On Hecke modules generated by eta-quotients of weight one

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Modular forms of weight one

Modular forms of weight one and Galois representations

$$\left\{ \begin{array}{c} \text{elliptic and cuspidal} \\ \text{Hecke eigenforms of weight one} \end{array} \right\}$$

$$\underset{1 \text{ to } 1}{\longleftrightarrow} \left\{ \begin{array}{c} \rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C}) \\ : \text{odd, conti., irred.} \end{array} \right\} / \sim$$

via the coincidence of L-functions.
(Hecke, Weil, Deligne, Serre, Langlands, Tunnell, Khare, Wintenberger,....)

$\longrightarrow$ A weight one modular form has an information of certain Galois extension over $\mathbb{Q}$.

Modular forms of weight one

If $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is continuous and irreducible, then the image of ρ in $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to one of

- D_n: dihedral group of order $2n$ (dihedral type),
- A_4 (tetrahedral type),
- S_4 (octahedral type),
- A_5 (icosahedral type).

Property of dihedral representations

If $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is of dihedral type, then $\rho \cong \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \xi$ for some quadratic field F and ray class character ξ of F. Therefore, if f is a Hecke eigenform of weight one and the Galois representation ρ_f associated to f is of dihedral type, then there exist a quadratic field F and a ray class character ξ of F such that

$$f(\tau) = \Theta_{\xi}(\tau) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

Motivation

$\longrightarrow$ A dihedral type modular form has an information of certain abelian extention of quadratic field.

Motivation

Given an eta-quotient f of weight one, which is not necessary Hecke eigenform, we want to know an information on the number field associated to f.

$\longrightarrow$ It is natural to study the Hecke module generated by an eta-quotient.

Notation

For a finite group G,

- $G^* := \mathrm{Hom}(G, \mathbb{C})$,
- $\widetilde{G^*} := G^*/\sim$, where for $\chi_1, \chi_2 \in G^*$, $\chi_1 \sim \chi_2$ means that $\overline{\chi_2} = \chi_1$.

For a quadratic field K,

- Cl_K : the ideal class group of K,
- $h(K)$: the class number of K,
- $\mathcal{O}_K$: the ring of integers of K.

- $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$: Dedekind eta function ($q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$),
- $M_k(N, \chi)$: the space of weight k holomorphic modular forms of level N and character χ,
- $S_k(N, \chi) \subset M_k(N, \chi)$: the space of cusp forms.

Main Theorem (1)

Let λ be a square-free positive integer such that $\lambda \equiv 15 \pmod{24}$, and set

$$\varphi_\lambda(\tau) := \frac{\eta(3\tau)\eta(12\tau)\eta(2\lambda\tau)^3}{\eta(6\tau)\eta(\lambda\tau)\eta(4\lambda\tau)} \in S_1(12\lambda, (\tfrac{-12\lambda}{\cdot})).$$

Let

$$V_\lambda := \langle \varphi_\lambda | T_n \ ; \ n \geq 1 \rangle_{\mathbb{C}} \, .$$

Theorem 1

With the above notation, we have

$$\dim_{\mathbb{C}} V_\lambda = h(\mathbb{Q}(\sqrt{-\lambda})).$$

Main Theorem (2)

Let N be a square-free positive interger s.t. $N \equiv 23 \pmod{24}$, $K := \mathbb{Q}(\sqrt{-N})$ and $h_N := h(K)$. Let

$$g_N := \sharp(\mathrm{Cl}_K/\mathrm{Cl}_K^2) : \text{the number of quadratic character of } \mathrm{Cl}_K,$$
$$d_N := (h_N - g_N)/2,$$
$$\psi_N(\tau) := \eta(\tau)\eta(N\tau),$$
$$V_N := \langle \psi_N | T_n \ ; \ n \geq 1 \rangle_{\mathbb{C}}.$$

Theorem 2

We have

$$\dim_{\mathbb{C}} V_N = d_N - \sharp \left\{ \chi \in \widetilde{\mathrm{Cl}_K^*} \ ; \ \chi^2 \neq 1, \chi([\mathfrak{p}_2]) = \pm 1 \text{ or } \chi([\mathfrak{p}_3]) = \pm 1 \right\},$$

where $\mathfrak{p}_2$ (resp. $\mathfrak{p}_3$) is a prime ideal of $\mathbb{Q}(\sqrt{-N})$ lying above 2 (resp. 3).

Main Theorem (2)

Corollary 3

Let p be a prime number s.t. $p \equiv 23 \pmod{24}$ and $K := \mathbb{Q}(\sqrt{-p})$. Then the following assertions are equivalent;

(a) $\dim_{\mathbb{C}} V_p = d_p$,

(b) $\mathrm{Cl}_K = \langle [\mathfrak{p}_2] \rangle = \langle [\mathfrak{p}_3] \rangle$.

(Proof of Cor.) If $N = p$ in Theorem 2, then h_p is odd, thus

$$\dim_{\mathbb{C}} V_p = d_p - \sharp \left\{ \chi \in \widetilde{\mathrm{Cl}_K^*} \ ; \ \chi \neq 1, \chi([\mathfrak{p}_2]) = 1 \text{ or } \chi([\mathfrak{p}_3]) = 1 \right\}.$$

This shows that $\dim_{\mathbb{C}} V_p = d_p$ is equivalent to that

$$\text{“}\chi([\mathfrak{p}_2]) = 1 \Rightarrow \chi = 1\text{” and “}\chi([\mathfrak{p}_3]) = 1 \Rightarrow \chi = 1.\text{”}$$

$\square$

CM modular forms

Definition

Let

$f \in M_k(N, \chi)$: a Hecke eigenform,

$m \in \mathbb{Z}_{>0}$,

θ : a non-trivial quadratic character modulo m.

We say "f has CM by θ" if

$$\theta(p)a_p(f) = a_p(f)$$

(i.e., $f|T_p = 0$) for all $p \nmid Nm$.

- $M_k^{CM}(N, \chi; \theta)$: the space of CM forms by θ,
- $S_k^{CM}(N, \chi; \theta) := M_k^{CM}(N, \chi; \theta) \cap S_k(N, \chi)$.

CM modular forms

Theorem (Kani)

Let $F := \overline{\mathbb{Q}}^{\mathrm{Ker}(\theta)}$, f a Hecke eigenform and ρ_f the Galois representation associated to f. Then, the following assertions are equivalent:

(a) f has CM by θ,

(b) $\exists \xi \in \mathrm{Hom}(G_F, \mathbb{C}^*)$ s.t. $\rho_f \cong \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \xi$.

The assertion (b) in Theorem 3 is equivalent to

(c) $\exists \xi$: a ray class character of F s.t. $f(\tau) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) q^{N(\mathfrak{a})}$.

Theorem (Kani)

Let N be a positive number such that $-N$ is a fundamental discriminant. Then

$$\dim_{\mathbb{C}} S_1^{CM}(N, (\tfrac{-N}{\cdot})\,;(\tfrac{-N}{\cdot})) = d_N.$$

Proof of Theorem 1

We fix the following notation:

- $\lambda \equiv 15 \pmod{24}$: square-free,
- $K := \mathbb{Q}(\sqrt{-\lambda})$,
- $\mathfrak{p}_2 := 2\mathbb{Z} + \dfrac{1+\sqrt{-\lambda}}{2}\mathbb{Z} = \left\{ \dfrac{x+y\sqrt{-\lambda}}{2} \ ; \ x \equiv y \pmod{4} \right\}$,
- $\mathfrak{p}_3 := 3\mathbb{Z} + \dfrac{1+\sqrt{-\lambda}}{2}\mathbb{Z} = \left\{ \dfrac{3x+y\sqrt{-\lambda}}{2} \ ; \ x \equiv y \pmod{2} \right\}$,
- $\overline{\mathfrak{p}_2}\mathfrak{p}_3 = \left\{ \dfrac{3x+y\sqrt{-\lambda}}{2} \ ; \ x \equiv y \pmod{4} \right\}$.

Recall

$$\varphi_\lambda(\tau) := \frac{\eta(3\tau)\eta(12\tau)\eta(2\lambda\tau)^3}{\eta(6\tau)\eta(\lambda\tau)\eta(4\lambda\tau)}.$$

Proof of Theorem 1

$$\vartheta_1(\tau) := \frac{\eta(\tau)\eta(4\tau)}{\eta(2\tau)} = \sum_{n=1}^{\infty} \left(\frac{8}{n}\right) q^{\frac{n^2}{8}},$$

$$\vartheta_2(\tau) := \frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)} = \sum_{n=1}^{\infty} \left(\frac{24}{n}\right) q^{\frac{n^2}{24}}.$$

Then,

$$\varphi_\lambda(\tau) = \vartheta_1(3\tau)\vartheta_2(\lambda\tau) = \sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{24}{n}\right) q^{\frac{3m^2+dn^2}{8}},$$

where $d = \lambda/3$.

Proof of Theorem 1

Proposition

We have

$$\varphi_\lambda(\tau) = \sum_{\substack{\mathfrak{a}\in[\overline{\mathfrak{p}}_2\mathfrak{p}_3]\ :\ \text{integral} \\ (\mathfrak{a},\overline{\mathfrak{p}}_2\mathfrak{p}_3)=1}} \left(\frac{8}{m_\mathfrak{a} n_\mathfrak{a}}\right)\left(\frac{3}{n_\mathfrak{a}}\right) q^{N(\mathfrak{a})},$$

where $m_\mathfrak{a}$ and $n_\mathfrak{a}$ is the rational integers such that

$$\mathfrak{a} = \left(\frac{3m_\mathfrak{a} + n_\mathfrak{a}\sqrt{-\lambda}}{2}\right)(\mathfrak{p}_2\mathfrak{p}_3)^{-1}.$$

Proposition

If p is a prime such that $\left(\frac{-\lambda}{p}\right) = -1$, then $\varphi_\lambda | T_p = 0$.

$\longrightarrow V_\lambda \subset S_1^{CM}(12\lambda, \left(\frac{-12\lambda}{\cdot}\right); \left(\frac{-\lambda}{\cdot}\right))$.

Proof of Theorem 1

For every Hecke eigenform $g \in V_\lambda$,

$$\exists \xi \ : \ \text{ray class character with some conductor } \mathfrak{f} \text{ s.t.}$$

$$g(\tau) = \Theta_\xi(\tau) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \xi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

Considering the level and character of V_λ, we have

Proposition

$$\mathfrak{f} = \overline{\mathfrak{p}_2}^2 \mathfrak{p}_3.$$

- $\mathrm{Cl}_K(\mathfrak{f})$: ray class group with modulus $\mathfrak{f} = \overline{\mathfrak{p}_2}^2 \mathfrak{p}_3$,
- $H_K(\mathfrak{f}) := \mathrm{Ker}(\mathrm{Cl}_K(\mathfrak{f}) \to \mathrm{Cl}_K)$.

Proof of Theorem 1

Consider the exact sequence

$$1 \longrightarrow \mathrm{Cl}_K^* \xrightarrow{\pi} \mathrm{Cl}_K(\mathfrak{f})^* \longrightarrow H_K(\mathfrak{f})^* \longrightarrow 1.$$

Since $(\mathcal{O}_K/\mathfrak{f})^\times/\{\pm 1\} \cong (\mathbb{Z}/12\mathbb{Z})^\times/\{\pm 1\} = \{1,5\}$, we have

$$\sharp H_K(\mathfrak{f}) = 2, \text{ thus } \sharp \mathrm{Cl}_K(\mathfrak{f})^* = 2h(-\lambda)$$

Let $T := \mathrm{Cl}_K^* \setminus \mathrm{Im}(\pi) = \{\xi \in \mathrm{Cl}_K(\mathfrak{f})^* \,;\, \xi(H_K(\mathfrak{f})) = \{\pm 1\}\}$ and

$$W_\lambda := \langle \Theta_\xi(\tau) \,;\, \xi \in T \rangle_{\mathbb{C}}.$$

Lemma

$$\Theta_\xi \in \begin{cases} S_1(12\lambda, (\frac{-\lambda}{\cdot})) & \text{if } \xi \in \mathrm{Im}(\pi), \\ S_1(12\lambda, (\frac{-12\lambda}{\cdot})) & \text{otherwise, i.e., } \xi \in T. \end{cases}$$

Proof of Theorem 1

Thus $V_\lambda \subset W_\lambda$.

Proposition

The set $\{\Theta_\xi(\tau)\ ;\ \xi \in T\}$ is a basis of W_λ. Thus,

$$\dim_{\mathbb{C}} W_\lambda = h(-\lambda).$$

For the proof of this proposition, note that for $\xi_1, \xi_2 \in T$ we have

$$\Theta_{\xi_1} = \Theta_{\xi_2} \iff \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \xi_1 \cong \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \xi_2.$$

Therefore, it is enough to show

$$\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \xi_1 \cong \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \xi_2 \implies \xi_1 = \xi_2$$

for $\xi_1, \xi_2 \in T$.

Proof of Theorem 1

Let $h := h(-\lambda)$ and

$$T = \{\xi_i \ ; \ i = 1, \ldots, h\},$$
$$\mathrm{Cl}_K = \{\mathcal{C}^{(j)} \ ; \ j = 1, \ldots, h\}.$$

For $\mathcal{C}^{(j)} \in \mathrm{Cl}_K$, let

$$\mathcal{C}_1^{(j)}, \ \mathcal{C}_2^{(j)} \in \mathrm{Cl}_K(\mathfrak{f})$$

be the lifts of $\mathcal{C}^{(j)}$ to $\mathrm{Cl}_K(\mathfrak{f})$, and

$$\Theta_{\mathcal{C}^{(j)}}(\tau) := \sum_{\mathfrak{a} \in \mathcal{C}_1^{(j)}} q^{N(\mathfrak{a})} - \sum_{\mathfrak{a} \in \mathcal{C}_2^{(j)}} q^{N(\mathfrak{a})}.$$

Note that

$$\xi(\mathcal{C}_1^{(j)}) = -\xi(\mathcal{C}_2^{(j)}) \quad \text{for } \xi \in T,$$

because ξ is non-trivial on $H_K(\mathfrak{f})$.

Proof of Theorem 1

Then, for $\xi \in T$,

$$\begin{aligned}
\Theta_\xi(\tau) &= \sum_{j=1}^{h} \left\{ \xi(\mathcal{C}_1^{(j)}) \sum_{\mathfrak{a}\in\mathcal{C}_1^{(j)}} q^{N(\mathfrak{a})} + \xi(\mathcal{C}_2^{(j)}) \sum_{\mathfrak{a}\in\mathcal{C}_2^{(j)}} q^{N(\mathfrak{a})} \right\} \\
&= \sum_{j=1}^{h} \xi(\mathcal{C}_1^{(j)}) \left(\sum_{\mathfrak{a}\in\mathcal{C}_1^{(j)}} q^{N(\mathfrak{a})} - \sum_{\mathfrak{a}\in\mathcal{C}_2^{(j)}} q^{N(\mathfrak{a})} \right) \\
&= \sum_{j=1}^{h} \xi(\mathcal{C}_1^{(j)}) \Theta_{\mathcal{C}^{(j)}}(\tau).
\end{aligned}$$

Fix $\xi \in T$. For each $\xi_i \in T$,

$$\exists \psi_i \in \mathrm{Im}(\pi) \ \text{ s.t. } \ \xi_i = \xi\psi_i$$

(recall $\pi : \mathrm{Cl}_K^* \to \mathrm{Cl}_K(\mathfrak{f})^*$).

Proof of Theorem 1

For $\ell \in \{1, \ldots, h\}$, set

$$c_i^{(\ell)} := \overline{\psi_i(\mathcal{C}^{(\ell)})}.$$

Then, by the orthogonality relation for characters, we have

$$\sum_{i=1}^{h} c_i^{(\ell)} \Theta_{\xi_i}(\tau) = h\xi(\mathcal{C}_1^{(\ell)})\Theta_{\mathcal{C}^{(\ell)}}(\tau).$$

Proposition

The forms $\Theta_{\mathcal{C}^{(1)}}, \ldots, \Theta_{\mathcal{C}^{(h)}}$ form a basis of W_λ.

Set

$$\mathcal{A} := [\overline{\mathfrak{p}_2}\mathfrak{p}_3] \in \mathrm{Cl}_K,$$
$$\mathcal{B} := [\mathfrak{p}_2\mathfrak{p}_3] \in \mathrm{Cl}_K.$$

Proof of Theorem 1

Theorem

We have $\varphi_\lambda = \pm\Theta_{\mathcal{A}}$ or $\varphi_\lambda = \pm\Theta_{\mathcal{B}}$

Corollary

There exist $c_1, \ldots, c_h \in \mathbb{C}^\times$ s.t. $\varphi_\lambda = \sum_{i=1}^{h} c_i \Theta_{\xi_i}$.

Corollary

We have $V_\lambda = W_\lambda$.

Proof of Theorem 2

Let

- N : a square-free positive integer s.t. $N \equiv 23 \pmod{24}$,
- $K := \mathbb{Q}(\sqrt{-N})$,
- $\psi_N(\tau) := \eta(\tau)\eta(N\tau) \in S_1^{CM}(N, (\frac{-N}{\cdot})\,;(\frac{-N}{\cdot}))$,
- $V_N := \langle \psi_N | T_n \;;\; n \geq 1 \rangle_{\mathbb{C}} \subset S_1^{CM}(N, (\frac{-N}{\cdot})\,;(\frac{-N}{\cdot}))$,

It is known that

$$\psi_N(\tau) = \frac{1}{2}\left(\sum_{x,y \in \mathbb{Z}} q^{6x^2+xy+\frac{N+1}{24}y^2} - \sum_{x,y \in \mathbb{Z}} q^{6x^2-5xy+\frac{N+25}{24}y^2} \right).$$

Let

- $\mathfrak{p}_2 := 2\mathbb{Z} + \dfrac{1+\sqrt{-N}}{2}\mathbb{Z}$: a prime in K lying above 2,
- $\mathfrak{p}_3 := 3\mathbb{Z} + \dfrac{1+\sqrt{-N}}{2}\mathbb{Z}$: a prime in K lying above 3.

Proof of Theorem 2

Then,

- $\mathfrak{p}_2\mathfrak{p}_3 = \left\{ \dfrac{12x + y + y\sqrt{-N}}{2} \ ; \ x, y \in \mathbb{Z} \right\}$,
- $\overline{\mathfrak{p}_2}\mathfrak{p}_3 = \left\{ \dfrac{12x - 5y + y\sqrt{-N}}{2} \ ; \ x, y \in \mathbb{Z} \right\}$.

Noting that

$$\frac{N((12x + y + y\sqrt{-N})/2)}{N(\mathfrak{p}_2\mathfrak{p}_3)} = 6x^2 + xy + \frac{N+1}{24}y^2,$$

$$\frac{N((12x - 5y + y\sqrt{-N})/2)}{N(\overline{\mathfrak{p}_2}\mathfrak{p}_3)} = 6x^2 - 5xy + \frac{N+25}{24}y^2,$$

Proof of Theorem 2

we have

Proposition

$$\psi_N(\tau) = \sum_{\mathfrak{a}\in[\mathfrak{p}_2\mathfrak{p}_3]} q^{N(\mathfrak{a})} - \sum_{\mathfrak{a}\in[\overline{\mathfrak{p}_2}\mathfrak{p}_3]} q^{N(\mathfrak{a})}.$$

Let

- $h := \sharp \mathrm{Cl}_K$,
- $\mathrm{Cl}_K^* := \{\chi_1, \dots, \chi_h\}$.

By the orthogonality relation of characters,

Proposition

For every $\mathcal{C} \in \mathrm{Cl}_K$, we have

$$\frac{1}{h}\sum_{i=1}^{h} \overline{\chi_i(\mathcal{C})}\Theta_{\chi_i}(\tau) = \sum_{\mathfrak{a}\in\mathcal{C}} q^{N(\mathfrak{a})}.$$

Proof of Theorem 2

Set

$$\mathcal{C}_1 := [\mathfrak{p}_2\mathfrak{p}_3], \ \ \mathcal{C}_2 := [\overline{\mathfrak{p}_2}\mathfrak{p}_3] \ \in \mathrm{Cl}_K.$$

Then,

$$\begin{aligned} h\psi_N(\tau) &= \sum_{i=1}^{h} \left(\overline{\chi_i(\mathcal{C}_1)} - \overline{\chi_i(\mathcal{C}_2)}\right)\Theta_{\chi_i}(\tau) \\ &= \sum_{\substack{\chi\in\widetilde{\mathrm{Cl}_K^*} \\ \chi^2\neq 1}} \left(\overline{\chi(\mathcal{C}_1)} - \overline{\chi(\mathcal{C}_2)} + \chi(\mathcal{C}_1) - \chi(\mathcal{C}_2)\right)\Theta_{\chi}(\tau) \\ &= \sum_{\substack{\chi\in\widetilde{\mathrm{Cl}_K^*} \\ \chi^2\neq 1}} \left(\chi([\mathfrak{p}_2]) - \overline{\chi([\mathfrak{p}_2])}\right)\left(\chi([\mathfrak{p}_3]) - \overline{\chi([\mathfrak{p}_3])}\right)\Theta_{\chi}(\tau). \end{aligned}$$

Recalling that $d_N = (h - g_N)/2 = \sharp\left\{\chi \in \widetilde{\mathrm{Cl}_K^*} \ ; \ \chi^2 \neq 1\right\}$,

Proof of Theorem 2

we have

$$\dim_{\mathbb{C}} V_N = d_N - \sharp\left\{\chi \in \widetilde{\mathrm{Cl}_K^*}\ ;\ \chi^2 \neq 1,\ \chi([\mathfrak{p}_2]) = \pm 1 \text{ or } \chi([\mathfrak{p}_3]) = \pm 1\right\},$$

as desired. $\square$

Further problems

- Describe the CM property and the connection to the ideal class group for various eta-quotients of weight one.
  $\longrightarrow$ There is a recent result by Berkovich and Patane on the expressions of some eta-quotients of weight one by theta series of positive definite binary quadratic forms.
- Can we study the Hecke action on $\eta(\tau)\eta(N\tau)$ without using the expression by binary quadratic forms?
  $\longrightarrow$ A study of the structure of ideal class groups of imaginary quadratic fields in terms of the Hecke action on modular forms.

Thank you very much for your attention.