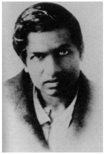
# Special functions and numbers related to mock modular forms

Ken Ono (Emory University)

## "Ramanujan's Death bed letter"

*Dear Hardy,*

*"I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call "Mock" $\vartheta$-functions. Unlike the "False" $\vartheta$-functions (partially studied by Rogers), they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."*



January 12, 1920.

# The first example

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

# Strange Conjecture

## Conjecture (Ramanujan)

*Consider the* **mock theta** $f(q)$ *and the* **modular form** $b(q)$*:*

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2},$$

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots \times \left(1 - 2q + 2q^4 - 2q^9 + \cdots\right).$$

# Strange Conjecture

## Conjecture (Ramanujan)

*Consider the* **mock theta** $f(q)$ *and the* **modular form** $b(q)$*:*

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2},$$

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots \times \left(1 - 2q + 2q^4 - 2q^9 + \cdots\right).$$

*If* $q$ *approaches an even order* $2k$ *root of unity, then*

$$f(q) - (-1)^k b(q) = O(1).$$

# Numerics

## Numerics

As $q \to -1$, we have

$$f(-0.994) \sim -1{\cdot}10^{31},\ f(-0.996) \sim -1{\cdot}10^{46},\ f(-0.998) \sim -6{\cdot}10^{90},$$

# Numerics

As $q \to -1$, we have

$$f(-0.994) \sim -1{\cdot}10^{31},\ f(-0.996) \sim -1{\cdot}10^{46},\ f(-0.998) \sim -6{\cdot}10^{90},$$

$$f(-0.998185) \sim -\text{Googol}$$

# Numerics continued...

# Numerics continued...

Amazingly, Ramanujan's guess gives:

| $q$ | $-0.990$ | $-0.992$ | $-0.994$ | $-0.996$ | $-0.998$ |
|---|---|---|---|---|---|
| $f(q) + b(q)$ | $3.961\ldots$ | $3.969\ldots$ | $3.976\ldots$ | $3.984\ldots$ | $3.992\ldots$ |

.

## Numerics continued...

Amazingly, Ramanujan's guess gives:

| $q$ | $-0.990$ | $-0.992$ | $-0.994$ | $-0.996$ | $-0.998$ |
|---|---|---|---|---|---|
| $f(q)+b(q)$ | $3.961\ldots$ | $3.969\ldots$ | $3.976\ldots$ | $3.984\ldots$ | $3.992\ldots$ |

.

This suggests that

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

## As $q \to i$

# As $q \to i$

| $q$ | $0.992i$ | $0.994i$ | $0.996i$ |
|---|---|---|---|
| $f(q)$ | $2 \cdot 10^6 - 4.6 \cdot 10^6 i$ | $2 \cdot 10^8 - 4 \cdot 10^8 i$ | $1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$ |
| $f(q) - b(q)$ | $\sim 0.05 + 3.85i$ | $\sim 0.04 + 3.89i$ | $\sim 0.03 + 3.92i$ |

# As $q \to i$

| $q$ | $0.992i$ | $0.994i$ | $0.996i$ |
|---|---|---|---|
| $f(q)$ | $2 \cdot 10^6 - 4.6 \cdot 10^6 i$ | $2 \cdot 10^8 - 4 \cdot 10^8 i$ | $1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$ |
| $f(q) - b(q)$ | $\sim 0.05 + 3.85i$ | $\sim 0.04 + 3.89i$ | $\sim 0.03 + 3.92i$ |

This suggests that

$$\lim_{q \to i} (f(q) - b(q)) = 4i.$$

# Is Ramanujan's correct?

# Is Ramanujan's correct?

**Questions**

*What are the numbers such as* $4$ *and* $4i$ *in general?*

# Is Ramanujan's correct?

**Questions**

*What are the numbers such as* $4$ *and* $4i$ *in general?*
*What is going on?*

# What are Ramanujan's mock thetas?

# What are Ramanujan's mock thetas?

**"Theorem" (Zwegers, 2002)**

*Ramanujan's mock theta functions are holomorphic parts of weight 1/2 harmonic Maass forms.*

# What are Ramanujan's mock thetas?

**"Theorem" (Zwegers, 2002)**

*Ramanujan's mock theta functions are holomorphic parts of weight 1/2 harmonic Maass forms.*

**Remark**

*More on this shortly...*

# Lots of recent applications...

- $q$-series and partitions
- Modular $L$-functions (e.g. BSD numbers)
- Eichler-Shimura Theory
- Probability models
- Generalized Borcherds Products
- *Moonshine* for affine Lie superalgebras and $M_{24}$
- Donaldson invariants
- Black holes

# Two topics this lecture....

## Two topics this lecture....

**I.** The Strange Conjecture (with A. Folsom and R. Rhoades).

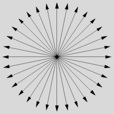
## Two topics this lecture....

**I.** The Strange Conjecture (with A. Folsom and R. Rhoades).

**II.** Elliptic curves $E/\mathbb{Q}$ (with C. Alfes, M. Griffin, and L. Rolen).

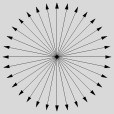
# The last letter

- Asymptotics, near roots of unity, of "Eulerian" modular forms.



# The last letter

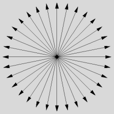
- Asymptotics, near roots of unity, of "Eulerian" modular forms.



- Raises **one** question and conjectures the answer.

# The last letter

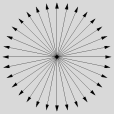
- Asymptotics, near roots of unity, of "Eulerian" modular forms.



- Raises **one** question and conjectures the answer.
- "Strange Conjecture".

# The last letter

- Asymptotics, near roots of unity, of "Eulerian" modular forms.



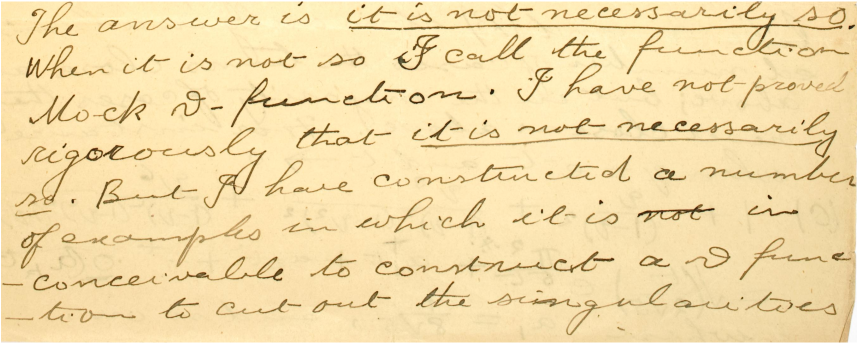
- Raises **one** question and conjectures the answer.
- "Strange Conjecture".
- Concludes with a list of his mock theta functions.

# Ramanujan's question

**Question (Ramanujan)**

*Must power series with "modular-like" asymptotics be the sum of a modular form and a function which is $O(1)$ at all roots of unity?*

## Ramanujan's Answer



The answer is <u>it is not necessarily so</u>. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that <u>it is not necessarily so</u>. But I have constructed a number of examples in which it is ~~not~~ in-conceivable to construct a ϑ function to cut out the singularities

# Ramanujan's last words

*"it is inconceivable to construct a $\vartheta$-function to cut out the singularities of a mock theta function..."*

Srinivasa Ramanujan

## Ramanujan's last words

*"it is inconceivable to construct a $\vartheta$-function to cut out the singularities of a mock theta function..."*

Srinivasa Ramanujan

**Theorem (Griffin-O-Rolen (2012))**

*A mock theta function and a modular form* **never** *cut out exactly the same singularities.*

# Strange Conjecture is a "Near Miss"

Define the **mock theta** $f(q)$ and the **modular form** $b(q)$ by

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots \times \left(1 - 2q + 2q^4 - 2q^9 + \cdots\right).$$

# Strange Conjecture is a "Near Miss"

Define the **mock theta** $f(q)$ and the **modular form** $b(q)$ by

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2},$$

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots \times \left(1 - 2q + 2q^4 - 2q^9 + \cdots\right).$$

**Conjecture (Ramanujan)**

*If $q$ approaches an even order $2k$ root of unity, then*

$$f(q) - (-1)^k b(q) = O(1).$$

# **Special values** of **special functions**.

# Special values of special functions.

**Theorem (Folsom-O-Rhoades)**

*If $\zeta$ is an even $2k$ order root of unity, then*

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

# Special Functions

# Special Functions

$$R(w;q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n} \qquad \text{(Dyson's Mock $\vartheta$-function)}$$

$$C(w;q) := \frac{(q;q)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty} \qquad \text{(Weierstrass MF)}$$

$$U(w;q) := \sum_{n=0}^{\infty} (wq;q)_n(w^{-1}q;q)_n q^{n+1} \qquad \text{(Quantum mod. form)}$$

# Special Functions

$$R(w;q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n} \qquad \text{(Dyson's Mock } \vartheta\text{-function)}$$

$$C(w;q) := \frac{(q;q)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty} \qquad \text{(Weierstrass MF)}$$

$$U(w;q) := \sum_{n=0}^{\infty} (wq;q)_n(w^{-1}q;q)_n q^{n+1} \qquad \text{(Quantum mod. form)}$$

Here we use that

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

# General Theorem

**Theorem (F-O-R)**

*If* $\zeta_b = e^{\frac{2\pi i}{b}}$ *and* $1 \leq a < b$, *then for every suitable root of unity* $\zeta$ *there is an explicit integer* $c$ *for which*

$$\lim_{q \to \zeta} \left( R(\zeta_b^a; q) - \zeta_{b^2}^c C(\zeta_b^a; q) \right) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) U(\zeta_b^a; \zeta).$$

# General Theorem

**Theorem (F-O-R)**

*If $\zeta_b = e^{\frac{2\pi i}{b}}$ and $1 \leq a < b$, then for every suitable root of unity $\zeta$ there is an explicit integer $c$ for which*

$$\lim_{q \to \zeta} \left( R(\zeta_b^a; q) - \zeta_{b^2}^c C(\zeta_b^a; q) \right) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) U(\zeta_b^a; \zeta).$$

**Remark**

*The first theorem is when $a = 1$ and $b = 2$ using the identities*

$$R(-1; q) = f(q) \quad \text{and} \quad C(-1; q) = b(q).$$

# What's going on?

## What's going on?

Loosely speaking, these theorems say that

$$\lim_{q \to \zeta} \left(\text{Mock } \vartheta - \epsilon_{\zeta} \text{MF}\right) = \text{Quantum MF}.$$

## What's going on?

Loosely speaking, these theorems say that

$$\lim_{q \to \zeta} \left(\text{Mock } \vartheta - \epsilon_{\zeta}\text{MF}\right) = \text{Quantum MF}.$$

**Question**

*What is a quantum modular form?*

# Upper and lower half-planes

# Upper and lower half-planes

## Example

For Ramanujan's $f(q)$, amazingly we have

$$f(q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{-n^2}}{(1+q^{-1})^2(1+q^{-2})^2 \cdots (1+q^{-n})^2}$$
$$= \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n^2} = 1 + q - q^2 + 2q^3 - 4q^4 + \ldots$$

# Upper and lower half-planes

## Example

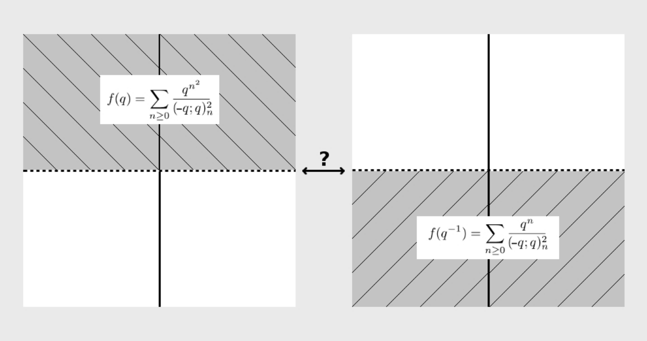
For Ramanujan's $f(q)$, amazingly we have

$$f(q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{-n^2}}{(1+q^{-1})^2(1+q^{-2})^2\cdots(1+q^{-n})^2}$$
$$= \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n^2} = 1 + q - q^2 + 2q^3 - 4q^4 + \dots$$

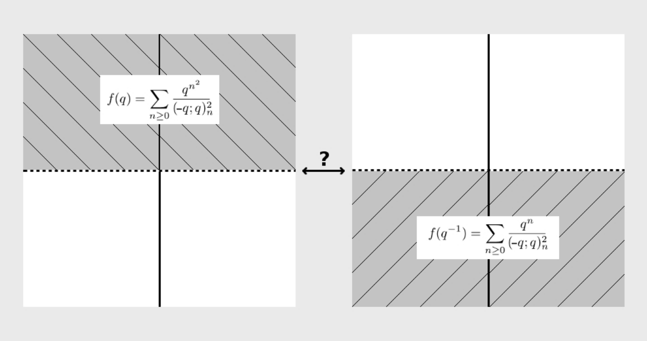
## Remark

*Under $z \leftrightarrow q = e^{2\pi i z}$, this means that $f(q)$ is defined on both $\mathbb{H}^{\pm}$.*

# We have the following....



# We have the following....




## Remark

*At rationals $z = h/2k$ these "meet" thanks to $U(-1;q)$.*

# Quantum modular forms

# Quantum modular forms

**Definition (Zagier)**

A **weight** $k$ **quantum modular form** is a complex-valued function $f$ on $\mathbb{Q} \setminus S$ for some set $S$, such that

# Quantum modular forms

## Definition (Zagier)

A **weight** $k$ **quantum modular form** is a complex-valued function $f$ on $\mathbb{Q} \setminus S$ for some set $S$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the function

# Quantum modular forms

### Definition (Zagier)

A **weight** $k$ **quantum modular form** is a complex-valued function $f$ on $\mathbb{Q} \setminus S$ for some set $S$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the function

$$h_\gamma(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a "suitable" property of continuity or analyticity.

# History of quantum modular forms

# History of quantum modular forms

**Remark**

*Zagier defined them at the 2010 Clay Conference at Harvard.*

# History of quantum modular forms

**Remark**

*Zagier defined them at the 2010 Clay Conference at Harvard.*

Zagier offered a few examples related to:

- Dedekind sums.
- $q$-series defined by Andrews, Dyson, and Hickerson.
- Quadratic polynomials of fixed discriminant.
- Jones polynomials in knot theory.
- Kontsevich's strange function $F(q)$.

# A new quantum modular form

**"Theorem" (2012, Bryson-O-Pitman-R)**

*The function*

$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(1; e^{2\pi i x})$$

*is a weight* $3/2$ *quantum modular form, which is defined on* $\mathbb{H} \cup \mathbb{R}$.

# A new quantum modular form

**"Theorem" (2012, Bryson-O-Pitman-R)**

*The function*

$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(1; e^{2\pi i x})$$

*is a weight* $3/2$ *quantum modular form, which is defined on* $\mathbb{H} \cup \mathbb{R}$.

**Remark**

*We have observed the phenomenon*

$$\lim_{q \to \zeta} \left(\text{Mock } \vartheta\text{-function} - \epsilon_\zeta \text{MF}\right) = \text{QMF}.$$

# A new quantum modular form

**"Theorem" (2012, Bryson-O-Pitman-R)**

*The function*

$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(1; e^{2\pi i x})$$

*is a weight* $3/2$ *quantum modular form, which is defined on* $\mathbb{H} \cup \mathbb{R}$.

**Remark**

*We have observed the phenomenon*

$$\lim_{q \to \zeta} (\text{Mock } \vartheta\text{-function} - \epsilon_\zeta \text{MF}) = \text{QMF}.$$

*The obstruction to modularity for QMFs are "period integrals" of modular forms.*

# Mock thetas are harmonic Maass forms

# Mock thetas are harmonic Maass forms

**Notation.** Throughout, let

- $\tau = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$.

# Mock thetas are harmonic Maass forms

**Notation.** Throughout, let

- $\tau = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$.

**Hyperbolic Laplacian.**

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

# Weak Maass forms

“Definition”

*A harmonic Maass form is any smooth function $f$ on $\mathbb{H}$ satisfying:*

# Weak Maass forms

## "Definition"

*A harmonic Maass form is any smooth function $f$ on $\mathbb{H}$ satisfying:*

1. *For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, we have*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \ f(\tau).$$

# Weak Maass forms

## "Definition"

*A harmonic Maass form is any smooth function $f$ on $\mathbb{H}$ satisfying:*

1. *For all* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, *we have*
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \ f(\tau).$$
2. *We have that* $\Delta_k f = 0$.

# Harmonic Maass forms have two parts ($q := e^{2\pi i\tau}$)

## Fundamental Lemma

*If $f \in H_{2-k}$ and $\Gamma(a,x)$ is the incomplete $\Gamma$-function, then*

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n.$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

**Holomorphic part** $f^+$ **Nonholomorphic part** $f^-$

# Harmonic Maass forms have two parts ($q := e^{2\pi i\tau}$)

## Fundamental Lemma

*If $f \in H_{2-k}$ and $\Gamma(a,x)$ is the incomplete $\Gamma$-function, then*

$$f(\tau) = \underbrace{\sum_{n \gg -\infty} c_f^+(n) q^n}_{\substack{\updownarrow \\ \textbf{Holomorphic part } f^+}} + \underbrace{\sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n}_{\substack{\updownarrow \\ \textbf{Nonholomorphic part } f^-}}.$$

## Remark

*Ramanujan's mock theta functions are examples of $f^+$.*

# Relation with classical modular forms

## Fundamental Lemma

*If* $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{\tau}}}$, *then we have a* **surjective** *map*

$$\xi_{2-k} : H_{2-k} \longrightarrow S_k,$$

# Relation with classical modular forms

## Fundamental Lemma

*If* $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{\tau}}}$, *then we have a* **surjective** *map*

$$\xi_{2-k} : H_{2-k} \longrightarrow S_k,$$

*Moreover, it has* **infinite** *kernel and satisfies*

$$\xi_{2-k}(f) = \xi_{2-k}(f^-).$$

# Relation with classical modular forms

## Fundamental Lemma

*If* $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{\tau}}}$*, then we have a* **surjective** *map*

$$\xi_{2-k} : H_{2-k} \longrightarrow S_k,$$

*Moreover, it has* **infinite** *kernel and satisfies*

$$\xi_{2-k}(f) = \xi_{2-k}(f^-).$$

## Question

*Are there* **canonical** *preimages for cusp forms of elliptic curves?*

# Relation with classical modular forms

## Fundamental Lemma

*If* $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{\tau}}}$, *then we have a* **surjective** *map*

$$\xi_{2-k} : H_{2-k} \longrightarrow S_k,$$

*Moreover, it has* **infinite** *kernel and satisfies*

$$\xi_{2-k}(f) = \xi_{2-k}(f^{-}).$$

## Question

*Are there* **canonical** *preimages for cusp forms of elliptic curves?*
*If so, what do the* $f^{+}$ **tell us***?*

# Weierstrass

# Weierstrass

**Notation.** Let $E$ be an elliptic curve.

- $E \cong \mathbb{C}/\Lambda_E$

# Weierstrass

**Notation.** Let $E$ be an elliptic curve.

- $E \cong \mathbb{C}/\Lambda_E$
- $E \ : \quad y^2 = 4x^3 - 60G_4(\Lambda_E)x - 140G_6(\Lambda_E).$

# Weierstrass

**Notation.** Let $E$ be an elliptic curve.

- $E \cong \mathbb{C}/\Lambda_E$
- $E \ : \quad y^2 = 4x^3 - 60G_4(\Lambda_E)x - 140G_6(\Lambda_E).$
- In terms of the Weierstrass $\wp$-function, we have

$$z \to P_z = (\wp(\Lambda_E; z), \wp'(\Lambda_E; z)).$$

## Weierstrass zeta-function

# Weierstrass zeta-function

**Definition**

The **Weierstrass zeta-function** for $E$ is

$$\zeta(\Lambda_E; z) := \frac{1}{z} + \sum_{w \in \Lambda_E \setminus \{0\}} \left( \frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right).$$

# Weierstrass zeta-function

**Definition**

The **Weierstrass zeta-function** for $E$ is

$$\zeta(\Lambda_E; z) := \frac{1}{z} + \sum_{w \in \Lambda_E \setminus \{0\}} \left( \frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right).$$

**Remark**

*It is easy to compute because*

$$\zeta(\Lambda_E; z) = \frac{1}{z} - \sum_{k=1}^{\infty} G_{2k+2}(\Lambda_E) z^{2k+1}.$$

# Elliptic curves and $\zeta(\Lambda_E; z)$

# Elliptic curves and $\zeta(\Lambda_E; z)$

## Remarks

- *We have the well-known "addition law"*

$$\zeta(\Lambda_E; z_1+z_2) = \zeta(\Lambda_E; z_1)+\zeta(\Lambda_E; z_2)+\frac{1}{2}\,\frac{\wp'(\Lambda_E; z_1) - \wp'(\Lambda_E; z_2)}{\wp(\Lambda_E; z_1) - \wp(\Lambda_E; z_2)}.$$

# Elliptic curves and $\zeta(\Lambda_E; z)$

## Remarks

- *We have the well-known "addition law"*

$$\zeta(\Lambda_E; z_1+z_2) = \zeta(\Lambda_E; z_1)+\zeta(\Lambda_E; z_2)+\frac{1}{2}\frac{\wp'(\Lambda_E; z_1) - \wp'(\Lambda_E; z_2)}{\wp(\Lambda_E; z_1) - \wp(\Lambda_E; z_2)}.$$

- *For $E$ with CM, Birch and Swinnerton-Dyer famously proved*

$$L(E,1) = \text{"finite sum of special values of } \zeta(\Lambda_E, s)\text{"}.$$

# Elliptic curves and $\zeta(\Lambda_E; z)$

## Remarks

- *We have the well-known "addition law"*

$$\zeta(\Lambda_E; z_1+z_2) = \zeta(\Lambda_E; z_1)+\zeta(\Lambda_E; z_2)+\frac{1}{2}\,\frac{\wp'(\Lambda_E; z_1) - \wp'(\Lambda_E; z_2)}{\wp(\Lambda_E; z_1) - \wp(\Lambda_E; z_2)}.$$

- *For $E$ with CM, Birch and Swinnerton-Dyer famously proved*

$$L(E,1) = \text{“finite sum of special values of } \zeta(\Lambda_E, s)\text{”}.$$

## Question

*Does $\zeta(\Lambda_E; z)$ reveal arithmetic information for all $E/\mathbb{Q}$?*

# Elliptic curves $E/\mathbb{Q}$

# Elliptic curves $E/\mathbb{Q}$

- There is a $F_E(\tau) = \sum a_E(n) q^n \in S_2(N_E)$ with

$$L(F_E, s) = L(E, s).$$

# Elliptic curves $E/\mathbb{Q}$

- There is a $F_E(\tau) = \sum a_E(n)q^n \in S_2(N_E)$ with

$$L(F_E, s) = L(E, s).$$

- We have the **modular parameterization**

$$\phi_E : X_0(N_E) \to \mathbb{C}/\Lambda_E \cong E.$$

# Elliptic curves $E/\mathbb{Q}$

- There is a $F_E(\tau) = \sum a_E(n)q^n \in S_2(N_E)$ with

$$L(F_E, s) = L(E, s).$$

- We have the **modular parameterization**

$$\phi_E : X_0(N_E) \to \mathbb{C}/\Lambda_E \cong E.$$

- This theory makes use of the **Eichler integral** of $F_E$ is

$$\mathcal{E}_E(\tau) = -2\pi i \int_\tau^{i\infty} F_E(z)dz = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} \cdot q^n.$$

# Inspired by a recent paper of Guerzhoy...

# Inspired by a recent paper of Guerzhoy...

**Theorem (Alfes, Griffin, O, Rolen)**

*We define*

# Inspired by a recent paper of Guerzhoy...

**Theorem (Alfes, Griffin, O, Rolen)**

*We define*

$$\mathfrak{Z}_E(z) := \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\deg(\phi_E)}{4\pi||F_E||^2} \cdot \overline{z},$$

# Inspired by a recent paper of Guerzhoy...

**Theorem (Alfes, Griffin, O, Rolen)**

*We define*

$$\mathfrak{Z}_E(z) := \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\deg(\phi_E)}{4\pi||F_E||^2} \cdot \overline{z},$$

*where*

$$S(\Lambda_E) := \lim_{s \to 0} \sum_{w \in \Lambda_E \setminus \{0\}} \frac{1}{w^2 |w|^{2s}}.$$

# Inspired by a recent paper of Guerzhoy...

**Theorem (Alfes, Griffin, O, Rolen)**

*We define*

$$\mathfrak{Z}_E(z) := \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\deg(\phi_E)}{4\pi||F_E||^2} \cdot \overline{z},$$

*where*

$$S(\Lambda_E) := \lim_{s\to 0} \sum_{w\in\Lambda_E\setminus\{0\}} \frac{1}{w^2|w|^{2s}}.$$

*Then* $\widehat{\mathfrak{Z}}_E(\tau) := \mathfrak{Z}_E(\mathcal{E}_E(\tau))$ *is a* **harmonic Maass function**.

# Inspired by a recent paper of Guerzhoy...

**Theorem (Alfes, Griffin, O, Rolen)**

*We define*

$$\mathfrak{Z}_E(z) := \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\deg(\phi_E)}{4\pi||F_E||^2} \cdot \overline{z},$$

*where*

$$S(\Lambda_E) := \lim_{s\to 0} \sum_{w\in\Lambda_E\setminus\{0\}} \frac{1}{w^2|w|^{2s}}.$$

*Then* $\widehat{\mathfrak{Z}}_E(\tau) := \mathfrak{Z}_E(\mathcal{E}_E(\tau))$ *is a* **harmonic Maass function**.

**Remark**

*There is a canonical modular function* $M_E(\tau)$ *for which* $\widehat{\mathfrak{Z}}_E^+(\tau) - M_E(\tau)$ *is holomorphic on* $\mathbb{H}$.

## Remarks

1. We refer to $\widehat{\mathfrak{Z}}_E^+(\tau)$ as the **Weierstrass mock modular form**.

# Remarks

1. We refer to $\widehat{\mathfrak{Z}}_E^+(\tau)$ as the **Weierstrass mock modular form**.
2. The coefficients of $\widehat{\mathfrak{Z}}_E^+(\tau)$ are $\mathbb{Q}$-rational if $E$ has CM,

# Remarks

1. We refer to $\widehat{\mathfrak{Z}}_E^+(\tau)$ as the **Weierstrass mock modular form**.
2. The coefficients of $\widehat{\mathfrak{Z}}_E^+(\tau)$ are $\mathbb{Q}$-rational if $E$ has CM, and are presumably transcendental otherwise.

# Remarks

1. We refer to $\widehat{\mathfrak{Z}}_E^+(\tau)$ as the **Weierstrass mock modular form**.
2. The coefficients of $\widehat{\mathfrak{Z}}_E^+(\tau)$ are $\mathbb{Q}$-rational if $E$ has CM, and are presumably transcendental otherwise.
3. Divisors of $\deg(\phi_E)$ are **congruence primes** for $F_E(\tau)$.

# Remarks

1. We refer to $\widehat{\mathfrak{Z}}_E^+(\tau)$ as the **Weierstrass mock modular form**.
2. The coefficients of $\widehat{\mathfrak{Z}}_E^+(\tau)$ are $\mathbb{Q}$-rational if $E$ has CM, and are presumably transcendental otherwise.
3. Divisors of $\deg(\phi_E)$ are **congruence primes** for $F_E(\tau)$.
4. The expansion at the cusp 0 encodes $L(F_E, 1)$, and gives information about $\mathbb{Q}$-rational torsion.

## The Congruent Number Problem (CNP)

**Problem (Open)**

*Determine the integers which are areas of rational right triangles.*

# The Congruent Number Problem (CNP)

**Problem (Open)**

*Determine the integers which are areas of rational right triangles.*

**Example**

1. The number 6 is congruent since it is the area of $(3, 4, 5)$.

# The Congruent Number Problem (CNP)

## Problem (Open)

*Determine the integers which are areas of rational right triangles.*

## Example

1. The number 6 is congruent since it is the area of $(3, 4, 5)$.
2. The number 3 is not congruent. **Prove it.**

# The Congruent Number Problem (CNP)

## Problem (Open)

*Determine the integers which are areas of rational right triangles.*

## Example

1. The number 6 is congruent since it is the area of $(3,4,5)$.
2. The number 3 is not congruent. **Prove it.**
3. The number 157 is congruent, since it is the area of

$$\left(\frac{411340519227716149383203}{21666555693714761309610}, \frac{680\cdots540}{411\cdots203}, \frac{224\cdots041}{891\cdots830}\right).$$

# A Classical Diophantine Criterion

**Theorem (Easy)**

*An integer $D$ is congruent if and only if the elliptic curve*

$$E_D: \quad Dy^2 = x^3 - x$$

*has positive rank.*

# A Classical Diophantine Criterion

**Theorem (Easy)**

*An integer $D$ is congruent if and only if the elliptic curve*

$$E_D : \quad Dy^2 = x^3 - x$$

*has positive rank.*

**Remark**

*Tunnell (1983) gave a conditional solution to the CNP using work of Coates and Wiles on BSD for CM elliptic curves.*

# Ranks of quadratic twists

### Definition

Let $E/\mathbb{Q}$ be the elliptic curve

$$E: \; y^2 = x^3 + Ax + B.$$

# Ranks of quadratic twists

## Definition

Let $E/\mathbb{Q}$ be the elliptic curve

$$E: \ y^2 = x^3 + Ax + B.$$

If $\Delta$ is a fund. disc., then the $\Delta$-quadratic twist of $E$ is

$$E(\Delta): \quad \Delta y^2 = x^3 + Ax + B.$$

# Birch and Swinnerton-Dyer Conjecture

## Conjecture

*If $E/\mathbb{Q}$ is an elliptic curve and $L(E,s)$ is its L-function, then*

$$\mathrm{ord}_{s=1}(L(E,s)) \;=\; \text{rank of } E(\mathbb{Q}).$$

# Birch and Swinnerton-Dyer Conjecture

**Conjecture**

*If $E/\mathbb{Q}$ is an elliptic curve and $L(E,s)$ is its L-function, then*

$$\mathrm{ord}_{s=1}(L(E,s)) \ = \ \text{rank of } E(\mathbb{Q}).$$

**A good question.** How does one compute $\mathrm{ord}_{s=1}(L(E,s))$?

# Kolyvagin's Theorem

**Theorem (Kolyvagin)**

*If* $\mathrm{ord}_{s=1}(L(E,s)) \leq 1$, *then*

$$\mathrm{ord}_{s=1}(L(E,s)) = \text{ rank of } E.$$

# Kolyvagin's Theorem

**Theorem (Kolyvagin)**

*If* $\mathrm{ord}_{s=1}(L(E,s)) \leq 1$, *then*

$$\mathrm{ord}_{s=1}(L(E,s)) = \text{ rank of } E.$$

**Question**

*How does one compute* $L(E,1)$ *and* $L'(E,1)$*?*

# Formulas for $L(E(\Delta), 1)$

**Theorem (Shimura-Kohnen/Zagier-Waldspurger)**

*There is a weight 3/2 modular form*

$$g(z) = \sum_{n=1}^{\infty} b_E(n) q^n$$

*such that if* $\epsilon(E_D) = 1$, *then*

$$L(E(\Delta), 1) = \alpha_E(\Delta) \cdot b_E(|\Delta|)^2.$$

# The Gross-Zagier Theorem

**Question**

*What about derivatives?*

# The Gross-Zagier Theorem

### Question

*What about derivatives?*

### Theorem (Gross and Zagier)

*Under suitable conditions, $L'(E(\Delta), 1)$ is given in terms of heights of Heegner points.*

# Natural Question

**Question**

*Find an extension of the Kohnen-Waldspurger theorem giving both*

$$L(E(\Delta),1) \quad \textit{and} \quad L'(E(\Delta),1).$$

# A new theta lift

# A new theta lift

Using recent work of Hövel, we define a "theta lift"

$$\mathcal{I} : H_0 \longrightarrow H_{\frac{1}{2}}.$$

# A new theta lift

Using recent work of Hövel, we define a "theta lift"

$$\mathcal{I} : H_0 \longrightarrow H_{\frac{1}{2}}.$$

**"Theorem" (Alfes, Griffin, O, Rolen)**

*We have the Hecke equivariant commutative diagram:*

$$\begin{array}{ccc} H_0(N_E) & \longrightarrow_{\xi_0} & S_2(N_E) \\ \downarrow_{\mathcal{I}} & & \downarrow_{Shintani} \\ H_{\frac{1}{2}}(4N_E) & \longrightarrow_{\xi_{\frac{1}{2}}} & S_{\frac{3}{2}}(4N_E). \end{array}$$

# $L$-values and derivatives

# $L$-values and derivatives

**Theorem (Alfes, Griffin, O, Rolen)**

*Define let $f_E(\tau) := \mathcal{I}(\widehat{\mathfrak{Z}}_E(\tau))$, and let*

$$f_E(\tau) = \sum_{n \gg -\infty} c_E^+(n) q^n + \sum_{n<0} c_E^-(n) \Gamma(1/2, 4\pi|n|y) q^{-n}.$$

*If $E/\mathbb{Q}$ has prime conductor $p$ and $\epsilon(E) = -1$, then we have:*

# $L$-values and derivatives

**Theorem (Alfes, Griffin, O, Rolen)**

*Define let* $f_E(\tau) := \mathcal{I}(\widehat{\mathfrak{Z}}_E(\tau))$, *and let*

$$f_E(\tau) = \sum_{n \gg -\infty} c_E^+(n) q^n + \sum_{n<0} c_E^-(n) \Gamma(1/2, 4\pi|n|y) q^{-n}.$$

*If* $E/\mathbb{Q}$ *has prime conductor* $p$ *and* $\epsilon(E) = -1$, *then we have:*

1. *If* $\Delta < 0$ *and* $\left(\frac{\Delta}{p}\right) = 1$, *then*

$$L(E(\Delta), 1) = \widetilde{\alpha_E(\Delta)} \cdot c_E^-(\Delta)^2.$$

# $L$-values and derivatives

**Theorem (Alfes, Griffin, O, Rolen)**

*Define let* $f_E(\tau) := \mathcal{I}(\widehat{\mathfrak{Z}}_E(\tau))$, *and let*

$$f_E(\tau) = \sum_{n \gg -\infty} c_E^+(n) q^n + \sum_{n<0} c_E^-(n)\Gamma(1/2, 4\pi|n|y)q^{-n}.$$

*If* $E/\mathbb{Q}$ *has prime conductor* $p$ *and* $\epsilon(E) = -1$, *then we have:*

1. *If* $\Delta < 0$ *and* $\left(\frac{\Delta}{p}\right) = 1$, *then*

$$L(E(\Delta), 1) = \widetilde{\alpha_E(\Delta)} \cdot c_E^-(\Delta)^2.$$

2. *If* $\Delta > 0$ *and* $\left(\frac{\Delta}{p}\right) = 1$, *then*

$$L'(E(\Delta), 1) = 0 \iff c_E^+(\Delta) \text{ is algebraic.}$$

# Example for $E: \quad y^2 = x^3 + 10x^2 - 20x + 8.$

| $\Delta$ | $c^+(-\Delta)$ | $L'(E_\Delta, 1)$ | $\mathrm{rk}(E_\Delta(\mathbb{Q}))$ |
|---|---|---|---|
| 1 | $-2.817617849\ldots$ | $0.30599977\ldots$ | 1 |
| 12 | $-4.88527238\ldots$ | $4.2986147986\ldots$ | 1 |
| 21 | $-1.727392572\ldots$ | $9.002386800\ldots$ | 1 |
| 28 | $6.781939953\cdots$ | $4.327260249\ldots$ | 1 |
| 33 | $5.663023201\ldots$ | $3.6219567911\ldots$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 1489 | 9 | 0 | 3 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 4393 | $\frac{33}{5}$ | 0 | 3 |

# Remarks

## Remarks

1. Have a general thm for all $E$ (in terms of vector-valued forms).

# Remarks

1. Have a general thm for all $E$ (in terms of vector-valued forms).
2. Explicit formulas for $c_E^+(\Delta)$ as CM traces of $\widehat{\mathfrak{Z}}_E(\tau)$.

# Remarks

1. Have a general thm for all $E$ (in terms of vector-valued forms).
2. Explicit formulas for $c_E^+(\Delta)$ as CM traces of $\widehat{\mathfrak{Z}}_E(\tau)$.
3. By Bruinier the $c_E^+(\Delta)$ are "periods of diff's of the 3rd kind".

# Remarks

1. Have a general thm for all $E$ (in terms of vector-valued forms).
2. Explicit formulas for $c_E^+(\Delta)$ as CM traces of $\widehat{\mathfrak{Z}}_E(\tau)$.
3. By Bruinier the $c_E^+(\Delta)$ are "periods of diff's of the 3rd kind".
4. The commutative diagram requires careful normalizations.

# Remarks

1. Have a general thm for all $E$ (in terms of vector-valued forms).
2. Explicit formulas for $c_E^+(\Delta)$ as CM traces of $\widehat{\mathfrak{Z}}_E(\tau)$.
3. By Bruinier the $c_E^+(\Delta)$ are "periods of diff's of the 3rd kind".
4. The commutative diagram requires careful normalizations.
5. Proof of the last theorem is a "canonical choice" of a weight 1/2 harmonic Maass form in previous work of Bruinier and O.

# I. Ramanujan's Strange Conjecture...

# I. Ramanujan's Strange Conjecture...

**Theorem (Folsom-O-Rhoades)**

*If $\zeta$ is an even $2k$ order root of unity, then*

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

# I. Ramanujan's Strange Conjecture...

### Theorem (Folsom-O-Rhoades)

*If $\zeta$ is an even $2k$ order root of unity, then*

$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = -4 \sum_{n=0}^{k-1} (1+\zeta)^2 (1+\zeta^2)^2 \cdots (1+\zeta^n)^2 \zeta^{n+1}.$$

### Remark

*A special case of a thm on Dyson's mock theta, Weierstrass' $\sigma$-function, and quantum mod forms.*

# II. Weierstrass mock modular form

# II. Weierstrass mock modular form

Armed with $F_E(\tau)$ and $\zeta(\Lambda_E; z)$, we directly obtain:

# II. Weierstrass mock modular form

Armed with $F_E(\tau)$ and $\zeta(\Lambda_E; z)$, we directly obtain:

1. Canonical weight 0 harmonic Maass form $\widehat{\mathfrak{Z}}_E(\tau)$.

## II. Weierstrass mock modular form

Armed with $F_E(\tau)$ and $\zeta(\Lambda_E; z)$, we directly obtain:

1. Canonical weight 0 harmonic Maass form $\widehat{\mathfrak{Z}}_E(\tau)$.
2. Canonical weight 1/2 harmonic Maass form $f_E := \mathcal{I}(\widehat{\mathfrak{Z}}_E; \tau)$.

# Special BSD numbers...

**Theorem (Alfes, Griffin, O, Rolen)**

*If $E/\mathbb{Q}$ has prime conductor $p$ and sfe $-1$, then we have:*

1. *If $\Delta < 0$ and $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L(E(\Delta),1) = \widetilde{\alpha_E(\Delta)} \cdot c_E^-(\Delta)^2.$$

2. *If $\Delta > 0$ and $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L'(E(\Delta),1) = 0 \iff c_E^+(\Delta) \text{ is algebraic.}$$