Ramanujan's last prophecy: quantum modular forms

Ken Ono (Emory University)

"Death bed letter"

Dear Hardy,

*"I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call "***Mock"** ϑ**-functions***. Unlike the "False" ϑ-functions (partially studied by Rogers), they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."*

Ramanujan, January 12, 1920.

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"Theorem" (Zwegers, 2002)

Ramanujan's mock theta functions are holomorphic parts of weight 1/2 harmonic Maass forms.

Defining Maass forms

Notation. Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$.

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

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1. *For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ we have*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \ f(z).$$

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2. *We have that $\Delta_k f = 0$.*

Remark

Modular forms *are holomorphic functions which satisfy (1).*

HMFs have two parts ($q := e^{2\pi i z}$)

Fundamental Lemma

If $f \in H_{2-k}$ and $\Gamma(a, x)$ is the incomplete Γ-function, then

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n.$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

Holomorphic part f^+ **Nonholomorphic part** f^-

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$$f(z) = \underbrace{\sum_{n \gg -\infty} c_f^+(n) q^n}_{\textbf{Holomorphic part } f^+} + \underbrace{\sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y) q^n}_{\textbf{Nonholomorphic part } f^-}.$$

Remark

The mock theta functions are examples of f^+.

So many recent applications

- q-series and partitions
- Modular L-functions (e.g. BSD numbers)
- Eichler-Shimura Theory
- Probability models
- Generalized Borcherds Products
- *Moonshine* for affine Lie superalgebras and M_{24}
- Donaldson invariants
- Black holes
- ...

What did Ramanujan have in mind?

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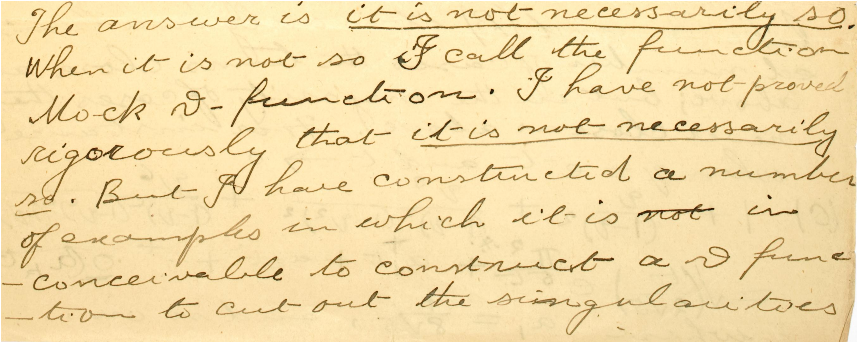
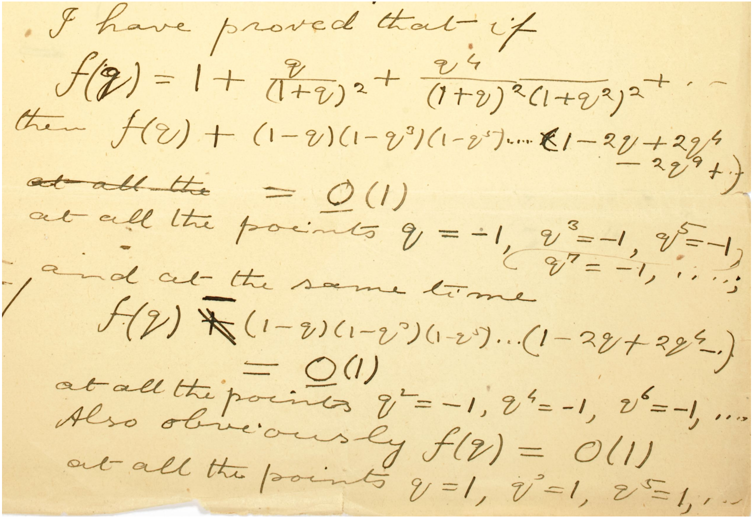
Question (Ramanujan)

Must Eulerian series with "similar asymptotics" be the sum of a modular form and a function which is $O(1)$ at all roots of unity?

Ramanujan's Speculation


The answer is it is not necessarily so.
When it is not so I call the function
Mock θ-function. I have not proved
rigorously that it is not necessarily
so. But I have constructed a number
of examples in which it is ~~not~~ in
-conceivable to construct a θ func
-tion to cut out the singularities

Ramanujan's "Example"



I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$

~~at all the~~ $= O(1)$

at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, $\ldots$;

and at the same time

$$f(q) \cancel{+}\!- (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots) = O(1)$$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, $\ldots$

Also obviously $f(q) = O(1)$

at all the points $q=1$, $q^3=1$, $q^5=1$, $\ldots$

Strange Conjecture

Conjecture (Ramanujan)

Consider the **mock theta** $f(q)$ *and the* **modular form** $b(q)$*:*

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots \times \left(1 - 2q + 2q^4 - 2q^9 + \cdots\right).$$

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If q approaches an even order $2k$ root of unity, then

$$f(q) - (-1)^k b(q) = O(1).$$

Numerics

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As $q \to -1$, we have

$$f(-0.994) \sim -1{\cdot}10^{31},\; f(-0.996) \sim -1{\cdot}10^{46},\; f(-0.998) \sim -6{\cdot}10^{90},$$

$$f(\text{-}0.998185) \sim -\text{Googol}$$

Numerics continued...

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Amazingly, Ramanujan's guess gives:

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\ldots$	$3.969\ldots$	$3.976\ldots$	$3.984\ldots$	$3.992\ldots$

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This suggests that

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

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q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$2 \cdot 10^6 - 4.6 \cdot 10^6 i$	$2 \cdot 10^8 - 4 \cdot 10^8 i$	$1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.05 + 3.85i$	$\sim 0.04 + 3.89i$	$\sim 0.03 + 3.92i$

As $q \to i$

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I. Ramanujan's Speculation (with M. Griffin and L. Rolen).

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II. $O(1)$ numbers and Quantum Modular Forms
(with A. Folsom and R. Rhoades)

Ramanujan's last words

"it is inconceivable to construct a ϑ-function to cut out the singularities of a mock theta function..."

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"...it has not been **proved** *that* **any** *of Ramanujan's mock theta functions really are mock theta functions according to his definition."* Bruce Berndt (2012)

Resolution

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Theorem (Griffin-O-Rolen (2012))

Ramanujan's examples satisfy his own definition.

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Theorem (Griffin-O-Rolen (2012))

Ramanujan's examples satisfy his own definition. More precisely, a mock theta function and a modular form never cut out exactly the same singularities.

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- A harmonic Maass form satisfies $F(z) = F^{-}(z) + F^{+}(z)$.

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- A harmonic Maass form satisfies $F(z) = F^{-}(z) + F^{+}(z)$.
- The function $F^{+}(z)$ is the holomorphic part.
- Ramanujan's alleged mock thetas are examples of $F^{+}(z)$.
- ...and $F^{-}(z)$ is a **period integral** of a **unary theta** function.

Big Fact

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Remark

Bruinier and Funke extended Petersson's scalar product to $\{\bullet,\bullet\} : M_k \times H_{2-k} \to \mathbb{C}$ *by*

$$\{g,F\}_k := (g,\xi(F))_k.$$

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Fundamental Fact

If $F(z) = F^-(z) + F^+(z) \in H_{2-k}$ *with* $F^-(z) \not\equiv 0$, *then* $F^+(z)$ *has* **infinitely** *many exponential singularities at roots of unity.*

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• The nonvanishing above and this combinatorial formula **implies** that $F^{+}(z)$ has some poles at some **cusp**.

• Exponential decay of $F^{-}(z)$ at cusps and **modularity** applied to $F(z)$ gives infinitely many exponential singularities for $F^{+}(z)$. $\square$

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•* Using **Kloostermania**, we find that a **positive** proportion of the coefficients of $M(z)$ and $\widehat{F}(z)$ agree and **are nonzero**, and so $\widehat{F}(z)$ has singularities.

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- The nonholomorphic part of $h(z)$ is $F^-(z) \not\equiv 0$.
- **Fundamental Fact** implies that $F^+(z) - g(z)$ has infinitely many exponential singularities at roots of unity.
- **Contradiction!**

Ramanujan's "Strange Conjecture"

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$$C(w;q) := \frac{(q;q)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty} \qquad \text{(Weierstrass MF)}$$

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Here we use that

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

General "Near Misses"

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Theorem (F-O-R)

If $\zeta_b = e^{\frac{2\pi i}{b}}$ *and* $1 \leq a < b$, *then for every suitable root of unity* ζ *there is an explicit integer* c *for which*

$$\lim_{q \to \zeta} \left(R(\zeta_b^a; q) - \zeta_{b^2}^c C(\zeta_b^a; q) \right) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) U(\zeta_b^a; \zeta).$$

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Remark

Ramanujan's "Strange Conjecture" is when $a = 1$ and $b = 2$.

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1. *What special properties do* **these** *mock ϑs enjoy?*

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$$\lim_{q\to\zeta}\left(\text{Mock }\vartheta - \epsilon_\zeta \text{MF}\right) = \text{Quantum MF}.$$

Two questions

1. *What special properties do* **these** *mock ϑs enjoy?*
2. *What is a quantum modular form?*

Upper and lower half-planes

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Example

For Ramanujan's $f(q)$, amazingly we have

$$f(q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{-n^2}}{(1+q^{-1})^2(1+q^{-2})^2 \cdots (1+q^{-n})^2}$$

$$= \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n^2} = 1 + q - q^2 + 2q^3 - 4q^4 + \ldots$$

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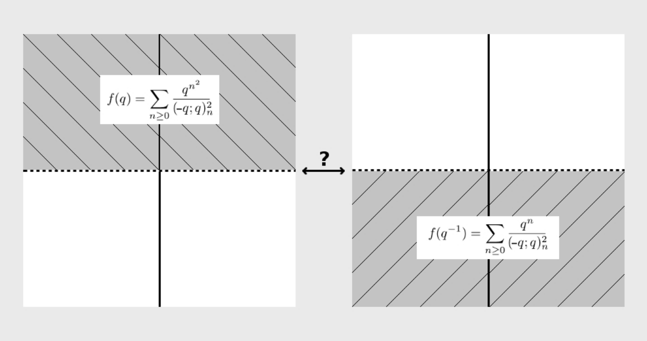
Remark

Under $z \leftrightarrow q = e^{2\pi iz}$, this means that $f(q)$ is defined on both $\mathbb{H}^{\pm}$.

We have the following....



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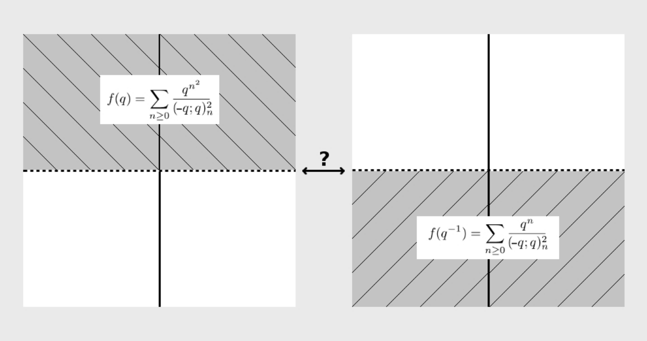

Remark

At rationals $z = h/2k$ these "meet" thanks to $U(-1;q)$.

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$$h_\gamma(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a "suitable" property of continuity or analyticity.

History of quantum modular forms

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Remark

Zagier defined them in his 2010 Clay Prize lecture at Harvard.

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Zagier offered a few examples related to:

- Dedekind sums.
- q-series defined by Andrews, Dyson, and Hickerson.
- Quadratic polynomials of fixed discriminant.
- Jones polynomials in knot theory.
- Kontsevich's strange function $F(q)$.

A new quantum modular form

"Theorem" (2012, Bryson-O-Pitman-R)

The function

$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(1; e^{2\pi i x})$$

is a weight $3/2$ *quantum modular form, which is defined on* $\mathbb{H} \cup \mathbb{R}$.

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Question

We have observed the phenomenon

$$\lim_{q \to \zeta} (\text{Mock } \vartheta\text{-function} - \epsilon_\zeta \text{MF}) = \text{QMF}.$$

A new quantum modular form

"Theorem" (2012, Bryson-O-Pitman-R)

The function

$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(1; e^{2\pi i x})$$

is a weight $3/2$ *quantum modular form, which is defined on* $\mathbb{H} \cup \mathbb{R}$.

Question

We have observed the phenomenon

$$\lim_{q \to \zeta} (\text{Mock } \vartheta\text{-function} - \epsilon_\zeta \text{MF}) = \text{QMF}.$$

How do **QMFs** *arise naturally from mock* ϑ*-functions?*

Rogers-Fine q-hypergeometric function

Definition (Rogers-Fine q-hypergeometric function)

$$F(\alpha, \beta, t; q) := \sum_{n=0}^{\infty} \frac{(\alpha q; q)_n t^n}{(\beta q; q)_n}.$$

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Lemma (EZ)

We have the "half" theta functions:

$$\frac{1}{1+w} \cdot F(wq^{-1}, -w, w; q) := \frac{1}{1+w} \cdot \sum_{n=0}^{\infty} \frac{(w; q)_n w^n}{(-wq; q)_n}$$

$$= \sum_{n=0}^{\infty} (-1)^n w^{2n} q^{n^2}.$$

Two families of specializations

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Definition

We define $G(a, b; z)$ and $H(a, b; z)$ by

$$G(a, b; z) := \frac{q^{\frac{a^2}{b^2}}}{1 - q^{\frac{a}{b}}} \cdot F\left(-q^{\frac{a}{b}-1}, q^{\frac{a}{b}}, -q^{\frac{a}{b}}; q\right),$$

$$H(a, b; z) := q^{\frac{1}{8}} \cdot F\left(\zeta_b^{-a} q^{-1}, \zeta_b^{-a}, \zeta_b^{-a} q; q^2\right).$$

q-identities

q-identities

Lemma

We have the following non-modular q-identities:

$$G(a,b;z) = \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{a}{b})^2},$$

$$H(a,b;z) = \sum_{n=0}^{\infty} \zeta_b^{-an} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}.$$

QMFs arising from Rogers-Fine

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Theorem (F-O-R)

For $a, b \in \mathbb{Z}^+$ with b even, we define a set of rationals $Q_{a,b}$.

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The following are true:

1. *The functions $G(a, b; z)$ and $H(a, b; z)$ converge for $z \in \mathbb{H}^+ \cup \mathbb{H}^-$.*
2. *For $x \in Q_{a,b} \cup \mathbb{H}^+$, we have that*

$$G(a, b; -x) + \frac{e^{-\frac{\pi i a}{b}}}{\sqrt{2ix}} \cdot H\left(a, b; \frac{1}{2x}\right)$$
$$= \text{“integral of a theta function”}.$$

In particular....

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Corollary (F-O-R)

Assuming the notation above, $G(a,b;x)$ and $H(a,b;x)$ are weight 1/2 **quantum modular forms** *on $Q_{a,b} \cup \mathbb{H}^{+}$.*

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$$G\left(a, b; \frac{-h}{k} + \frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r, c_G) \cdot \frac{(-t)^r}{r! \cdot b^{2r}},$$

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Remark

The L-functions $L(s, c_G)$ and $L(s, c_H)$ are explicit linear combinations of Hurwitz zeta-functions.

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4. ...**and** *Ramanujan's own mock ϑs make it happen* **:-)** *!*

Rogers-Fine and Quantum Modularity

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"Theorem"

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3. q-manipulations to relate these to $G(a, b; z)$ and $H(a, b; z)$.
4. Quantum modularity follows by lengthy direct calculations...

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Proof.

1. The RF gives "asymptotic" expansions at roots of unity.
2. Lemma of Lawrence and Zagier also gives asymptotics.

Ramanujan Hit Parade
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1. **Four of the top 5** *involve ranks, cranks, mock ϑs, and QMFs.*

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Amusing Remarks

1. **Four of the top 5** *involve ranks, cranks, mock ϑs, and QMFs.*
2. *The importance of each instrument was found* **independently**.
3. *We show they form a* **harmonious quantum orchestra.**