ of §6.4 and the additional generators a_{ϵ_i} $(i=1,\dots,r-1,\ a=0,1,\infty)$ with the relations $0_{\epsilon_i}1_{\epsilon_i}\infty_{\epsilon_i}=1$, $\infty_{\epsilon_i}=0_{\epsilon_{i+1}}^{-1}$ $(i=1,\dots,r-1)$ and $\infty_{\epsilon_{r-1}}=z_1^{-1}$ $(\in\Pi_{g,2})$. So if we (re)set $z_1:=0_{\epsilon_1}$, $z_2:=1_{\epsilon_1},\cdots,z_r:=1_{\epsilon_{r-1}}$, then we obtain a standard presentation:

$$\Pi_{g,r+1}=\left\langle \begin{matrix} x_i,y_i,z_j \\ 1\le i\le g, 0\le j\le r\end{matrix} \;\middle|\; [x_1,y_1]\cdots[x_g,y_g]z_1\cdots z_r z_0=1\right\rangle$$

on which $G_{\mathbb{Q}}$ acts through the tangential base point $\tilde{\nu}_1$. We conclude this section by describing the limit Galois representation in this standard $\Pi_{g,r+1}$. In the following formulae, note also that $0_{\epsilon_j}=z_1\cdots z_j$, $1_{\epsilon_j}=z_{j+1}$ $(j=1,\dots,r-1)$.

THEOREM 6.8. *Under the above presentation of*

$$\Pi_{g,r+1}=\pi_1(Y_{g,r+1}\otimes\Omega\smallsetminus\{Q_{\kappa_0},Q_{\nu_i}\}_{i=1}^r,\tilde{\nu_1})$$

via the generator system $\{x_1,y_1\dots,x_g,y_g,z_0,z_1,\dots,z_r\}$, *the limit Galois representation can be described by the following formula* $(\sigma\in G_{\mathbb{Q}})$:

$$\begin{aligned}
\sigma(x_i)&=\tilde{\mathfrak{f}}_{2i}(\sigma)^{-1}\mathfrak{f}_\sigma^{\overrightarrow{\infty0}}(0_{\lambda_{2i-1}},1_{\lambda_{2i-1}})^{-1}x_i^{\chi(\sigma)}\mathfrak{f}_\sigma^{\overrightarrow{\infty0}}(0_{\lambda_{2i-1}},1_{\lambda_{2i-1}})\tilde{\mathfrak{f}}_{2i}(\sigma),\\
\sigma(y_i)&=\tilde{\mathfrak{f}}_{2i+1}(\sigma)^{-1}\mathfrak{f}_\sigma^{\overrightarrow{\infty0}}(0_{\lambda_{2i}},1_{\lambda_{2i}})\cdot y_i\cdot\mathfrak{f}_\sigma^{\overrightarrow{\infty1}}(0_{\lambda_{2i-1}},1_{\lambda_{2i-1}})\tilde{\mathfrak{f}}_{2i}(\sigma),\qquad(i=1,\dots,g);\\
\sigma(z_1)&=z_1^{\chi(\sigma)},\\
\sigma(z_j)&=\mathfrak{f}_{j-1}^T(\sigma)^{-1}\mathfrak{f}_\sigma(0_{\epsilon_{j-1}},1_{\epsilon_{j-1}})^{-1}z_j^{\chi(\sigma)}\mathfrak{f}_\sigma(0_{\epsilon_{j-1}},1_{\epsilon_{j-1}})\mathfrak{f}_{j-1}^T(\sigma),\quad(j=2,\dots,r);\\
\sigma(z_0)&=\tilde{\mathfrak{f}}_1(\sigma)^{-1}z_0^{\chi(\sigma)}\tilde{\mathfrak{f}}_1(\sigma).
\end{aligned}$$

Here, we define $\mathfrak{f}_j^T(\sigma):=\prod_{s=1}^{j-1}\mathfrak{f}_\sigma^{\overrightarrow{\infty1}}(0_{\epsilon_{j-s}},1_{\epsilon_{j-s}})$ *and* $\tilde{\mathfrak{f}}_i(\sigma):=\mathfrak{f}_i(\sigma)\mathfrak{f}_r^T(\sigma)$ *with* $\mathfrak{f}_i(\sigma)$ *being as in Theorem 6.5.*

PROOF. The formulae are obtained only by combining those in Example 3.18 and Theorem 6.5. Or it follows from a direct application of Theorem 3.15. Here we just remind readers that $\vec{v}(\mu_i/\epsilon_i)=\vec{v}(\kappa_{2g}/\epsilon_{r-1})=\overrightarrow{\infty1}$, $\vec{v}(\mu_i/\epsilon_{i+1})=\vec{v}(\kappa_{2g}/\lambda_{2g})=\overrightarrow{01}$ for $i=1,\dots r-2$. (In Part I, p.341, line 1, we should have typeset $\overrightarrow{01}$ and $\overrightarrow{\infty1}$ reversely.) □

§7. Galois representations in $\pi_1(M_{g,1})$, $\pi_1(M_{g,2})$.

7.1. Let $M_{g,2}$ be the moduli stack over $\mathbb{Q}$ of the smooth projective curves of genus g with 2 marked points (Q,Q'). The generic fibre of the curve $(Y;Q_{\kappa_0},Q_{\kappa_{2g}})$ constructed in §5 gives a $\mathbb{Q}((q))$-rational point $\vec{a}'$ of $M_{g,2}$ with $Q=Q_{\kappa_0}$, $Q'=Q_{\kappa_{2g}}$. Considering the natural forgetful morphism $p:M_{g,2}\to M_{g,1}$ with respect to the second marked point Q', we obtain a tangential base point $\vec{a}=p(\vec{a}')$ on $M_{g,1}$.

On the other hand, in [N97], we introduced two other tangential base points $\vec{b}$ and $\vec{v}$ on $M_{g,1}$, and determined how the induced Galois representations act on certain Dehn twist generators of $\hat{\Gamma}_g^1$. In this section, we compare $\vec{a}$ with these $\vec{b},\vec{v}$ and examine the corresponding Galois representation.

7.2. We first recall the construction of $\vec{v}$ in [N97], which comes from deformation of a certain hyperelliptic stable curve starting from a "$\mathbf{P}^1_{0\pm1\infty}$-diagram" with tangential structure. The skeleton diagram is, in fact, the same as Y^0 considered in §6.1 except that each marked component $Y^0_{\lambda_i}$ has an extra marked point to be isomorphic to $(\mathbf{P}^1; 0, \pm1, \infty)$. We shall use the same notations as §6.1 to designate the components and double points of Y^0, and shall not prepare new notations for the added marked points as they play only superfluous roles below. We cover Y^0 by the new collection of coordinates:

$$\mathcal{S} := \left\{ s_{\kappa_{2j}/\lambda_{2j}},\, s_{\kappa_{2j-2}/\lambda_{2j-1}},\, s_{\kappa_{\pm(2j-1)}/\lambda_{2j}},\, s_{\kappa_{\pm(2j-1)}/\lambda_{2j-1}} \;\middle|\; j = 1, \ldots, g \right\},$$

where each $s_{\kappa/\lambda}$ is a coordinate of Y^0_λ which takes values $\{0, \pm1, \infty\}$ at the distinguished points in such a way that $s_{\kappa/\lambda}(P^0_\kappa) = 0$. More precisely, we give it by:

$$\text{(7.2.1)} \qquad \begin{cases} s_{\kappa_{2j}/\lambda_*} = \frac{t_{\kappa_{2j}/\lambda_*}}{t_{\kappa_{2j}/\lambda_*} - 2} & (* = 2j, 2j-1), \\ s_{\kappa_{\pm(2j-1)}/\lambda_*} = t_{\kappa_{\pm(2j-1)}/\lambda_*} & (* = 2j, 2j-1), \end{cases}$$

for $j = 1, \ldots, g$. Then, the pair $(Y^0, \mathcal{S})$ acquires the hyperelliptic involution i interchanging $s_{\kappa_{2j}} \leftrightarrow -s_{\kappa_{2j}}$, $s_{\kappa_{2j-1}} \leftrightarrow s_{\kappa_{-(2j-1)}}$. The quotient by i becomes then naturally a $\mathbf{P}^1_{01\infty}$-diagram — a linear chain of $\mathbf{P}^1$'s (cf. [N97] §3).

Prepare variables q_κ corresponding to the edges

$$\kappa \in M = \{\kappa_{\pm(2i-1)}, \kappa_{2j}\}_{\substack{1 \le i \le g \\ 1 \le j \le g-1}}.$$

As shown in [IN], one can patch formal schemes of the form

$$\operatorname{Spf} \mathbb{Q}[t_*, \frac{1}{1 - t_*}] [\![q_\kappa]\!]_{\kappa \in M}$$

along the diagram Y^0 by the relations $t_{\kappa/\lambda} t_{\kappa/\lambda'} = q_\kappa$ $(\kappa \in M; \kappa/\lambda, \lambda'; \lambda \neq \lambda')$ to obtain the universal deformation Y^{univ} of Y^0 over the power series ring $\mathbb{Q}[\![q_\kappa]\!]_{\kappa \in M}$. On the other hand, in [N97], using the modified tangential structure $\mathcal{S}$, we patch formal schemes of the form

$$\operatorname{Spf} \mathbb{Q}[s_*, \frac{1}{1 \pm s_*}] [\![q]\!]$$

along Y^0 with the equations $s_{\kappa/\lambda} s_{\kappa/\lambda'} = q$ $(\kappa \in M; \kappa/\lambda, \lambda'; \lambda \neq \lambda')$ to get a deformation scheme $Y_{\vec{v}}$ over $\mathbb{Q}[\![q]\!]$ (written just "Y" in loc.cit.) By the universality of Y^{univ}, there is induced a morphism $\operatorname{Spec} \mathbb{Q}[\![q]\!] \to \operatorname{Spec} \mathbb{Q}[\![q_\kappa]\!]_{\kappa \in M}$. We then observe Taylor expansions of RHS of (7.2.1) to see that this morphism is given by

$$\text{(7.2.2)} \qquad \begin{cases} q_{\kappa_{2i}} \mapsto 4q(1 + O(q)), & (i = 1, \ldots, g-1), \\ q_{\kappa_{\pm(2j-1)}} \mapsto q(1 + O(q)), & (j = 1, \ldots, g), \end{cases}$$

where $1 + O(q)$ means a power series in $\mathbb{Q}[\![q]\!]$ with constant term 1.

7.3. Now, the profinite Teichmüller modular group $\hat{\Gamma}^1_g = \pi_1(M_{g,1} \otimes \overline{\mathbb{Q}}, \vec{a})$ has a standard generator system of the Dehn twists D_c for circles $c = a_i, e_j, d_{\pm k}$ $(1 \le i \le 2g,\ 1 \le j \le g-1,\ 1 \le k \le g)$ on the corresponding analytic model (see §5) indicated in Figure 7.1, where we understand $d_1 = d_{-1} = a_1$.

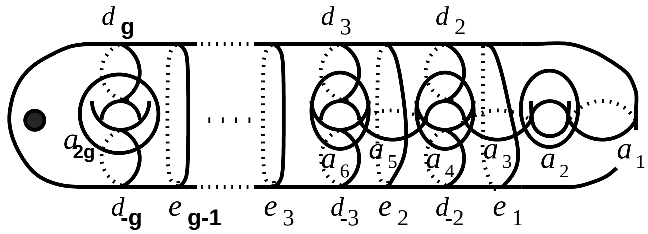

FIGURE 7.1

Let $\mathcal{D}$ be the divisor of Spec $\mathbb{Q}[\![q_\kappa]\!]_{\kappa\in M}$ defined by $\prod_{\kappa\in M} q_\kappa = 0$. Then the local fundamental group $\pi_1^{\mathcal{D}}(\mathrm{Spec}\,\overline{\mathbb{Q}}[\![q_\kappa]\!]_{\kappa\in M}, \vec{a}) \cong \hat{\mathbb{Z}}^{3g-1}$ is naturally mapped into $\hat{\Gamma}_g^2$ so that the image of each generator D_κ ($\kappa \in M$) (characterized by the transformations $q_\kappa^{1/n} \mapsto \zeta_n^{-1} q_\kappa^{1/n}$, $q_{\kappa'}^{1/n} \mapsto q_{\kappa'}^{1/n}$ ($\kappa' \neq \kappa$) in $\overline{\mathbb{Q}}[\![q_\kappa^{1/n}]\!]_{\kappa\in M}$) via the forgetful homomorphism $\hat{\Gamma}_g^2 \to \hat{\Gamma}_g^1$ is $D_{d_{\pm j}}$, D_{e_j} according as $\kappa = \kappa_{\pm(2j-1)}$, κ_{2j} respectively. Under this correspondence (together with the *real* standard chain connecting $\vec{a}$ and $\vec{v}$), we see in view of (7.2.2) that the difference of two Galois sections $s_{\vec{a}}, s_{\vec{v}} : G_{\mathbb{Q}} \to \pi_1(M_{g,1}, \vec{a})$ ($\cong \pi_1(M_{g,1}, \vec{v})$) is given by

$$(7.3.1) \qquad s_{\vec{a}}(\sigma) = \Big(\prod_{j=1}^{g-1} D_{e_j}^{2\rho_2(\sigma)}\Big) s_{\vec{v}}(\sigma) = \Big(\prod_{j=1}^{g-1} w_{2j+1}^{4\rho_2(\sigma)}\Big) s_{\vec{v}}(\sigma) \qquad (\sigma \in G_{\mathbb{Q}}).$$

Here, in the latter expression, we used the notations $w_i = (D_{a_1} \cdots D_{a_{i-1}})^i$ ($i \geq 2$), for which the relations $w_{2k} = D_{d_k} D_{d_{-k}}$, $w_{2k+1}^2 = e_k$ ($k = 1, 2, \ldots$) are known to hold.

7.4. In [N97], we considered a $\mathbf{P}^1_{01\infty}$-tree X^0 consisting of $2g$ components $X^0_{\lambda_i} \cong \mathbf{P}^1_{\mathbb{Q}}$ with $2g-1$ double points $P^0_{\mu_i}$ ($i = 1, \ldots, 2g-1$) and $2g+2$ marked points $Q^0_{\nu_i}$ ($i = 1, \ldots, 2g+2$) attached as in Figure 7.2.

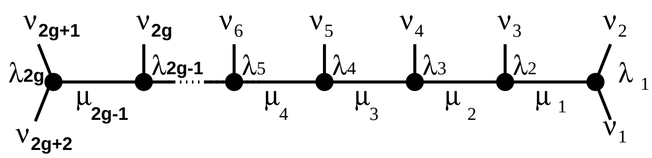

FIGURE 7.2

We then associated with it a tangential structure $\mathcal{R} = \{r_{\mu/\lambda}\}_{\mu/\lambda}$ satisfying the condition $r_{\mu_i/\lambda_{i+1}}(Q^0_{\nu_{i+2}}) = 1$, $r_{\mu_i/\lambda_i}(Q^0_{\nu_{i+1}}) = 1$ ($i = 1, \ldots, 2g-1$) and deformed it over $\mathbb{Q}[\![q_1, \ldots, q_{2g-1}]\!]$ by the equations $r_{\mu_i/\lambda_{i+1}} r_{\mu_i/\lambda_i} = q_i$. The resulting tangential base point on $M_{0,2g+2}$ we denote by $\vec{b}'$. Let $\varphi : M_{0,2g+2} \to M_{g,1}$ be the morphism obtained by associating to $(\mathbf{P}^1; 0, a_1, \ldots, a_{2g-1}, 1, \infty)$ the hyperelliptic curve "$Y^2 = X(X-1)\prod_{i=1}^{2g-1}(X - a_i)$" with the distinguished marked point ∞. Then, it is shown in [N97] that the image $\vec{b} = \varphi(\vec{b}')$ gives a tangential base point on $M_{g,1}$ whose associated Galois section $s_{\vec{b}} : G_{\mathbb{Q}} \to \pi_1(M_{g,1})$ acts on the Dehn twists D_{a_i} ($i = 1, \ldots, 2g$) by:

$$(7.4.1) \qquad s_{\vec{b}}(\sigma) D_{a_i} s_{\vec{b}}(\sigma)^{-1} = f_\sigma(D_{a_i}^2, w_i) D_{a_i}^{\chi(\sigma)} f_\sigma(w_i, D_{a_i}^2) \qquad (\sigma \in G_{\mathbb{Q}}).$$

On the other hand, by construction,

$$(7.4.2) \qquad \begin{cases} s_{\vec{v}}(\sigma)D_{d_{\pm i}}s_{\vec{v}}(\sigma)^{-1} = D_{d_{\pm i}}^{\chi(\sigma)} & (1 \le i \le g), \\ s_{\vec{v}}(\sigma)D_{e_j}s_{\vec{v}}(\sigma)^{-1} = D_{e_j}^{\chi(\sigma)} & (1 \le j \le g-1), \end{cases}$$

and the difference between $s_{\vec{v}}$ and $s_{\vec{b}}$ was estimated in loc.cit. as

$$(7.4.3) \qquad s_{\vec{b}}(\sigma) = w_{2g+1}^{c_\sigma}\Big(\prod_{\substack{j=2\\ even}}^{2g} w_j^{4\rho_2(\sigma)}\Big)s_{\vec{v}}(\sigma) \qquad (\sigma \in G_{\mathbb{Q}})$$

where some ambiguous parameter c_σ was involved (cf. loc.cit. p.169).

7.5. By applying the new tool of Harbater-Stevenson [HS] of formal patching (cf. also Wewers [W]), one can show $c_\sigma = 0$ in (7.4.3), which together with (7.3.1) completes the comparison of three tangential base points $\vec{a}, \vec{v}, \vec{b}$ on $M_{g,1}$. First of all, recall that we introduced in [N97] §3 another tangential structure $\{t_{\mu/\lambda}\}_{\mu/\lambda}$ on X^0 such that

$$t_{\mu_i/\lambda} = \begin{cases} r_{\mu_i/\lambda}, & (i = odd,\ \lambda = \lambda_{i+1}), \\ \frac{r_{\mu_i/\lambda}}{r_{\mu_i/\lambda}-1}, & (i = odd,\ \lambda = \lambda_i;\ i = even,\ \lambda = \lambda_i, \lambda_{i+1}), \end{cases}$$

for $1 \le i \le 2g-1$. It is easy to see that if X^0 is deformed by this tangential structure with equations $t_{\mu_i/\lambda_{i+1}}t_{\mu_i/\lambda_i} = q, q^2$ according as $i =$ odd, even over $\mathbb{Q}[\![q]\!]$, then the resulting tangential base point $\vec{b}^*$ is equivalent to $\vec{b}'$ in the sense of §5.9. In particular, the Galois section $s_{\vec{b}^*} : G_{\mathbb{Q}} \to \pi_1(M_{0,2g+2})$ is the same as $s_{\vec{b}'}$. We write $X^*/\mathbb{Q}[\![q]\!]$ for the deformation of $X^0/\mathbb{Q}$ in this way.

Consider now the skeleton double cover $Y^0 \to X^0$ which appeared in §7.2, where each $Y^0_{\lambda_i}$ is a double cover of $X^0_{\lambda_i}$ $(i = 1, \dots, 2g)$. Let us cover those irreducible components of Y^0 and X^0 by the coordinates in the tangential structures $\mathcal{S}$ and the above $\{t_{\mu/\lambda}\}_{\mu/\lambda}$ respectively (cf. §7.2, 7.4). The morphism $Y^0 \to X^0$ is given by $t_{\mu/\lambda} = s^2_{\kappa/\lambda}$ for $\mu = \mu_{2j}$, $\kappa = \kappa_{2j}$ $(j = 0, \dots, g)$, where we understand $\mu_0 = \nu_1$, $\mu_{2g} = \nu_{2g+2}$ and that t_{μ_0/λ_1}, $t_{\mu_{2g}/\lambda_{2g}}$ are distinguished coordinates such that $t_{\mu_0/\lambda_1}(Q^0_{\nu_2}) = \infty$, $t_{\mu_{2g}/\lambda_{2g}}(Q^0_{\nu_{2g+1}}) = \infty$. Easy calculations show that for $\mu = \mu_{2j-1}$, $\kappa = \kappa_{\pm(2j-1)}$ $(j = 1, \dots, g)$, $t_{\mu/\lambda}$ should correspond to $f(s_{\kappa/\lambda})$, where $f(T) = 4T(1+T)^{-2}$. Now, for each $\kappa \in M$, we prepare a complete ring R_κ by

$$R_\kappa = \begin{cases} \mathbb{Q}[\![s, s', q]\!]/(f(s)f(s') - q), & (\kappa = \kappa_{\pm(2j-1)}), \\ \mathbb{Q}[\![s, s', q]\!]/(ss' - q), & (\kappa = \kappa_{2j}), \end{cases}$$

where $s = s_{\kappa/\lambda}$, $s' = s_{\kappa/\lambda'}$ $(\lambda \ne \lambda')$ from $\mathcal{S}$. Moreover, for each $P^0_\kappa \in Y^0$ mapped to $P^0_\mu \in Y^0$, we fix an algebra injection $\iota_\kappa : \hat{O}_{X^*, P^0_\mu} \hookrightarrow R_\kappa$ as follows. First let $t = t_{\mu/\lambda}$, $t' = t_{\mu/\lambda'}$ $(\lambda \ne \lambda')$; if $\kappa = \kappa_{\pm(2j-1)}$, then ι_κ is just $\mathbb{Q}[\![t, t', q]\!]/(tt' - q) \to \mathbb{Q}[\![s, s', q]\!]/(f(s)f(s') - q)$ with $t \mapsto f(s)$, $t' \mapsto f(s')$; if $\kappa = \kappa_{2j}$, then ι_κ is $\mathbb{Q}[\![t, t', q]\!]/(tt' - q^2) \to \mathbb{Q}[\![s, s', q]\!]/(ss' - q)$ with $t \mapsto s^2$, $t' \mapsto (s')^2$. Then, the collection of data $\{Y^0 \to X^0 \subset X^*, \{\iota_\kappa : \hat{O}_{X^*, P^0_\mu} \hookrightarrow R_\kappa\}_{\kappa \in M}\}$ forms a "$\mathbb{Z}/2\mathbb{Z}$-Galois relative thickening problem" for $(Y^0, \{P^0_\kappa\}_\kappa)$ relative to $Y^0 \to X^0$ and $j : X^0 \hookrightarrow X^*$ in the sense of [HS], and according to [HS] Theorem 2, this problem has a solution $Y^*/\mathbb{Q}[\![q]\!]$.

By virtue of the existence of Y^*, we see that the tangential base point $\vec{b}$ on $M_{g,1}$ is centered at the same cusp as $\vec{v}$. In fact, the argument goes in the commutative diagram of [N97] (2.1) (cf. also §7.8 below):

$$
(7.5.1)\qquad
\begin{array}{ccccc}
\mathbf{A}_v^{2g+1}\setminus\Delta & \longrightarrow & M_{0,2g+2} & = & M_{0,2g+2} \\
\downarrow & & \varphi\downarrow & & \downarrow \\
\mathbf{A}_u^{2g+1}\setminus D & \longrightarrow & \mathcal{H}_{g,1} & \longrightarrow & M_{0,2g+2}/S_{2g+1},
\end{array}
$$

where newly inserted is the morphism φ defined in §7.4. In loc.cit., the tangential base point $\vec{b}$ was defined originally on the affine space (minus the weak diagonals) $\mathbf{A}_v^{2g+1}\setminus\Delta$ so that its image in the hyperelliptic locus $\mathcal{H}_{g,1}\subset M_{g,1}$ coincides with our $\vec{b}=\varphi(\vec{b}')\approx\varphi(\vec{b}^*)$. But it was not obvious from that construction whether this image lies in the local neighborhood of a maximally degenerate stable hyperelliptic curve. The above existence of Y^* as a double cover of the deformed chain of $\mathbf{P}^1$'s guarantees that $\vec{b}$ lies in the local neighborhood of the locus of Y^*.

Especially the difference between the two sections $s_{\vec{b}}, s_{\vec{v}}: G_{\mathbb{Q}}\to\pi_1(M_{g,1})$ is lying in the abelian group generated by $D_{d_{\pm i}}, D_{e_j}$ $(1\le i\le g,\ 1\le j\le g-1)$. But since the conjugation by w_{2g+1} interchanges, say, D_{d_g} and $D_{d_{-g}}$, it follows that w_{2g+1} does not lie in this abelian group. This concludes that c_σ must vanish in the expression (7.4.3). (Recall also $w_{2g+1}^2=1$ in $\hat{\Gamma}_g^1$.)

THEOREM 7.6. *Let $s_{\vec{a}}: G_{\mathbb{Q}}\to\pi_1(M_{g,1},\vec{a})$ be the Galois section at the tangential base point $\vec{a}$ (§7.1). Then, for each $\sigma\in G_{\mathbb{Q}}$, the conjugate action by $s_{\vec{a}}(\sigma)$ $(*\mapsto s_{\vec{a}}(\sigma)*s_{\vec{a}}(\sigma)^{-1})$ on the Dehn twist generators D_{a_*}, $D_{d_{\pm *}}$, D_{e_*} (*cf. §7.3) *of $\hat{\Gamma}_g^1=\pi_1(M_{g,1}\otimes\overline{\mathbb{Q}},\vec{a})$ are given by*

$$
\begin{cases}
D_{a_{2i-1}}\mapsto w_{2i-1}^{4\rho_2}\mathfrak{f}_\sigma(D_{a_{2i-1}}^2,w_{2i-1})D_{a_{2i-1}}^{\chi(\sigma)}\mathfrak{f}_\sigma(w_{2i-1},D_{a_{2i-1}}^2)w_{2i-1}^{-4\rho_2},\\
D_{a_{2i}}\mapsto w_{2i}^{-4\rho_2}\mathfrak{f}_\sigma(D_{a_{2i}}^2,w_{2i})D_{a_{2i}}^{\chi(\sigma)}\mathfrak{f}_\sigma(w_{2i},D_{a_{2i}}^2)w_{2i}^{4\rho_2},\\
D_{d_i}\mapsto D_{d_i}^{\chi(\sigma)},\ D_{d_{-i}}\mapsto D_{d_{-i}}^{\chi(\sigma)},\ D_{e_j}\mapsto D_{e_j}^{\chi(\sigma)},
\end{cases}
$$

where $1\le i\le g$, $1\le j\le g-1$.

PROOF. These formulae follow immediately from combination of the Galois actions (7.4.1) and (7.4.2) restated in terms of $s_{\vec{a}}(\sigma)$ via (7.3.1), (7.4.3) (where $c_\sigma=0$ by §7.5). □

7.7. By a two point marked hyperelliptic curve, we mean a tuple (Y,i,Q_1,Q_2) where Y is a hyperelliptic curve with the hyperelliptic involution i interchanging two rational marked points Q_1, Q_2 on it. Let $\mathcal{H}_{g,2}$ be the moduli stack over $\mathbb{Q}$ classifying those two point marked hyperelliptic curves. We call the canonical image of $\mathcal{H}_{g,2}$ in $M_{g,2}$ the hyperelliptic locus in $M_{g,2}$. In the remaining part of this section, we shall construct a tangential base point on $\mathcal{H}_{g,2}$ and examine the induced Galois action on the Dehn twists in $\pi_1(M_{g,2})$ along the circles indicated in the following Figure 7.3. Later in Appendix, we will give a proof that these Dehn twists generate $\hat{\Gamma}_g^2:=\pi_1(M_{g,2}\otimes\overline{\mathbb{Q}})$.

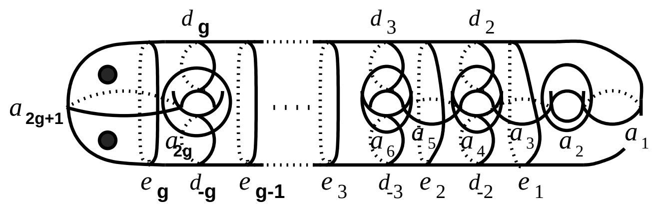

FIGURE 7.3

7.8. In starting our strategy, we first introduce the affine v-space $\mathbf{A}_v^{2g+2}$ minus the weak diagonals Δ and the affine u-space $\mathbf{A}_u^{2g+2}$ minus the discriminant locus D. Here $\Delta = \bigcup_{i\neq j} \Delta_{ij}$ with $\Delta_{ij} = \{v = (v_1, \dots, v_{2g+2}) \in \mathbf{A}_v^{2g+2} \mid v_i = v_j\}$, and D is the locus of the points $u = (u_1, \dots, u_{2g+2}) \in A_u^{2g+2}$ such that the equation $f_u(x) = x^{2g+2} + u_1 x^{2g+1} + \cdots + u_{2g+2}$ has duplicate roots. There is a natural etale map $\mathbf{A}_v^{2g+2} \setminus \Delta \to \mathbf{A}_u^{2g+2} \setminus D$ by the correspondence $v \mapsto f_u(x) = (x - v_1)\cdots(x - v_{2g+2})$.

Let $u = (u_1, \dots, u_{2g+2}) \in \mathbf{A}_u^{2g+2} \setminus D$, and consider the hyperelliptic curve Y_u birationally defined by the equation $y^2 = f_u(x)$. It is naturally a double cover over the x-line branched at the $2g+2$ zeros of $f_u(x)$. We look at the fibre of the point $x = \infty$. The projective model $z^{2g}y^2 = x^{2g+2} + u_1 x^{2g+1} z + \cdots + u_{2g+2} z^{2g+2}$ is singular at $(x : y : z) = (0 : 1 : 0)$, so we cannot use it directly to detect the fibre in the smooth model Y_u. To desingularize it, we turn back to the original equation $y^2 = f_u(x)$, and make birational variable substitutions $\eta = yx^{-g-1}$, $\xi = x^{-1}$ to get

$$\eta^2 = 1 + u_1\xi + \cdots + u_{2g+2}\xi^{2g+2}.$$

Then, we may identify the affine neighborhood of Y_u near the fibre over $x = \infty$ with the above affine curve in (ξ, η)-plane. Consequently, we detect the two rational points Q_1, Q_2 of Y_u lying over $x = \infty$ whose (ξ, η)-coordinates are $(0, 1)$, $(0, -1)$ respectively. Obviously the hyperelliptic involution i of Y_u interchanges Q_1 and Q_2, so the above procedure produces the point (Y_u, i, Q_1, Q_2) of $\mathcal{H}_{g,2}$. Thus, we have a natural morphism $\mathbf{A}_u^{2g+2} \setminus D \to \mathcal{H}_{g,2}$.

Meanwhile, given a point (Y, i, Q_1, Q_2) of $\mathcal{H}_{g,2}$, we may realize Y as a double cover of $\mathbf{P}^1$ so that the branch locus forms a degree $2g+2$ effective divisor on $\mathbf{P}^1$ disjoint from the distinguished rational point over which lie the marked points Q_1, Q_2. This determines a point of $M_{0,2g+3}/S_{2g+2}$. Summarizing our arguments so far, we obtain a commutative diagram similar to (7.5.1):

$$\begin{array}{ccccc} \mathbf{A}_v^{2g+2} \setminus \Delta & \longrightarrow & M_{0,2g+3} & = & M_{0,2g+3} \\ \downarrow & & \varphi\downarrow & & \downarrow \\ \mathbf{A}_u^{2g+2} \setminus D & \longrightarrow & \mathcal{H}_{g,2} & \longrightarrow & M_{0,2g+3}/S_{2g+2}. \end{array} \tag{7.8.1}$$

7.9. Now, let us construct a tangential base point on $M_{g,2}$ with deformation method. This time, we consider the $\mathbf{P}^1_{01\infty}$-diagram

$$\Big(Y^0 = \bigcup_{i=1}^{2g+1} Y^0_{\lambda_i}, \{P^0_{\kappa_{\pm(2i-1)}}, P^0_{\kappa_{2j}}\}_{\substack{1\le i\le g \\ 1\le j\le g}}, \{Q^0_{\kappa_0}, Q^0_{\kappa_{\pm(2g+1)}}\}\Big)$$

whose dual graph consists of the vertex set $\Lambda = \{\lambda_i \mid 1 \le i \le 2g+1\}$, the edge set $M = \{\kappa_{\pm(2i-1)}, \kappa_{2j} \mid 1 \le i \le g, 1 \le j \le g\}$ and the half edge set $N = \{\kappa_0, \kappa_{\pm(2g+1)}\}$ connected as in Figure 7.4.

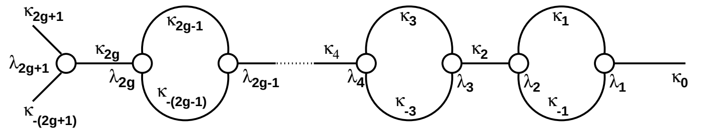

FIGURE 7.4

We associate with it the tangential structure $\mathcal{T}' = \{t_{\mu/\lambda} \mid \mu/\lambda,\ \mu \in M, \lambda \in \Lambda\}$ extending $\mathcal{T}$ of §6.1 with additional members $t_{\kappa_{2g}/\lambda_{2g}}$, $t_{\kappa_{2g}/\lambda_{2g+1}}$ such that

$$t_{\kappa_{2g}/\lambda_{2g}}(P^0_{\kappa_{-(2g-1)}}) = t_{\kappa_{2g}/\lambda_{2g+1}}(P^0_{\kappa_{-(2g+1)}}) = 1.$$

As usual, we deform $Y^0/\mathbb{Q}$ over $\mathbb{Q}[\![q]\!]$ according to the equations $t_{\mu/\lambda}t_{\mu/\lambda'} = q$ for all adjacent pairs $\mu/\lambda, \lambda'$ $(\lambda \neq \lambda')$. Let $Y/\mathbb{Q}[\![q]\!]$ be the resulting scheme and Q_{κ_0}, $Q_{\kappa_{\pm(2g+1)}}$ be the accompanying sections extending the marked points $Q^0_{\kappa_0}$, $Q^0_{\kappa_{\pm(2g+1)}}$ respectively. We define the tangential base point $\vec{a}_2$ on $M_{g,2}$ to be that which is defined by the (generic) fibre of $(Y; Q_{\kappa_{2g+1}}, Q_{\kappa_{-(2g+1)}})$ over $\mathbb{Q}(\!(q)\!)$. Then,

THEOREM 7.10. *Let $s_{\vec{a}_2} : G_{\mathbb{Q}} \to \pi_1(M_{g,2}, \vec{a}_2)$ be the Galois section at the tangential base point $\vec{a}_2$. Then, for each $\sigma \in G_{\mathbb{Q}}$, the conjugate action by $s_{\vec{a}_2}(\sigma)$ $(* \mapsto s_{\vec{a}_2}(\sigma) * s_{\vec{a}_2}(\sigma)^{-1})$ on the Dehn twists D_{a_*}, $D_{d_{\pm *}}$, D_{e_*}* (cf. §7.7) *of $\hat{\Gamma}^2_g = \pi_1(M_{g,2} \otimes \overline{\mathbb{Q}}, \vec{a}_2)$ are given by*

$$\begin{cases} D_{a_{2i-1}} \mapsto w_{2i-1}^{4\rho_2}\mathfrak{f}_\sigma(D^2_{a_{2i-1}}, w_{2i-1})D^{\chi(\sigma)}_{a_{2i-1}}\mathfrak{f}_\sigma(w_{2i-1}, D^2_{a_{2i-1}})w_{2i-1}^{-4\rho_2} \\ D_{a_{2j}} \mapsto w_{2j}^{-4\rho_2}\mathfrak{f}_\sigma(D^2_{a_{2j}}, w_{2j})D^{\chi(\sigma)}_{a_{2j}}\mathfrak{f}_\sigma(w_{2j}, D^2_{a_{2j}})w_{2j}^{4\rho_2} \\ D_{d_j} \mapsto D^{\chi(\sigma)}_{d_j},\ D_{d_{-j}} \mapsto D^{\chi(\sigma)}_{d_{-j}},\ D_{e_j} \mapsto D^{\chi(\sigma)}_{e_j}, \end{cases}$$

where $1 \le i \le g+1$, $1 \le j \le g$, and $w_k = (D_{a_1} \cdots D_{a_{k-1}})^k$ $(2 \le k \le 2g)$.

7.11. *Illustrations on the proof of Theorem 7.10.* The proof of Theorem 7.10 goes in an exactly similar way to that of Theorem 7.6. In fact, it is only a "$g\frac{1}{2}$-version" of the latter proof, and to fill the full details will be left to the readers. However, since the materials needed for the proof of Theorem 7.6 were scattered among [IN], [N97] and §6, let us illustrate how we may combine necessary items to complete the proof of Theorem 7.10.

We first prepare the universal deformation Y^{univ} of Y^0 over $\mathbb{Q}[\![q_\kappa]\!]_{\kappa \in M}$ with regard to $\mathcal{T}'$, i.e., by the equations $t_{\kappa/\lambda}t_{\kappa/\lambda} = q_\kappa$ $(\kappa \in M;\ \kappa/\lambda, \lambda',\ \lambda \neq \lambda')$, which provides a local coordinate system of the formal neighborhood of the locus of Y^0 in $\mathcal{M}_{g,3}$. This also determines the tangential base point $\vec{a}_2$ on $M_{g,2}$ (after taking the generic fibre and projecting it by the forgetful morphism $M_{g,3} \to M_{g,2}$ with regard to Q_{κ_0}.)

Meanwhile, we cover Y^0 by another tangential structure $\mathcal{S}' = \{s_{\mu/\lambda} \mid \mu/\lambda,\ \mu \in M, \lambda \in \Lambda\}$ of $\mathbf{P}^1_{0\pm1\infty}$-type given by the similar formulae to (7.2.1). Then, $(Y^0, \mathcal{S}')$ has a natural involution i which makes it a degree 2 Galois cover of the tangential structured $\mathbf{P}^1_{01\infty}$-tree $(X^0, \{t_{\mu_i/\lambda_i}, t_{\mu_i/\lambda_{i+1}}\}_i)$ having one more component than what appeared in §7.5. Deforming $(Y^0, \mathcal{S}')$ and $(X^0, \{t_{\mu_i/\lambda_i}, t_{\mu_i/\lambda_{i+1}}\}_i)$ together over $\mathbb{Q}[\![q]\!]$, and taking their generic fibres over $\mathbb{Q}(\!(q)\!)$, we obtain a hyperelliptic cover $Y_{\vec{v}_2} \to \mathbf{P}^1_{\mathbb{Q}(\!(q)\!)}$ with three distinguished points $Q_{\kappa_{\pm(2g+1)}}$, Q_{κ_0}, and then a tangential base point $\vec{v}_2$ on $\mathcal{H}_{g,2}$ (dropping the last marked point Q_{κ_0}).

In the formal level, one can compare deformation data between $(Y^0, \mathcal{T}')$ and $(Y^0, \mathcal{S}')$ to see exact difference of two Galois sections arising from $\vec{a}_2$ and $\vec{v}_2$ (as in (7.3.1); cf. also [N97] Lemma 4.3). In these tangential base points, the Galois group $G_{\mathbb{Q}}$ acts on the Dehn twists along $d_{\pm i}$, e_j by the cyclotomic character. On the other hand, consideration in the diagram (7.8.1) enables one to measure the difference between $\vec{v}_2$ and the standard tangential base point $\vec{b}_2$ of $\mathbf{A}_v^{2g+2} \setminus \Delta$ detected by Drinfeld [Dr], Ihara-Matsumoto [Ih-Ma] (cf.(7.4.3); since $w_{2g+2} = 1$ in $\hat{\Gamma}_g^2$, we may skip the process corresponding to §7.5). We know that at the latter tangential base point, $G_{\mathbb{Q}}$ acts on the Dehn twists along a_i's by the standard formulae well known now in the braid group. Combining these two types of information on Galois actions, we conclude Theorem 7.10.

§8. Coupling Galois representations in $\pi_1(M_{g,r})$.

8.1. Let $M_{g,r}$ (resp. $\mathcal{M}_{g,r}$) denote the moduli stack over $\mathbb{Q}$ of the proper smooth (resp. stable) marked curves of genus g with r-marked points. In §6.6, we introduced a $\mathbf{P}^1_{01\infty}$-diagram $Y^0_{g,r+1}$ and deformed it into the stable marked curve $Y_{g,r+1}$ over the ring $\mathbb{Q}[\![q]\!]$ with respect to a certain tangential structure $\mathcal{T}_{g,r+1}$. This defines a morphism $\operatorname{Spec}\mathbb{Q}[\![q]\!] \to \mathcal{M}_{g,r+1}$ and its restriction to the generic fibre gives a tangential base point $\vec{a}'$ on $M_{g,r+1}$. Note here that the $(r+1)$ distinguished points on the curves parametrized by the points of $\mathcal{M}_{g,r+1}$ are now marked by the indices $\kappa_0, \nu_1, \dots, \nu_r$. Our principal subject here is to observe the image $\vec{a}$ of $\vec{a}'$ by the forgetful morphism $M_{g,r+1} \to M_{g,r}$ obtained by forgetting the marked point Q_{κ_0}. Especially, we investigate the behavior of $\vec{a}$ and the associated Galois action $\varphi_{\vec{a}} : G_{\mathbb{Q}} \to \operatorname{Aut}(\pi_1(M_{g,r} \otimes \overline{\mathbb{Q}}))$ under various moduli theoretical manipulations.

In [Kn], F.Knudsen considered the morphism $\mathcal{M}_{g,r+1} \to \mathcal{M}_{g,r}$ obtained by forgetting one marked point with contracting possibly appearing unstable rational component, and showed that $\mathcal{M}_{g,r+1}$ forms a universal family of marked stable curves over the moduli stack $\mathcal{M}_{g,r}$. We call this morphism the forgetful morphism with respect to a prescribed marked point. Note that one has $r+1$ choices of the marked points to be forgotten, so that the way of regarding $\mathcal{M}_{g,r+1}$ as the universal family over $\mathcal{M}_{g,r}$ is not unique. This simple fact, actually, causes unexpected varieties of the theory of the Galois-Teichmüller tower as indicated in our previous works [N94,96,...]. Unlike our previous papers, in this paper we do not renumber the remaining marks after forgetting a marked point; hence the target moduli spaces may parameterize differently labeled marked curves according to the choice of the forgotten marked point.

8.2. Assume $r \geq 3$, and fix a decomposition $r = s + s'$ $(s, s' \geq 0)$. Consider the tangential structured $\mathbf{P}^1_{01\infty}$-diagram $(Y^0_s, \mathcal{T}_s) \cong (Y^0_{g,r}, \mathcal{T}_{g,r})$ which is obtained from $(Y^0_{g,r+1}, \mathcal{T}_{g,r+1})$ by contracting the component carrying the marked point $Q^0_{\nu_s}$. In the contraction process, there arises a new edge (or a leg if $s = 1, 2$) in the dual graph of Y^0_s which we simply write μ_* (See Figure 8.1). Deforming $(Y^0_s, \mathcal{T}_s)$ into the r pointed stable curve $Y_s/\mathbb{Q}[\![q]\!]$, we naturally obtain a commutative diagram

$$\begin{array}{ccccc} Y_s \otimes \mathbb{Q}(\!(q)\!) & \longrightarrow & Y_s & \longrightarrow & \mathcal{M}_{g,r+1} \\ \downarrow & & \downarrow & & \downarrow f_s \\ \operatorname{Spec}\mathbb{Q}(\!(q)\!) & \longrightarrow & \operatorname{Spec}\mathbb{Q}[\![q]\!] & \longrightarrow & \mathcal{M}_{g,r}, \end{array} \tag{8.2.1}$$

where $f_s : \mathcal{M}_{g,r+1} \to \mathcal{M}_{g,r}$ is the forgetful morphism with respect to ν_s. In this situation, we have two remarkable tangential base points. One is that on $M_{g,r}$ defined by the lower horizontal arrow of (8.2.1). We shall denote it by $\vec{a}_s$. On the other hand, on Y_s, we have a standard tangential base point $\tilde{\mu}_*$ which was introduced in [IN] in the case when μ_* is an edge. When μ_* is a leg (adjacent to ϵ_2), i.e., $s = 1, 2$, we define $\tilde{\mu}_*$ as follows. Choose the distinguished coordinate t_{μ_*/ϵ_2} so that $t_{\mu_*/\epsilon_2}(Q^0_{\nu_3}) = \infty, 1$ according as $s = 1, 2$ respectively. Then, we define $\tilde{\mu}_*$ to be the tangential base point determined by the Puiseux ring $\overline{\mathbb{Q}}[\![q^{1/n}, t^{1/n}_{\mu_*/\epsilon_2}]\!]_{n \geq 1}$ near the point $Q^0_{\mu_*}$. For any $s = 1, \ldots, r$, we thus obtain the image of $\tilde{\mu}_*$ in $M_{g,r+1}$ induced by the upper horizontal arrow of (8.2.1) which will be denoted also by $\tilde{\mu}_*$.

There arises a problem to compare the two tangential base points $\tilde{\mu}_*$ and $\vec{a}'$ on $M_{g,r+1}$ which are concentrated on the same cusp (= maximally degenerate locus) of $\mathcal{M}_{g,r+1}$. Nevertheless, before going to the problem, we should like to examine the relation between $\vec{a}'$ and $\vec{a}_s$ first, as noticing that the image of $\tilde{\mu}_*$ (resp. $\vec{a}'$) by the forgetful morphism f_s is (resp. is not) apriori $\vec{a}_s$.

8.3. The strategy we take here to compare $\vec{a}'$ and $\vec{a}_s$ is to construct explicitly a contraction morphism from $Y_{g,r+1}$ to a deformation of Y^0_s over $\mathbb{Q}[\![q]\!]$. Suppose first that $s \geq 3$ so that the dual graphs of the contraction morphism $Y^0_{g,r+1} \to Y^0_s$ looks as in Figure 8.1:

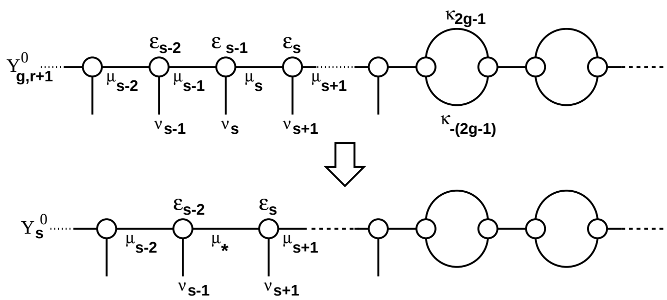

FIGURE 8.1

The schemes $Y^0_{g,r+1}$, Y^0_s are covered by the affine open sets U^0_μ, U^0_ν, where μ (resp. ν) runs over the indices of edges (resp. legs) of their dual graphs, which can be described explicitly by using distinguished coordinates (cf. §5.5) in such a way that

$$(8.3.1) \qquad \begin{cases} U^0_\mu = \operatorname{Spec} \mathbb{Q}[t, t', \frac{1}{1-t}, \frac{1}{1-t'}]/(tt'), \\ U^0_\nu = \operatorname{Spec} \mathbb{Q}[t_{\nu/\lambda}, \frac{1}{1-t_{\nu/\lambda}}] \end{cases}$$

with $t = t_{\mu/\lambda}$, $t' = t_{\mu/\lambda'}$ $(\mu/\lambda, \lambda'; \lambda \neq \lambda')$ lying in the tangential structures given on $Y^0_{g,r+1}$, Y^0_s. These affine schemes are patched along the smaller affine pieces

$$(8.3.2) \qquad U^0_\lambda = \operatorname{Spec} \mathbb{Q}[t, \frac{1}{t}, \frac{1}{1-t}] \qquad (\lambda : \text{vertices}),$$

where $t = t_{\mu/\lambda}$ (μ : any adjacent edge of λ). Using this system of Zariski coverings, we can detect the contraction morphism $c^0 : Y^0_{g,r+1} \to Y^0_s$ as a compatible collection of ring homomorphisms which are identity except on $U^0_{\mu_{s-1}}$ and $U^0_{\mu_s}$, on which they are given by

$$(8.3.3)\qquad c^0:\begin{cases} U^0_{\mu_*}\leftarrow U^0_{\mu_{s-1}} & (t_{\mu_*/\epsilon_{s-2}}\mapsto t_{\mu_{s-1}/\epsilon_{s-2}},\ t_{\mu_*/\epsilon_s}\mapsto 0),\\ U^0_{\mu_*}\leftarrow U^0_{\mu_s} & (t_{\mu_*/\epsilon_{s-2}}\mapsto 0,\ t_{\mu_*/\epsilon_s}\mapsto t_{\mu_s/\epsilon_s}),\\ U^0_{\mu_*}\leftarrow U^0_{\nu_s} & (t_{\mu_*/\epsilon_{s-2}}\mapsto 0,\ t_{\mu_*/\epsilon_s}\mapsto 0).\end{cases}$$

We shall fatten the contraction morphism $c^0 : Y^0_{g,r+1} \to Y^0_s$ to the N-th stage deformation $c^N : Y^N_{g,r+1} \to Y^{\sharp N}_s$ over $\mathbb{Q}[q]/(q^N)$, where $\sharp$ indicates that the scheme $Y^{\sharp N}_s$ differs from Y^N_s used to produce the standard deformation $Y_s/\mathbb{Q}[\![q]\!]$. We first introduce the N-th stage deformation of the affine schemes $U^0_\mu, U^0_\nu, U^0_\lambda$ of (8.3.1-2) as follows:

$$(8.3.4)\qquad \begin{cases} U^N_\mu = \begin{cases}\operatorname{Spec}\mathbb{Q}[t,t',\frac{1}{1-t},\frac{1}{1-t'},q]/(tt'-q^2,q^N) & (\mu=\mu_*),\\ \operatorname{Spec}\mathbb{Q}[t,t',\frac{1}{1-t},\frac{1}{1-t'},q]/(tt'-q,q^N) & (\mu\neq\mu_*),\end{cases}\\ U^N_\nu = \operatorname{Spec}\mathbb{Q}[t_{\nu/\lambda},\dfrac{1}{1-t_{\nu/\lambda}},q]/(q^N),\\ U^N_\lambda = \operatorname{Spec}\mathbb{Q}[t,\dfrac{1}{t},\dfrac{1}{1-t},q]/(q^N).\end{cases}$$

As explained in [IN], these are patched together on the underlying spaces $Y^0_{g,r+1}$, Y^0_s to form well defined schemes $Y^N_{g,r+1}$, $Y^{\sharp N}_s$ over $\mathbb{Q}[\![q]\!]/(q^N)$. Let us define $c^N : Y^N_{g,r+1} \to Y^{\sharp N}_s$ by a compatible collection of homomorphisms between those corresponding pairs of rings of (8.3.4). Again, on almost corresponding parts between $Y^N_{g,r+1}$ and $Y^{\sharp N}_s$, the rings are identical, and we set identity maps on those parts. The only part we need to care about is that which is involved with $U^N_{\epsilon_{s-1}}$, $U^N_{\epsilon_s}$, where we define

$$(8.3.5)\qquad c^N:\begin{cases} U^N_{\mu_*}\leftarrow U^N_{\mu_{s-1}} & (t_{\mu_*/\epsilon_{s-2}}\mapsto t_{\mu_{s-1}/\epsilon_{s-2}},\ t_{\mu_*/\epsilon_s}\mapsto qt_{\mu_{s-1}/\epsilon_{s-1}}),\\ U^N_{\mu_*}\leftarrow U^N_{\mu_s} & (t_{\mu_*/\epsilon_{s-2}}\mapsto qt_{\mu_s/\epsilon_{s-1}},\ t_{\mu_*/\epsilon_s}\mapsto t_{\mu_s/\epsilon_s}),\\ U^N_{\mu_*}\leftarrow U^N_{\nu_s} & (t_{\mu_*/\epsilon_{s-2}}\mapsto q(1-t_{\nu_s/\epsilon_{s-1}}),\ t_{\mu_*/\epsilon_s}\mapsto q(1-t_{\nu_s/\epsilon_{s-1}})^{-1}).\end{cases}$$

To check the compatibility of these maps amounts to seeing the commutativity of the following diagram (8.3.6). We write $A^N_\#$ for the defining ring of the affine set $U^N_\#$, and write $\xrightarrow[t^{-1}]{}$ to designate the ring localization by adding t^{-1}.

$$(8.3.6)\qquad \begin{array}{ccccccccc} A^N_{\epsilon_{s-2}} & \xleftarrow{t^{-1}_{\mu_*/\epsilon_{s-2}}} & A^N_{\mu_*} & = & A^N_{\mu_*} & = & A^N_{\mu_*} & \xrightarrow{t^{-1}_{\mu_*/\epsilon_s}} & A^N_{\epsilon_s}\\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle (8.3.5)} & & & & {\scriptstyle (8.3.5)}\downarrow & & \downarrow{\scriptstyle id}\\ A^N_{\epsilon_{s-2}} & \xleftarrow[t^{-1}_{\mu_{s-1}/\epsilon_{s-2}}]{} & A^N_{\mu_{s-1}} & \xrightarrow[t^{-1}_{\mu_{s-1}/\epsilon_{s-1}}]{} & A^N_{\epsilon_{s-1}} & \xleftarrow[t^{-1}_{\mu_s/\epsilon_{s-1}}]{} & A^N_{\mu_s} & \xrightarrow[t^{-1}_{\mu_s/\epsilon_s}]{} & A^N_{\epsilon_s}. \end{array}$$

The left and right commutativities of (8.3.6) are almost automatic, while the middle commutativity can be checked easily after recalling the relation $t_{\mu_s/\epsilon_{s-1}}t_{\mu_{s-1}/\epsilon_{s-1}} = 1$ from our choice of the tangential structure $\mathcal{T}_{g,r+1}$.

Thus, we have constructed morphisms $c^N : Y^N_{g,r+1} \to Y^{\sharp N}_s$ for $N \geq 0$. Since these are compatible with respect to N, taking the projective limit we obtain a morphism of formal schemes $\mathcal{Y}_{g,r+1} \to \mathcal{Y}^\sharp_s$ (with marked sections), and its algebraization $c^\sharp : Y_{g,r+1} \to Y^\sharp_s$ over $\mathbb{Q}[\![q]\!]$. When $s = 1, 2$, no particular elaborations are needed to construct the fattened contraction $c^\sharp : Y_{g,r+1} \to Y_s^\sharp$. In fact, in these cases, we may take Y_s as $Y_s^\sharp$, and for rings and homomorphisms needed for defining c^N, it is enough to adopt rather obvious ones. So we omit the details of these cases.

8.4. The stable curve Y_s (resp. $Y_s^\sharp$) determines a morphism $\operatorname{Spec}\mathbb{Q}[\![q]\!] \to \mathcal{M}_{g,r}$ denoted by $\vec{a}_s$ (resp. $\vec{a}_s^\sharp$). Both $\vec{a}_s$ and $\vec{a}_s^\sharp$ are concentrated at the same cusp in $\mathcal{M}_{g,r}$. Let M_s be the edge set of the dual graph of Y_s^0 and let $Y_s^{univ}/\mathbb{Q}[\![q_\mu]\!]_{\mu\in M_s}$ be the universal deformation constructed with the equations $t_{\mu/\lambda}t_{\mu/\lambda} = q_\mu$ ($\mu \in M_s$; $\mu/\lambda, \lambda'$, $\lambda \neq \lambda'$). By comparing the local deformation data, we see that $\vec{a}_s$ is given by the diagonal specialization ${}^\forall q_\mu \mapsto q$, while $\vec{a}_s^\sharp$ corresponds to the specialization $q_\mu \mapsto q$ ($\mu \neq \mu_*$), $q_{\mu_*} \mapsto q^2$. In each of these specializations, principal coefficients of variable transformations are 1, hence $G_\mathbb{Q}$ actions on the Puiseux power series commute with those specializations. Thus, $\vec{a}_s$ and $\vec{a}_s^\sharp$ give "equivalent" tangential base points on $M_{g,r}$ in the sense of §5.9.

On the other hand, by our construction of the contraction morphism $c^\sharp$: $Y_{g,r+1} \to Y_s^\sharp$, it follows that the forgetful morphism $f_s : M_{g,r+1} \to M_{g,r}$ maps $\vec{a}'$ to $\vec{a}_s^\sharp$. Thus, as tangential base points, we conclude

$$f_s(\vec{a}') = \vec{a}_s^\sharp \approx \vec{a}_s = f_s(\vec{\mu}_*). \tag{8.4.1}$$

8.5. In the following, we need to enter into more global considerations in the moduli spaces of curves, i.e., we look closely at the fundamental groups of the tubular neighborhoods of certain divisors at infinity on the moduli spaces. We mainly use the theory of Grothendieck-Murre [GM] which relates them to the fundamental groups of those divisors minus deeper stratification. One crucial ingredient adopted in [N96] is to use the theory not only on a single moduli space but also on a pair of moduli spaces connected by a forgetful morphism. Let $\mathcal{D}$ (resp. D) be the sum of divisors at infinity of $\mathcal{M}_{g,r+1}$ (resp. $\mathcal{M}_{g,r}$), and define for $s \geq 3$ the irreducible components $\mathcal{D}_1^{(s)}, \mathcal{D}_2^{(s)} \subset \mathcal{D}$ and $D^{(s)} \subset D$ to be the closures of loci of stable curves whose degeneration type are Figure 8.2a, 8.2b, 8.2c respectively.

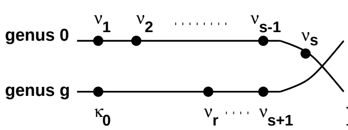

FIG. 8.2a

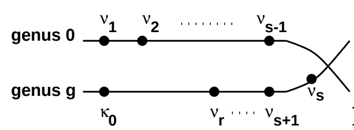

FIG. 8.2b

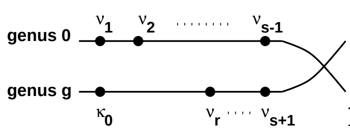

FIG. 8.2c

For these data, we consider the fundamental groups $\pi_1^{\mathcal{D}}((M_{g,r+1}/\mathcal{D}_i^{(s)})^\wedge)$ ($i = 1,2$), $\pi_1^D((M_{g,r}/D^{(s)})^\wedge)$ well defined, say, in the following sense (cf. also [N96] §3). Let $\mathcal{H}_g$ be the Hilbert moduli scheme over $\mathbb{Q}$ of the tri-canonically embedded stable curves of genus g introduced in Deligne-Mumford [DM], and let $\mathcal{H}_{g,n} \to \mathcal{M}_{g,n}$ be the pullback of the natural forgetful morphisms $\mathcal{H}_g \to \mathcal{M}_g$ and $\mathcal{M}_{g,n} \to \mathcal{M}_g$. Denote by $H_{g,n}$ the complement of all divisors at infinity in $\mathcal{H}_{g,n}$. We use the same notations for the pullbacked divisors at infinity on $\mathcal{H}_{g,r+1}$, $\mathcal{H}_{g,r}$ as those on $\mathcal{M}_{g,r+1}$, $\mathcal{M}_{g,r}$. Then, $\mathcal{D}_1^{(s)}, \mathcal{D}_2^{(s)} \subset \mathcal{H}_{g,r+1}$, $D \subset \mathcal{H}_{g,r}$ are smooth irreducible divisors ([Kn] Cor.3.9). After the Grothendieck-Murre theory [GM], one can think of the fundamental groups of formal schemes $\pi_1^{\mathcal{D}}((\mathcal{H}_{g,r+1}/\mathcal{D}_i^{(s)})^\wedge)$ ($i = 1,2$), $\pi_1^D((\mathcal{H}_{g,r}/D^{(s)})^\wedge)$

admitting tame ramifications along the (intersections with) divisors indicated in the shoulders of π_1. We understand $\pi_1^{\mathcal{D}}((\mathcal{M}_{g,r+1}/\mathcal{D}_i^{(s)})^\wedge)$ $(i=1,2)$, $\pi_1^{D}((\mathcal{M}_{g,r}/D^{(s)})^\wedge)$ as the natural images of the above corresponding ones by the canonical homomorphisms $\pi_1(H_{g,*}) \to \pi_1(M_{g,*})$. In what follows, we sometimes employ arguments where as if we were treating $\mathcal{M}_{g,*}$ as smooth schemes. This is just for simplifying our presentations and for stressing essential ideas rather than being involved with technical formalities; the readers will find easily that we keep more strict arguments in the background, say, using rigidification via Hilbert schemes as above.

As a quick application of the above formulation, we obtain the following.

THEOREM 8.6. *Apply the forgetful morphism $f_0 : \mathcal{M}_{g,r+1} \to \mathcal{M}_{g,r}$ with respect to κ_0 to $\vec{a}'$. Call its image $\vec{a}$. Let $s_{\vec{a}} : G_{\mathbb{Q}} \to \pi_1(M_{g,r}, \vec{a})$ be the Galois section at the tangential base point $\vec{a}$. Then, for each $\sigma \in G_{\mathbb{Q}}$, the conjugate action by $s_{\vec{a}}(\sigma)$ $(* \mapsto s_{\vec{a}}(\sigma) * s_{\vec{a}}(\sigma)^{-1})$ on the Dehn twist generators D_{a_*}, $D_{d_{\pm *}}$, D_{e_*}* (cf. §5) *of $\hat{\Gamma}_g^r := \pi_1(M_{g,r} \otimes \overline{\mathbb{Q}}, \vec{a})$ are given by*

$$\begin{cases} D_{a_{2i-1}} \mapsto w_{2i-1}^{4\rho_2} \mathfrak{f}_\sigma(D_{a_{2i-1}}^2, w_{2i-1}) D_{a_{2i-1}}^{\chi(\sigma)} \mathfrak{f}_\sigma(w_{2i-1}, D_{a_{2i-1}}^2) w_{2i-1}^{-4\rho_2} \\ D_{a_{2i}} \mapsto w_{2i}^{-4\rho_2} \mathfrak{f}_\sigma(D_{a_{2i}}^2, w_{2i}) D_{a_{2i}}^{\chi(\sigma)} \mathfrak{f}_\sigma(w_{2i}, D_{a_{2i}}^2) w_{2i}^{4\rho_2} \\ D_{d_i} \mapsto D_{d_i}^{\chi(\sigma)},\ D_{d_{-i}} \mapsto D_{d_{-i}}^{\chi(\sigma)},\ D_{e_j} \mapsto D_{e_j}^{\chi(\sigma)}, \end{cases}$$

where $1 \le i \le g$, $1 \le j \le g-1$.

PROOF. Letting $\mathcal{D}_0^{(r)}$, $\mathcal{D}_0^{(r-1)}$ be the irreducible divisors at infinity defined as the images of $\mathcal{D}_1^{(r)}$, $\mathcal{D}_1^{(r-1)}$ via f_0 respectively, consider the diagram

$$\begin{array}{ccccccccc} \mathcal{M}_{g,2} \times \mathcal{M}_{0,r+1} & \cong & \mathcal{D}_1^{(r)} & \subset \mathcal{M}_{g,r+1} \supset & \mathcal{D}_1^{(r-1)} & \cong & \mathcal{M}_{g,3} \times \mathcal{M}_{0,r} \\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \\ \mathcal{M}_{g,1} \times \mathcal{M}_{0,r+1} & \cong & \mathcal{D}_0^{(r)} & \subset \mathcal{M}_{g,r} \supset & \mathcal{D}_0^{(r-1)} & \cong & \mathcal{M}_{g,2} \times \mathcal{M}_{0,r}. \end{array}$$

The natural images of $\vec{a}$ in the first components of $\mathcal{D}_0^{(r)}$, $\mathcal{D}_0^{(r-1)}$ are the tangential base points $\vec{a}$, $\vec{a}_2$ considered in §7.1, §7.9. Below, we consider all fundamental groups as based at the images of $\vec{a}'$. Let c be any one of the circles issued in the statement of the theorem. In the two Grothendieck-Murre exact sequences

$$1 \to \langle D_{e_g} \rangle \to \pi_1^{\mathcal{D}}((\mathcal{M}_{g,r}/\mathcal{D}_0^{(r)})^\wedge) \to \pi_1(M_{g,1} \times M_{0,r+1}) \to 1,$$
$$1 \to \langle D_{k_{r-1}} \rangle \to \pi_1^{\mathcal{D}}((\mathcal{M}_{g,r}/\mathcal{D}_0^{(r-1)})^\wedge) \to \pi_1(M_{g,2} \times M_{0,r}) \to 1,$$

the $G_{\mathbb{Q}}$-action on the images of D_c in the right most groups are described by the same formula as in the theorem, as shown by Theorem 7.6 and Theorem 7.10 respectively. Still in the middle groups (which have natural maps to $\pi_1(M_{g,r})$), the $G_{\mathbb{Q}}$-actions may differ from the requested form but only by a factor lying in the intersection of two procyclic subgroups $\langle D_{e_g} \rangle \cap \langle D_{k_{r-1}} \rangle$. Since we know that this intersection is trivial, we conclude there occur no differences in the above liftings.□

8.7. Let us now discuss the problem raised in §8.2, i.e., about comparing two tangential base points $\vec{a}'$ and $\tilde{\mu}_*$ on $M_{g,r+1}$. In this paragraph, we take the natural images of $\vec{a}'$ as base points of fundamental groups. By virtue of (8.4.1), the images of $G_{\mathbb{Q}}$ in $\pi_1(M_{g,r+1})$ via $\vec{a}'$ and $\tilde{\mu}_*$ differ from each other by some power of $D_{k_s} D_{k_{s-1}}^{-1}$. Let $c_\sigma \in \hat{\mathbb{Z}}$ be its power for $\sigma \in G_{\mathbb{Q}}$.

Notations being as in §8.1-3, assume $s \geq 3$, and make $Y_s \to \mathbb{Q}[\![q]\!]$ fit into $\mathcal{M}_{g,r+1} \to \mathcal{M}_{g,r}$. Then, the pullback of $\mathcal{D}$ in Y_s is the sum of Y_s^0 and the marked sections Q_ν ($\nu \in \{\nu_1, \dots, \nu_r, \kappa_0\} \setminus \{\nu_s\}$) which we simply denote by $Y_s^0 + Q$. Moreover, we may regard the singular fibre Y_s^0 as a sum of the two divisors:

$$Y_1 = Y_{\epsilon_1}^0 + \dots + Y_{\epsilon_{s-2}}^0,$$
$$Y_2 = Y_{\epsilon_s}^0 + \dots + Y_{\epsilon_{r-1}}^0 + Y_{\lambda_{2g}}^0 + \dots + Y_{\lambda_1}^0,$$

with each the pullback of $\mathcal{D}_1^{(s)}$, $\mathcal{D}_2^{(s)}$ in Y_s^0 respectively. Apply the Grothendieck-Murre theory to obtain the commutative diagram of fundamental groups:

$$(8.7.1) \qquad \begin{array}{ccccc} \pi_1^{Y_s^0+Q}((Y_s/Y_1)^\wedge) & \to & \pi_1^{\mathcal{D}}((\mathcal{M}_{g,r+1}/\mathcal{D}_1^{(s)})^\wedge) & \to & \pi_1(M_{0,s+1} \times M_{g,s'+1}) \\ \downarrow & & \downarrow & & \downarrow \\ G_{\mathbb{Q}((q))} & \to & \pi_1^{D}((\mathcal{M}_{g,r}/D^{(s)})^\wedge) & \to & \pi_1(M_{0,s} \times M_{g,s'+1}). \end{array}$$

The kernels of the three vertical arrows are canonically isomorphic to $\hat{\Pi}_{0,s}$, the profinite fundamental group of an s point punctured Riemann sphere. We may regard the tangential base point $\tilde{\mu}_*$ as coming from the corresponding one in $(Y_s/Y_1)^\wedge$ which is isomorphic to the deformation of the tangential structured $\mathbf{P}^1_{01\infty}$-tree considered in I, Example (3.18). Meanwhile, the universal expression of the tangential base point $\vec{a}'$: $\operatorname{Spec} \mathbb{Q}[\![q_\kappa]\!]_{\kappa \in M} \to \mathcal{M}_{g,r+1}$ is, as the completion of $\mathcal{M}_{g,r+1}$ at the issued cusp, naturally lying in $(\mathcal{M}_{g,r+1}/\mathcal{D}_1^{(s)})^\wedge$. Projecting these two tangential base points to the right most part of (8.7.1), we see that the difference $(D_{k_s} D_{k_{s-1}}^{-1})^{c_\sigma}$ can be evaluated faithfully in genus 0 setting, namely in the fibre of $\pi_1(M_{0,s+1}) \to \pi_1(M_{0,s})$. But this case was already discussed in I,§3.19, where we showed the coincidence of sections $s_{\vec{a}'}(G_{\mathbb{Q}}) = s_{\mu_*}(G_{\mathbb{Q}})$ under the situation $\vec{a}' = \vec{b}_{n+1}$, $\mu_* = \nu_1$. From this follows that $c_\sigma = 0$. (In loc.cit., we should have written explicitly that the tangential base point $\tilde{\nu}_1$ is defined by the Puiseux ring $\overline{\mathbb{Q}}[\![t_{\nu_1/\lambda_1}^{1/n}, q^{1/n}]\!]_{n \geq 1}$ with t_{ν_1/λ_1} satisfying $t_{\nu_1/\lambda_1}(Q_{\nu_2}^0) = 1$. Actually, ν_1 should here be understood as a new leg "ν_*" adjacent to λ_2. Note however that, in loc.cit. we rearranged all indices after forgetting $Q_{\nu_2}^0$.) When $s = 1, 2$, we consider the diagram (8.7.1) of the case $s = r$ and apply the same argument of I,§3.19 in the "inverse way" to the above, i.e, view ν_n of loc.cit. as κ_{2g} here. When $s = 2$, the application is direct, and it follows immediately that $c_\sigma = 0$. When $s = 1$, we have to modify the argument, as, in loc.cit., we did not consider the forgetful morphism with respect to ν_1. In this case, the first two main generators for the limit Galois representation become $x_{12} = \sigma_1^2$ and $x_{13} = \sigma_1^{-1}\sigma_2^2\sigma_1$. Since the $G_{\mathbb{Q}}$-action at $\vec{a}'$ on them is given in the form $x_{12} \mapsto x_{12}^\chi$, $x_{13} \mapsto \sigma_1^{-\chi}\mathfrak{f}(\sigma_1^2, \sigma_2^2)^{-1}\sigma_2^{2\chi}\mathfrak{f}(\sigma_1^2, \sigma_2^2)\sigma_1^\chi$, this forces the base point "$\tilde{\nu}_*$" to turn around the issued cusp until reaching $\vec{a}'$ by an angle corresponding to the factor $(\sigma_1^2)^{\frac{\chi(\sigma)-1}{2}}$, i.e., a positive half rotation. (The fact that the two issued base points lie over the same "$\vec{b}_n$" is settled in our previous discussion in §8.3.) Actually, we made the definition of $\tilde{\mu}_*$ in §8.2 to reflect exactly this factor in the case $s = 1$; hence again $c_\sigma = 0$. Thus, we conclude $c_\sigma = 0$ in all cases. Consequently, as tangential base points, it follows that

$$(8.7.2) \qquad \vec{a}' \approx \tilde{\mu}_* \quad \text{on} \quad M_{g,r+1}.$$

By virtue of this fact, whenever we embed $\Pi_{g,r}$ onto $\ker(\pi_1(f_s)) \subset \pi_1(M_{g,r+1}, \vec{a}')$ by regarding Q_{ν_s} ($s = 1, \dots, r$) as a moving base point, we can interpret the limit

$G_{\mathbb{Q}}$-action (calculated for $s = 1$ in Theorem 6.8) as the restriction of the Galois representation in $\pi_1(M_{g,r+1}, \vec{a}')$ to the kernel of $\pi_1(f_s)$.

THEOREM 8.8. *Apply the forgetful morphism $f_0 : \mathcal{M}_{g,r+1} \to \mathcal{M}_{g,r}$ with respect to κ_0 to $\vec{a}'$. Call its image $\vec{a}$. Let $s_{\vec{a}} : G_{\mathbb{Q}} \to \pi_1(M_{g,r}, \vec{a})$ be the Galois section at the tangential base point $\vec{a}$. For $\sigma \in G_{\mathbb{Q}}$, let $\rho_{-1} = \frac{\chi(\sigma)-1}{2}$, $\mathfrak{f} = \mathfrak{f}_\sigma$. Then, the conjugate action by $s_{\vec{a}}(\sigma)$ on the Dehn twists $D_{u_{st}}$, D_{h_s} $(1 \le s (< t) \le r)$ of $\hat{\Gamma}_g^r = \pi_1(M_{g,r} \otimes \overline{\mathbb{Q}}, \vec{a})$ are given in the form $D_{h_s} \mapsto \mathfrak{F}_s D_{h_s}^{\chi(\sigma)} \mathfrak{F}_s^{-1}$, $D_{u_{st}} \mapsto \mathfrak{F}_{st} D_{u_{st}}^{\chi(\sigma)} \mathfrak{F}_{st}^{-1}$, where*

$$\mathfrak{F}_s = \begin{cases} \mathfrak{f}(D_{k_{23}}, D_{k_{12}})\mathfrak{f}(D_{k_{24}}, D_{k_{13}}) \cdots \mathfrak{f}(D_{k_{2,r}}, D_{k_{1,r-1}})\mathfrak{f}(D_{h_1}, D_{e_g}), \quad (s=1); \\ D_{k_s}^{\rho-1}\mathfrak{f}(D_{b_{s,s+1}}, D_{k_s})\mathfrak{f}(D_{b_{s,s+2}}, D_{k_{s+1}}) \cdots \mathfrak{f}(D_{b_{s,r}}, D_{k_{r-1}})\mathfrak{f}(D_{h_s}, D_{e_g}), \\ \hfill (s>1); \end{cases}$$

$$\mathfrak{F}_{st} = \begin{cases} \mathfrak{f}(D_{k_{23}}, D_{k_{12}})\mathfrak{f}(D_{k_{24}}, D_{k_{13}}) \cdots \mathfrak{f}(D_{k_{2,t-1}}, D_{k_{1,t-2}})D_{k_{1,t-1}}^{-\rho-1}\mathfrak{f}(D_{u_{1t}}, D_{k_{1,t-1}}), \\ \hfill (s=1); \\ D_{k_s}^{\rho-1}\mathfrak{f}(D_{b_{s,s+1}}, D_{k_s})\mathfrak{f}(D_{b_{s,s+2}}, D_{k_{s+1}}) \cdots \mathfrak{f}(D_{b_{s,t-1}}, D_{k_{t-2}}) \cdot \\ \hfill \cdot D_{k_{t-1}}^{-\rho-1}\mathfrak{f}(D_{u_{st}}, D_{k_{t-1}}), \quad (s>1). \end{cases}$$

Here, if a sequence part "$\cdots$" does not make sense, the portion should be understood to be 1.

PROOF. Fix $1 \le s \le r$. Notice that $D_{h_s}D_{d_g}^{-1}$ and $D_{u_{st}}$ are contained in the kernel of the forgetful homomorphism $\pi_1(f_s) : \pi_1(M_{g,r}) \to \pi_1(M_{g,r-1})$. Taking this and (8.7.2) into account, we shall fit in the situation of §8.2 and evaluate the $G_{\mathbb{Q}}$-action on (the natural lifts in $\pi_1(M_{g,r+1})$ of) $D_{h_s}D_{d_g}^{-1}$ and $D_{u_{st}}$ in the limit Galois representation in $\Pi_{g,r}$ at the base point $\tilde{\mu}_*$. Suppose first that $s > 1$. In this case, the Galois action is described in Theorem 6.8. Since $s < t$, we need only loops streaming towards κ_0, i.e.,we may regard μ_* as if it be ν_1 of §6.7. Then, $D_{h_s}D_{d_g}^{-1}$, $D_{u_{st}}$ correspond to $y_g x_g^{-1} y_g^{-1}$, $1_{\epsilon_{t-s}}$ respectively. It follows from Theorem 6.8 and the formula from I, Prop.2.11:

$$\mathfrak{f}_\sigma^{\overrightarrow{\infty 1}}(x,y)^{-1} = \mathfrak{f}_\sigma^{\overrightarrow{\infty 0}}(x,y)^{-1} z^{\rho-1} = x^{\rho-1}\mathfrak{f}(z,x)z^{\rho-1} \quad (xyz=1),$$

that the action of $\sigma \in G_{\mathbb{Q}}$ on $y_g x_g^{-1} y_g^{-1}$ is given by its χ-power conjugated by

$$\mathfrak{F}_s = \mathfrak{f}_\sigma^{\overrightarrow{\infty 1}}(0_{\epsilon_s}, 1_{\epsilon_s})^{-1} \cdots \mathfrak{f}_\sigma^{\overrightarrow{\infty 1}}(0_{\epsilon_{r-1}}, 1_{\epsilon_{r-1}})^{-1} \mathfrak{f}_\sigma^{\overrightarrow{\infty 0}}(0_{\lambda_{2g}}, 1_{\lambda_{2g}})^{-1}.$$

One can interpret loops appearing here as

$$\begin{cases} 0_{\epsilon_j} = D_{k_j}D_{b_{s,j}}^{-1}, \ 1_{\epsilon_j} = D_{u_{s,j+1}}, \ \infty_{\epsilon_j} = D_{b_{s,j+1}}D_{k_{s,j+1}}^{-1}, \ (j = s, \dots, r-1), \\ 0_{\lambda_{2g}} = D_{e_g}D_{b_{s,r}}^{-1}, \ \infty_{\lambda_{2g}} = D_{h_s}D_{d_g}^{-1}. \end{cases}$$

Putting them into the above and rewriting $\mathfrak{f}_\sigma^{\overrightarrow{\infty 1}}$, $\mathfrak{f}_\sigma^{\overrightarrow{\infty 0}}$ in terms of $\mathfrak{f}_\sigma$, we obtain the desired formula (Note that $D_{k_{s-1}}$, $D_{d_g}^{-1}$ commute with the other involved elements.) The case of u_{st} follows similarly from the formula for $z_{t-s+1} = 1_{\epsilon_{t-s}}$ of Theorem 6.8. Next we assume $s = 1$. This case also follows in the same way as the case $s > 1$, except that we have to take into consideration the modified definition of $\tilde{\mu}_*$ (§8.2) and its canonical turn to the base point "$\tilde{\nu}_*$" (§8.7). This reflects to the necessity of putting the left multiplication by $D_{k_{12}}^{-\rho-1}$ to the formulae of the limit Galois actions on $y_g x_g^{-1} y_g^{-1}$, $1_{\epsilon_{t-1}}$. We leave the readers to fill details of calculations. □

§9. Moves and associated pro-words.

9.1. In this section, we shall settle the proof of Theorem 5.8. The formulae in (1) follow from our construction of $\vec{a}$, and those in (2) are already shown in Theorem 8.6. To verify (3) and (4), we have to compare $\mathcal{F}_s$, $\mathcal{F}_{st}$ of Theorem 5.8 with $\mathfrak{F}_s$ and $\mathfrak{F}_{st}$ of Theorem 8.8. More precisely, we should prove

$$\mathcal{F}_s D_{h_s} \mathcal{F}_s^{-1} = \mathfrak{F}_s D_{h_s} \mathfrak{F}_s^{-1}, \qquad (1 \le s \le r), \tag{9.1.1}$$

$$\mathcal{F}_{st} D_{u_{st}} \mathcal{F}_{st}^{-1} = \mathfrak{F}_{st} D_{u_{st}} \mathfrak{F}_{st}^{-1}, \quad (1 \le s < t \le r). \tag{9.1.2}$$

Our strategy to prove them here is indirect. We utilize the framework of [NS], where we showed how one can compute the action of $\mathbb{I}\Gamma$ on any Dehn twist explicitly by moving along a sequence of pants decompositions to catch the issued circle. The obtained formula may take different forms according to the choice of the sequence of pants decompositions, but Proposition 8.1 of [NS] insures that the expressed element in the profinite Teichmüller modular group is independent of the choices. Therefore, it suffices to show that $\mathfrak{F}_s$, $\mathfrak{F}_{st}$ can be obtained by the algorithm of [NS] from particular sequences of pants decompositions (which would differ from those sequences used to get $\mathcal{F}_s$, $\mathcal{F}_{st}$ in [NS] §11).

9.2. For convenience of readers, we review briefly the algorithm of [NS] of producing formulas of the $\mathbb{I}\Gamma$-action on any Dehn twist. Given a quilt decomposition Q/P on a marked Riemann surface Σ, we introduced in loc.cit. a well defined representation $\rho_{Q/P} : \mathbb{I}\Gamma \to \mathrm{Aut}(\hat{\Gamma}(\Sigma))$ in the profinite completion of the (pure) mapping class group of Σ. Let c be any (homotopy class of a) circle on Σ, and let $\gamma : P = P_0 \to P_1 \to \cdots \to P_n$ be any sequence of pants decompositions with P_n containing the circle c such that each $P_i \to P_{i+1}$ is an "A-move" or "S-move". In this section we need only A-moves, so we shall assume that γ is a sequence of A-moves. This means that each step $P_i \to P_{i+1}$ replaces a circle c_i of P_i by a new circle c_i' of P_{i+1} $(i = 0, \ldots, n-1)$ in such a way that: (a) the geometric (minimum) intersection number of c_i and c_i' is two; (b) the algebraic (homological) intersection number of c_i and c_i' is zero. Moving along this sequence until getting $c_{n-1}' = c$, one can define sequences of quilts Q_i/P_i $(i = 0, \ldots, N)$ and of integers N_i $(i = 0, \ldots, n-1)$ characterized by the following conditions:

$$Q_0 = Q; \tag{9.2.1}$$

$$D_{c_i}^{N_i/2}(Q_i) \text{ is adjusted to } c_i'. \tag{9.2.2}$$

The latter condition means that, after N_i-times applying the half-twist along c_i to the seams of Q_i, we may reduce the geometric intersection number of those seams with c_i' $(i = 0, \ldots, n-1)$ to the minimum $(= 2)$. Let $(\chi, \mathfrak{f})$ be any element of $\mathbb{I}\Gamma$ defined by the parameter of the Grothendieck-Teichmüller group in $\hat{\mathbb{Z}}^\times \times \hat{F}_2$, and let $\rho_{-1} = \frac{\chi-1}{2}$. Then, as shown in [NS] Proposition 8.1, we can tell the action of $(\chi, \mathfrak{f}) \in \mathbb{I}\Gamma$ on the Dehn twist $D_c = D_{c_{n-1}'}$ in the form $D_c \mapsto \mathfrak{F} D_c^\chi \mathfrak{F}^{-1}$, where

$$\mathfrak{F} = D_{c_0}^{N_i \rho_{-1}} \mathfrak{f}(D_{c_0'}, D_{c_0}) \cdots D_{c_{n-1}}^{N_{n-1}\rho_{-1}} \mathfrak{f}(D_{c_{n-1}'}, D_{c_{n-1}}). \tag{9.2.3}$$

When some of the $P_i \to P_{i+1}$ is an S-move, which by definition replaces c_i by c_i' with geometric intersection number 1, one needs more elaborate descriptions to define Q_i, N_i and the corresponding factor of $\mathfrak{F}$. See [NS] §7.

9.3. Now, for each of $\mathfrak{F}_s$, $\mathfrak{F}_{st}$, we shall give a sequence of pants decompositions which gives it as the element $\mathfrak{F}$ described in the procedure of §9.2. We start from the standard quilt decomposition Q_0/P_0 given in §5.6. We leave the right half part of Figure 5.5 invariant, and move only circles in the left "tail" part. So we have only to consider the genus 0 subsurface obtained by cutting Figure 5.5 at $d_{-g} =: Q_{\nu_\infty}$, $d_g =: Q_{\nu_0}$. The resulting surface is illustrated in Figure 5.7. We shall start from the quilt Q_0/P_0 of Figure 9.1, where seams are drawn by dotted lines.

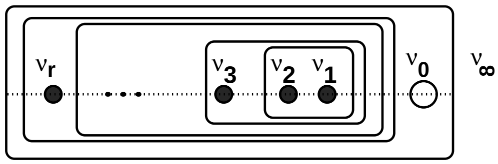

FIGURE 9.1

In the following we use the notation $\mathcal{C}(P)$ to designate the set of circles of the pants decomposition P.

Case 1: $\mathfrak{F}_1$. We define P_1 to be the pants decomposition obtained by replacing $k_{12} \in \mathcal{C}(P_0)$ by k_{23}, and inductively introduce P_j by replacing $k_{1,j+1} \in \mathcal{C}(P_{j-1})$ by $k_{2,j+2}$ $(j = 2, \ldots, r-2)$. Then, finally, define P_{r-1} by replacing $k_{1,r} = e_g \in \mathcal{C}(P_{r-2})$ by h_1. In this process of moves $Q/P_0 \to \cdots \to P_{r-1}$, the quilts Q_i are always adjusted to their next pants decompositions P_{i+1}. Hence, we need not apply half-twists on seams of the quilts. Thus we conclude

$$\mathfrak{F} = \mathfrak{F}_s = \mathfrak{f}(D_{k_{23}}, D_{k_{12}})\mathfrak{f}(D_{k_{24}}, D_{k_{13}}) \cdots \mathfrak{f}(D_{k_{2,r}}, D_{k_{1,r-1}})\mathfrak{f}(D_{h_1}, D_{e_g}).$$

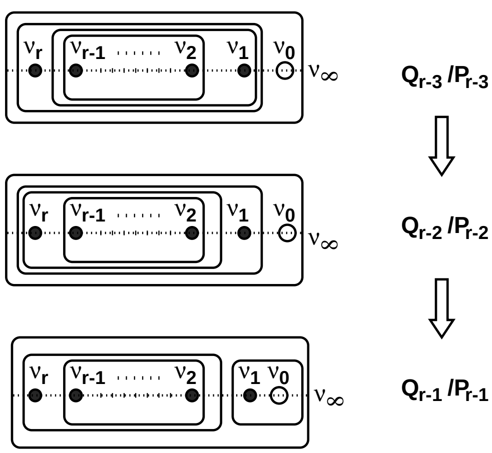

FIGURE 9.2

Case 2: $\mathfrak{F}_s$ $(2 \leq s \leq r)$. In this case, we first define P_1 by replacing $k_s \in \mathcal{C}(P_0)$ by $b_{s,s+1}$. Then, in order to get Q_1/P_1, we have to apply the half-twist along k_s to the seams as in Figure 9.3. Then, we define P_2 by replacing $k_{s+1} \in \mathcal{C}(P_1)$ by $b_{s,s+2}$,..., define P_{r-s} by replacing $k_{r-1} \in \mathcal{C}(P_{r-s-1})$ by $b_{s,r}$, and finally define P_{r-s+1} by replacing $k_r = e_g \in \mathcal{C}(P_{r-s})$ by h_s. In the latter process of moves

$Q_1/P_1 \to \cdots \to P_{r-s+1}$, the quilts are adjusted to their next pants decompositions, hence we need not apply half-twists on the seams. Thus, we see that

$$\mathfrak{F} = \mathfrak{F}_s = D_{k_s}^{\rho-1}\mathfrak{f}(D_{b_{s,s+1}}, D_{k_s})\mathfrak{f}(D_{b_{s,s+2}}, D_{k_{s+1}})\cdots\mathfrak{f}(D_{b_{s,r}}, D_{k_{r-1}})\mathfrak{f}(D_{h_s}, D_{e_g}).$$

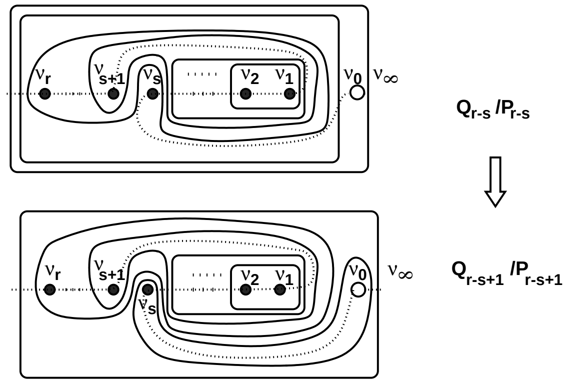

FIGURE 9.3

Case 3: $\mathfrak{F}_{1t}$ $(2 \le t \le r)$. We use the same P_i as Case 1 until $i = t-3$. Then we define P_{t-2} by replacing $k_{t-1} \in \mathcal{C}(P_{t-3})$ by u_{1t}. In the last step, we need to apply the reverse half-twist along k_{t-1} to the seams so that they are to be adjusted to u_{1t}. See Figure 9.4, where we understand $Q'_{t-3} = D_{k_{t-1}}^{-1/2}(Q_{t-3})$. Thus,

$$\mathfrak{F} = \mathfrak{F}_{1t} = \mathfrak{f}(D_{k_{23}}, D_{k_{12}})\mathfrak{f}(D_{k_{24}}, D_{k_{13}})\cdots\mathfrak{f}(D_{k_{2,t-1}}, D_{k_{1,t-2}})D_{k_{1,t-1}}^{-\rho-1}\mathfrak{f}(D_{u_{1t}}, D_{k_{1,t-1}}).$$

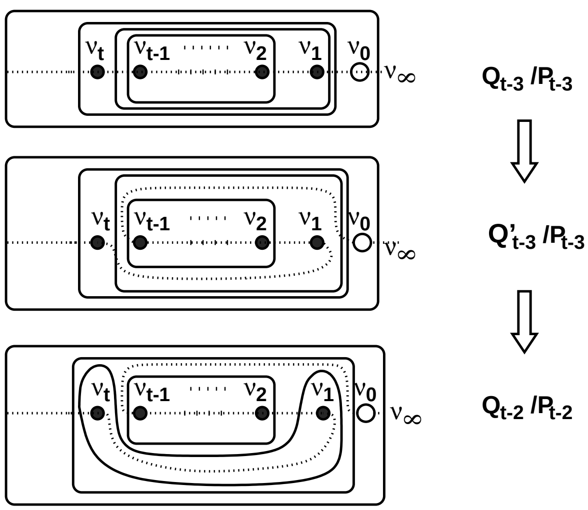

FIGURE 9.4

Case 4: $\mathfrak{F}_{st}$ $(2 \le s < t \le r)$. In this case, we use the same P_i as Case 2 until $i = t-s-1$. Then, define P_{t-s} by replacing $k_{t-1} \in \mathcal{C}(P_{t-s-1})$ by u_{st}. In the last step, we need to apply the reverse half twist along k_{t-1} so that the seams are to be adjusted to u_{st} as in Figure 9.5. This process produces

$$\begin{aligned}\mathfrak{F} &= \mathfrak{F}_{st} \\ &= D_{k_s}^{\rho-1}\mathfrak{f}(D_{b_{s,s+1}}, D_{k_s})\mathfrak{f}(D_{b_{s,s+2}}, D_{k_{s+1}})\cdots\mathfrak{f}(D_{b_{s,t-1}}, D_{k_{t-2}})D_{k_{t-1}}^{-\rho-1}\mathfrak{f}(D_{u_{st}}, D_{k_{t-1}}).\end{aligned}$$

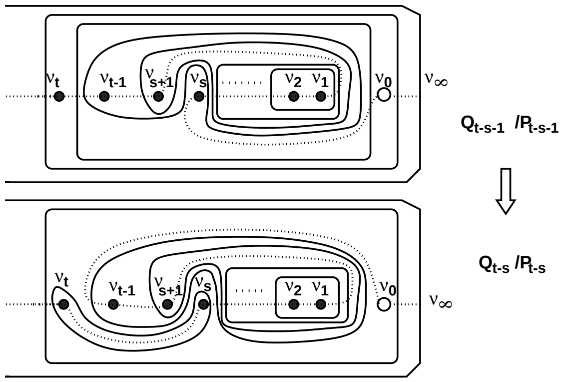

FIGURE 9.5

Thus the proof of Theorem 5.8 is completed.

§10. Coalescing tangential base points.

10.1. In this section, we discuss a generalization of our construction of the tangential base point $\vec{v}$ given in [N97] (see §7) in the context of Hurwitz moduli spaces. Let G be a finite group and $T : G \to S_n$ be a faithful transitive permutation representation. For each sequence $\mathcal{C} = (C_1, \dots, C_r)$ of conjugacy classes of G, the strict Nielsen set $SNi(G, \mathcal{C})$ is defined to be the collection of r-tuples $\mathbf{g} = (g_i) \in C_1 \times \cdots \times C_r$ such that $g_1 \cdots g_r = 1$ and $\langle g_1, \dots, g_r \rangle = G$. Pick $\mathbf{g} \in SNi(G, \mathcal{C})$ and let $sO(\mathbf{g})$ be the orbit of $\mathbf{g}$ under the natural action by the pure Hurwitz braid group with r strings. M.Fried [F] introduced a strict Hurwitz modular variety $H(sO(\mathbf{g}))$ as a finite cover over $(\mathbf{P}^1)^r - \Delta$ corresponding to this transitive permutation representation. This is a moduli space parameterizing covers of $\mathbf{P}^1$ with branch cycle data in $sO(\mathbf{g})$.

The aim of this section is to present a way of constructing a tangential base point $\vec{v}(\mathbf{g})$ on $H(sO(\mathbf{g}))$ which would be called the *coalescing tangential base point* associated to $\mathbf{g} \in SNi(G, \mathcal{C})$.

10.2. We first define $(r-2)$ triples (x_i, y_i, z_i) $(i = 1, \dots, r-2)$ by

$$\begin{cases} x_i &:= g_1 \cdots g_i, \\ y_i &:= g_{i+1}, \\ z_i &:= g_{i+2} \cdots g_r. \end{cases}$$

Then, $x_i y_i z_i = 1$, $z_i = x_{i+1}^{-1}$ hold. Each triple defines via T a not necessarily connected cover $Y_i^0 \to \mathbf{P}^1$ branched only over $\{0, 1, \infty\}$. Take a number filed $K/\mathbb{Q}$ such that

1. each connected component of Y_i^0 $(\forall i)$ is defined over K;
2. each branch point on Y_i^0 $(\forall i)$ over $0, 1, \infty$ is defined over K.

The above $(r-2)$ triples produce an admissible cover $f : Y^0 \to X^0$ over K in the sense of Harris-Mumford [HM] in such a way that

$$X^0 = \cup_{i=1}^{r-2} \mathbf{P}^1_{01\infty} / \sim,$$
$$Y^0 = \cup_{i=1}^{r-2} Y_i^0 / \sim,$$

where $\sim$ indicate identifications of pairs of points as nodes in the following manners. On Y^0, for each i, the branch point on Y_i^0 corresponding to a $\langle z_i \rangle$-orbit is identified with the branch point on Y_{i+1}^0 corresponding to the same $\langle x_{i+1} = z_i^{-1} \rangle$-orbit. To be compatible with such Y^0, the X^0 is to be the obvious chain of $\mathbf{P}^1$'s which looks as in Figure 10.1.

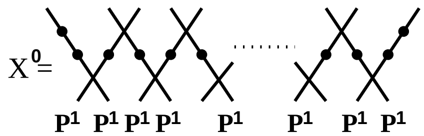

FIGURE 10.1

For a prototype example, suppose the case where $G = \mathbb{Z}/2\mathbb{Z}$ and $\mathbf{g} = (-1, \ldots, -1)$. Then, the admissible cover $Y^0 \to X^0$ associated to this $\mathbf{g}$ is nothing but the degenerate hyperelliptic cover considered in [N97]. We would like to give the definition of $\vec{v}(\mathbf{g})$ so that it gives in this special case the "hyperelliptic tangential base point" $\vec{v}$ on the moduli space of $M_{g,1}$ discussed there.

10.3. Example (Harbater-Mumford type). Another key example defined over $\mathbb{Q}$ is given in the paper [F] by M.Fried. Let

$$D_5 = \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & 1 \end{pmatrix} \mid \varepsilon = \pm 1,\ a \in \mathbb{Z}/5\mathbb{Z} \right\}$$

be the fifth dihedral group acting on $\mathbb{Z}/5\mathbb{Z}$ by $x \mapsto \varepsilon x + a$ $(x \in \mathbb{Z}/5\mathbb{Z})$ and consider $SNi(D_5; C_1, C_2, C_3, C_4)$ which contains the tuple

$$\mathbf{g} = \left(\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & -1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & -2 \\ 0 & 1 \end{smallmatrix}\right) \right).$$

Then the resulting admissible cover $Y^0 \to X^0$ may be illustrated as in Figure 10.2.

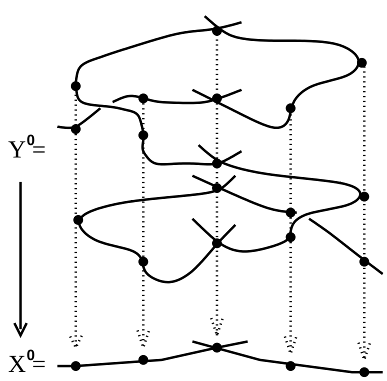

FIGURE 10.2

10.4. We shall construct a tangential base point $\vec{v}(\mathbf{g})$ by making use of the formal patching technique arranged by Harbater-Stevenson [HS]. Regard X_i^0 as the projective line $\mathbf{P}_K^1$ with the standard coordinate t_i $(1 \le i \le r-2)$ illustrated in Figure 10.3, and set $t_i' = t_i^{-1}$.

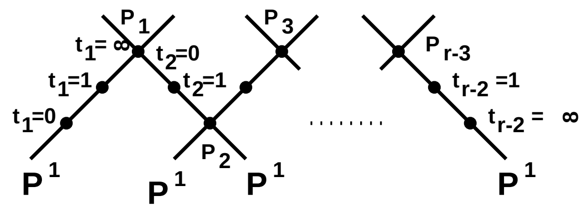

FIGURE 10.3

Let P_i $(i = 1, \dots, r-3)$ be double points with $t_i = \infty$ (i.e., $t_i' = 0$ or equivalently $t_{i+1} = 0$), and write the fiber set $f^{-1}(P_i)$ as $\{P_{i,1}, \dots, P_{i,m_i}\}$ which bijectively corresponds to the set of $\langle z_i \rangle$-orbits in $\{1, \dots, n\}$. Letting e_{ij} denote the ramification index at P_{ij} (which is the same as the length of the corresponding $\langle z_i \rangle$-orbit), we define

$$e_i := \text{l.c.m.}\{e_{i1}, \dots, e_{im_i}\}.$$

Then, as in [IN], one can make the deformation space $X^*/K[\![q]\!]$ of X^0/K by the equations $t_i' t_{i+1} - q^{e_i}$ $(1 \le i \le r-3)$. Next, choose uniformizing parameters s_{ij}' of $\hat{O}_{Y_i^0, P_{ij}}$ and $s_{i+1,j}$ of $\hat{O}_{Y_{i+1}^0, P_{ij}}$ such that $(s_{ij}')^{e_{ij}} = t_i'$, $(s_{i+1,j})^{e_{ij}} = t_{i+1}$, and introduce the local rings

$$R_{ij} := K[\![s_{ij}', s_{i+1,j}, q]\!]/(s_{ij}' s_{i+1,j} - q^{e_i/e_{ij}}) \quad (1 \le i \le r-3,\ 1 \le j \le m_i)$$

over $K[\![q]\!]$ which are equipped with natural homomorphisms $\hat{O}_{X^*, P_i^0} \hookrightarrow R_{ij}$. Then, the collection of data $\{Y^0 \to X^0 \subset X^*, \{\hat{O}_{X^*, P_i^0} \hookrightarrow R_{ij}\}_{ij}\}$ forms a relative thickening problem for $(Y^0, \{P_{ij}\})$ relative to $Y^0 \to X^0$ in the sense of Harbater-Stevenson [HS]. Thus, it follows from Theorem 2 of [HS] that there arises an admissible cover $Y^* \to X^*$ over $K[\![q]\!]$ whose generic fibre is a smooth connected cover over $\mathbf{P}^1_{K(\!(q)\!)}$ with branch cycle data $\mathbf{g} \in SNi(G, \mathcal{C})$. We define the K-rational tangential base point $\vec{v}(\mathbf{g})$ on $H(sO(\mathbf{g}))$ to be that which is induced from $Y^* \otimes K(\!(q)\!) \to \mathbf{P}^1_{K(\!(q)\!)}$.

Appendix.

In this appendix, we prove that the Dehn twists along the circles indicated in Figure 5.6 generate the mapping class group of the marked surface. The corresponding result should be essentially known, for example, in [M], [Bi1], but it would be convenient to present a concise proof here using only a simple technique utilized effectively by Humphries [H].

We write $\Sigma_{g,r}$ for the surface of genus g with r marked points $Q_1, \dots, Q_r$, and denote the mapping class group by $\Gamma(\Sigma_{g,r})$. There is an exact sequence

$$1 \longrightarrow \Pi_{g,0}^{(r)} \longrightarrow \Gamma(\Sigma_{g,r}) \longrightarrow \Gamma(\Sigma_{g,0}) \longrightarrow 1$$

obtained by forgetting all marked points. Here, the kernel $\Pi_{g,0}^{(r)}$ can be identified with the pure braid group on the closed surface Σ_g with r strings. It is known by M.Dehn, W.B.R.Lickorish that $\Gamma(\Sigma_{g,0})$ is generated by the images of $a_1, \ldots, a_{2g}$ and $d_{\pm i}$ $(i = 1, \ldots, g)$. So it suffices to consider the kernel. This is generated by the loops regarded as those $x_1^{(s)}, \ldots, x_g^{(s)}, y_1^{(s)}, \ldots, y_g^{(s)}, z_1^{(s)}, \ldots, z_r^{(s)}$ $(z_s^{(s)} = 1)$ starting from Q_s and running in the surface $\Sigma - \{Q_t \mid 1 \le s \le r,\ t \ne s\}$ as in Figure A1.

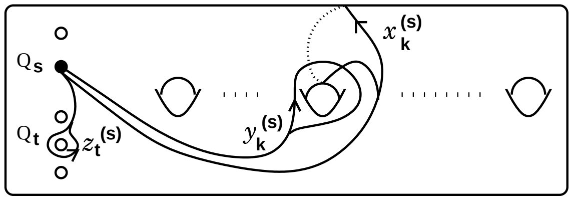

FIGURE A1

To represent each of these loops by Dehn twists, we employ usual technique that the loop can be interpreted as a quotient of two Dehn twists along circles lying along the loop "inside and outside". Figure A2 shows the typical situation where we should have $\ell = D_a D_b^{-1}$:

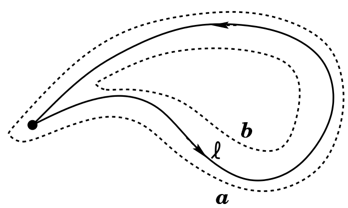

FIGURE A2

It is obvious, as the inside circle becomes trivial, that one can write $z_i^{(s)}$ as u_{is}. To produce $x_k^{(s)}$, $y_k^{(s)}$ $(1 \le k \le g)$, we introduce the sequence $\eta_g^{(s)}, \xi_g^{(s)}, \ldots, \eta_1^{(s)}, \xi_1^{(s)}$ by

$$\eta_g^{(s)} := (y_g^{(s)})^{-1}, \quad \xi_g^{(s)} := y_g^{(s)}(x_g^{(s)})^{-1}(y_g^{(s)})^{-1};$$
$$\eta_k^{(s)} := (y_k^{(s)})^{-1}\eta_{k+1}^{(s)}, \quad \xi_k^{(s)} := (\eta_k^{(s)})^{-1}(x_k^{(s)})^{-1}\eta_k^{(s)}, \quad (1 \le k \le g-1).$$

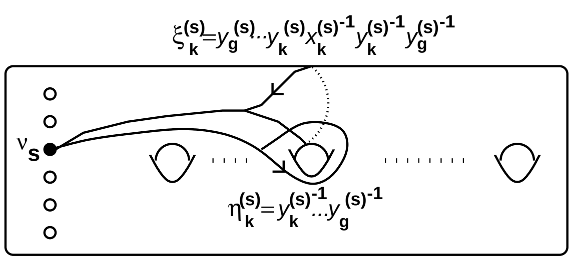

FIGURE A3

First, it follows from the above remark that $\xi_g^{(s)} = h_s$. We then apply the general formula $D_a D_b D_a^{-1} = D_{D_a(b)}$ repeatedly as in Figure A4. It is then clear that all $\xi_k^{(s)}$, $\eta_k^{(s)}$, hence all $x_k^{(s)}$, $y_k^{(s)}$ $(1 \le k \le g)$ can be produced by the Dehn twists along the circles indicated in Figure 5.6.

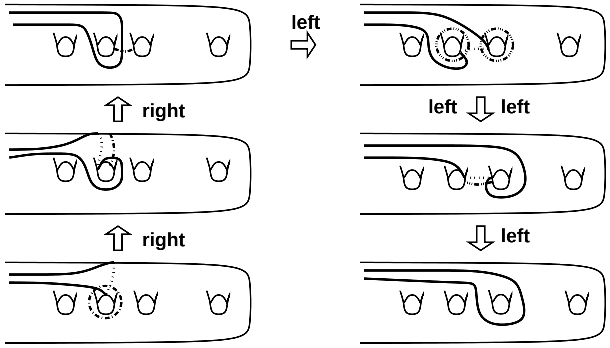

FIGURE A4

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