On the elliptic modular function $j(\tau)$

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Modular functions and Quadratic forms

— Number theoretic delights

December 21, 2013, Osaka

Introduction

The elliptic modular function $j(\tau)$ (Dedekind, Klein) :

$$
\begin{aligned}
j(\tau) &= \frac{(1+240\sum_{n=1}^{\infty}(\sum_{d|n} d^3)q^n)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} \\
&= \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \quad (q = e^{2\pi i\tau}).
\end{aligned}
$$

- holomorphic function on $\mathcal{H} := \{\tau \,|\, \tau \in \mathbf{C},\ \mathrm{Im}(\tau) > 0\}$
- $\mathrm{SL}_2(\mathbf{Z})$-invariant, i.e., $j(\frac{a\tau+b}{c\tau+d}) = j(\tau), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$
- simple pole at $i\infty$ with residue 1

$\Rightarrow$ Unique up to an additive constant.

— generates the field of $\mathrm{SL}_2(\mathbf{Z})$-invariant meromorphic functions on $\mathcal{H} \cup \{\infty\}$,

— any holomorphic or meromorphic modular forms of integral weight on $\mathrm{SL}_2(\mathbf{Z})$ are rational function of $j(\tau)$ and $j'(\tau)$:

$$E_4(\tau) = \frac{j'(\tau)^2}{j(\tau)(j(\tau) - 1728)}, \quad E_6(\tau) = -\frac{j'(\tau)^3}{j(\tau)^2(j(\tau) - 1728)}.$$

★ Two fascinating properties of $j(\tau)$:

— Complex Multiplication (Kronecker's "Jugendtraum")

Singular moduli (values of $j(\tau)$ at imaginary quadratics = elliptic points) generate certain abelian extensions (ring class fields) of the imaginary quadratic field.

Ex. $j(\sqrt{-1}) = 1728$, $j(\sqrt{-2}) = 8000$,

$$j\left(\frac{-1+\sqrt{-163}}{2}\right) = (-640320)^3$$

$(\Longrightarrow e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007\cdots)$

$$j(\sqrt{-5}) = 632000 + 282880\sqrt{5},$$

$$j(\tfrac{-1+\sqrt{-23}}{2}) = \alpha = -3493225.6999699333\ldots,$$

$$\alpha^3 + 3491750\alpha^2 - 5151296875\alpha + (5^3 \cdot 11 \cdot 17)^3 = 0.$$

— Monstrous Moonshine (Conway-Norton, Borcherds)

Fourier coefficients of $j(\tau)$ $\Longleftrightarrow$ the characters of the Monster simple group

(Fourier coefficients = behavior at parabolic points)

Ex.
$$\begin{aligned} 196884 &= 1 + 196883, \\ 21493760 &= 1 + 196883 + 21296876, \\ 864299970 &= 2\times 1 + 2\times 196883 + 21296876 + 842609326 \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

Freeman Dyson: "Unfashionable Pursuits" (1983)

I have to confess to you that I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the monster group, build in some unsuspected way into the structure of the universe.

$\Rightarrow$ Vertex Operator Algebra (String Theory)

(elliptic $\Leftrightarrow$ parabolic)

A formula for the Fourier coefficients of $j(\tau)$

in terms of singular moduli (K., 1996)

The coefficient c_n of q^n in $j(\tau)$ $\quad (n \geq 1)$

$$= \frac{1}{n} \sum_{r \in \mathbf{Z}} \left\{ \mathbf{t}(n - r^2) - \frac{(-1)^{n+r}}{4} \mathbf{t}(4n - r^2) + \frac{(-1)^r}{4} \mathbf{t}(16n - r^2) \right\}$$

$$= \frac{1}{n} \left\{ \sum_{r \in \mathbf{Z}} \mathbf{t}(n - r^2) + \sum_{r \geq 1, \text{odd}} \Big((-1)^n \mathbf{t}(4n - r^2) - \mathbf{t}(16n - r^2) \Big) \right\}.$$

Here, $\mathbf{t}(d)$ is a "trace" of $j(\tau_d) - 744$ $(\mathrm{disc}(\tau_d) = -d < 0)$.

Theorem (Zagier)

$$\sum_{d=-1}^{\infty} \mathbf{t}(d)q^d = -\frac{1}{q} + 2 - 248q^3 + 492q^4 - 4119q^7 + \cdots$$

is a modular form of weight 3/2 on $\Gamma_0(4)$.

The $\mathbf{t}(d)$'s satisfy the recursion

$$\begin{aligned} \mathbf{t}(4n-1) &= -240\sigma_3(n) - \sum_{2\le r\le\sqrt{4n+1}} r^2\mathbf{t}(4n-r^2), \\ \mathbf{t}(4n) &= -2 \sum_{1\le r\le\sqrt{4n+1}} \mathbf{t}(4n-r^2) \qquad (n\ge 0). \end{aligned}$$

$$\left(\sigma_3(n) = \textstyle\sum_{d|n} d^3 \ (n>0),\ \sigma_3(0) = \frac{1}{2}\zeta(-3) = \frac{1}{240}.\ \right)$$

$\mathbf{t}(-1) = -1$, $\mathbf{t}(0) = 2$, $\mathbf{t}(3) = -248$, $\mathbf{t}(4) = 492, \ldots$, and $\mathbf{t}(d) = 0$ if $d < -1$ or $-d < 0$ is not a discriminant.

Example (using the second formula)

$$
\begin{aligned}
c_1 &= 2\mathbf{t}(0) - \mathbf{t}(3) - \mathbf{t}(15) - \mathbf{t}(7) \\
&= 2 \times 2 - (-248) - (-192513) - (-4119) = 196884, \\
c_2 &= \frac{1}{2}(\mathbf{t}(7) + \mathbf{t}(-1) - \mathbf{t}(31) - \mathbf{t}(23) - \mathbf{t}(7)) \\
&= (\mathbf{t}(-1) - \mathbf{t}(31) - \mathbf{t}(23))\,/2 \\
&= (-1 - (-39493539) - (-3493982))\,/2 = 21493760, \\
c_3 &= \frac{1}{3}(\mathbf{t}(3) + 2\mathbf{t}(-1) - \mathbf{t}(11) - \mathbf{t}(3) \\
&\qquad -\mathbf{t}(47) - \mathbf{t}(39) - \mathbf{t}(23) - \mathbf{t}(-1)) \\
&= (\mathbf{t}(-1) - \mathbf{t}(11) - \mathbf{t}(47) - \mathbf{t}(39) - \mathbf{t}(23))\,/3 \\
&= (-1 - (-33512) - (-2257837845) \\
&\qquad -(-331534572) - (-3493982))\,/3 = 864299970.
\end{aligned}
$$

What about the "values" or behavior of $\boxed{j(\tau)}$

at hyperbolic points = *real* quadratic numbers ?

$\Rightarrow$ Recent works of Duke-Imamoḡlu-Tóth and K.

Cycle integrals of the j-function

Let $Q(X,Y) = AX^2 + BXY + CY^2$ be a primitive quadratic form of disciminant $D = B^2 - 4AC > 0$.

D.-I.-T. consider the generating function of the "trace" of the integral

$$\int_{\tau_0}^{\gamma_0\tau_0} j(\tau)\frac{\sqrt{D}\,d\tau}{Q(\tau,1)}$$

with $\tau_0 \in \mathcal{H}$ and γ_0 being a suitably chosen generator of the stabilizer of Q in $\mathrm{SL}_2(\mathbf{Z})$.

They showed:

Theorem (Duke-Imamoḡlu-Tóth, 2011)

The function

$$\sum_{D>0} \Big(\text{“trace”} \ \text{ of } \int_{\tau_0}^{\gamma_0\tau_0} j(\tau)\frac{\sqrt{D}\,d\tau}{Q(\tau,1)} \Big)\, q^D$$

is a mock modular form of weight $1/2$, its shadow being Zagier's

$$\sum_{d=-1}^{\infty} \mathbf{t}(d) q^d$$

(of weight $3/2$).

Proof uses Poincaré series, Kloosterman sum identity, etc.

‘Values’ of $j(\tau)$ at real quadratics

w: a real quadratic number of $\mathrm{disc}(w) = D > 0$.

If

$$\gamma w = \frac{aw+b}{cw+d} = w, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),$$

then

$$\frac{\gamma\tau - w}{\gamma\tau - w'} = \varepsilon^2 \, \frac{\tau - w}{\tau - w'} \qquad (w' = \text{conjugate of } w),$$

with ε a unit of norm 1 in $O_D = \mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$. (Assume for simplicity $w > w'$.)

Let ε_0: the fundamental unit of norm 1, $\gamma_0 \longleftrightarrow \varepsilon_0$.

Put

$$z := \frac{\tau - w}{\tau - w'} \in \mathcal{H} \qquad (\Longleftrightarrow \ \tau = \frac{w - w'z}{1-z}).$$

Then

$$j(\tau) = j\left(\frac{w - w'z}{1-z}\right) \text{ is invariant under } z \mapsto \varepsilon_0^2 z,$$

and hence by $z = e^u$, $0 < \mathrm{Im}(u) < \pi$,

$$j\left(\frac{w - w'e^u}{1-e^u}\right) \text{ is invariant under } u \mapsto u + 2\log\varepsilon_0.$$

$$\Downarrow$$

Fourier expansion

$$j\left(\frac{w-w'e^u}{1-e^u}\right)=\sum_{n=-\infty}^{\infty} a_n\, e^{2\pi i n\frac{u}{2\log\varepsilon_0}}.$$

Definition "Value" $\mathrm{val}(w)$ = constant term

$$\begin{aligned}
\mathrm{val}(w) &= a_0\\
&= \frac{1}{2\log\varepsilon_0}\int_{\sigma_0}^{\sigma_0+2\log\varepsilon_0} j\left(\frac{w-w'e^u}{1-e^u}\right)du.\\
&\qquad\qquad (\sigma_0\in\mathbf{C},\quad 0<\mathrm{Im}(\sigma_0)<\pi)\\
&= \frac{\sqrt{D}}{2\log\varepsilon_0}\int_{\tau_0}^{\gamma_0\tau_0}\frac{j(\tau)\,d\tau}{Q(\tau)}
\end{aligned}$$

$$(Q(\tau)=A\tau^2+B\tau+C=A(\tau-w)(\tau-w'),w=(-B+\sqrt{D})/2A)$$

Proposition (Basic properties)

1) $\mathsf{val}(w)$ is $\mathsf{SL}_2(\mathbf{Z})$-equivalent, i.e., $\mathsf{val}(\frac{aw+b}{cw+d}) = \mathsf{val}(w)$, $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbf{Z})$.

2) $\mathsf{val}(w) = \mathsf{val}(w')$.

3) $\overline{\mathsf{val}(w)} = \mathsf{val}(-w')$.

Corollary

1) Suppose $\mathsf{disc}(w) = D$ and let ε_D be the fundamental unit of O_D. Then, if $N(\varepsilon_D) := \varepsilon_D \varepsilon'_D = -1$, we have $\mathsf{val}(w) \in \mathbf{R}$.

2) If w and $-w'$ are $\mathsf{SL}_2(\mathbf{Z})$-equivalent, then $\mathsf{val}(w) \in \mathbf{R}$.

$(w \longleftrightarrow \mathcal{A} \in Cl^+(D) \Rightarrow -w' \longleftrightarrow \mathcal{A}^{-1})$

Observations (K., 2009, unproven)

Observation 1 The minimum among all real values of $\mathrm{val}(w)$ is realized at the golden ratio:

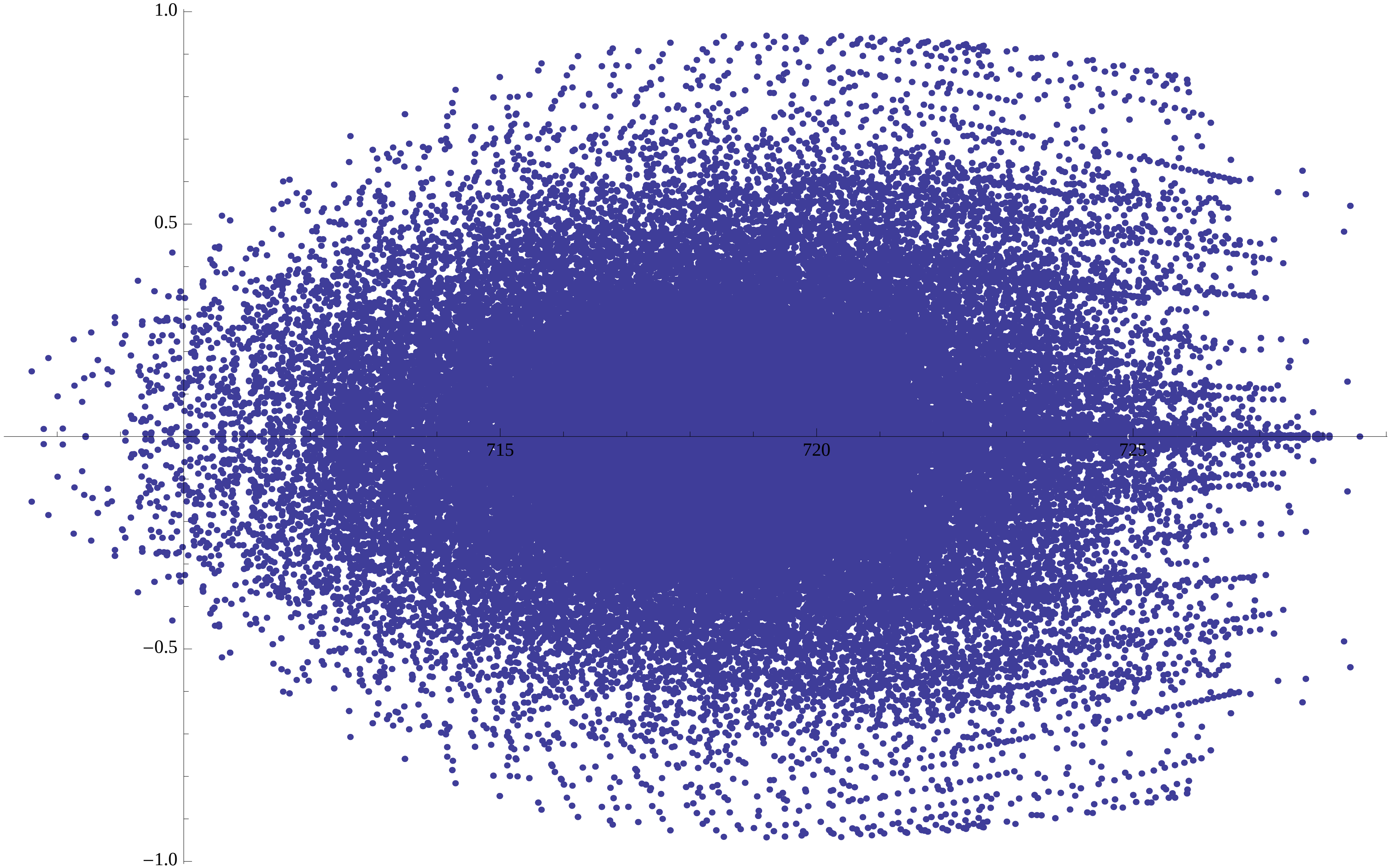
$$\mathrm{val}((1+\sqrt{5})/2) = 706.324813540\ldots.$$

All real parts of $\mathrm{val}(w)$ lie in

$$[706.3248\ldots, 744],$$

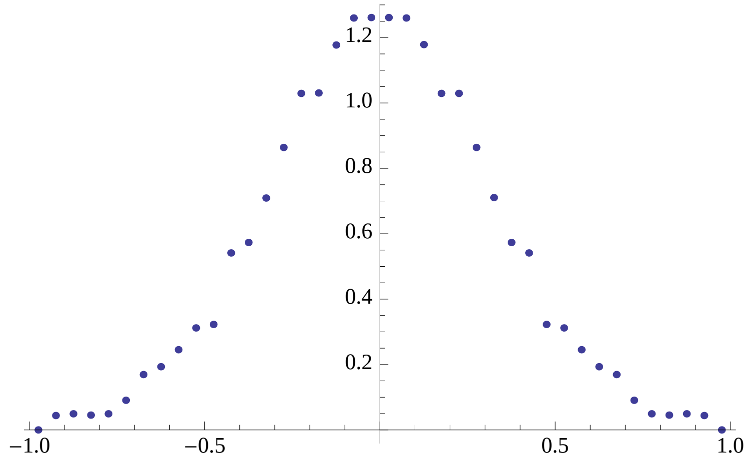
$\boxed{744}$ = const. term in the Fourier exp. of $j(\tau)$ at the cusp

Observation 2 The imaginary part of any $\mathrm{val}(w)$ lies in the interval $(-1, 1)$.

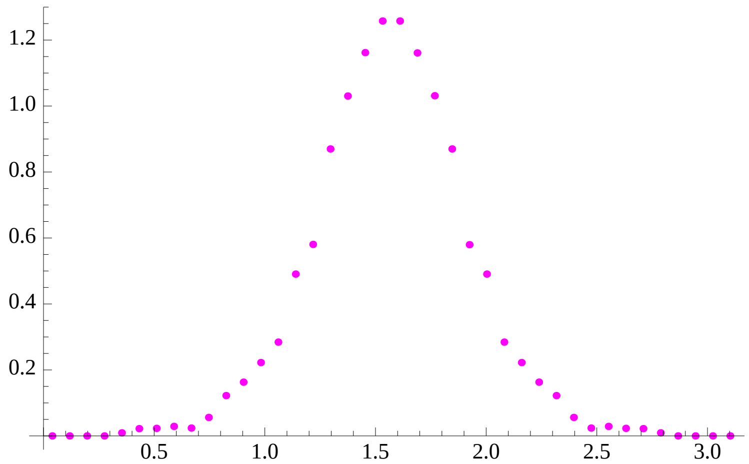


Non-real values of val(w) up to $D \leq 100000$

The distribution of $\mathrm{Im}\,(\mathrm{val}(w))$ looks like



Up to $d \sim 20000$, about 15000 data.



The distribution of θ with $\cos\theta = \mathrm{Im}\,(\mathrm{val}(w))$

w	D	$\mathrm{val}(w)$
$(12+\sqrt{34})/11$	136	$710.60045194400248945\ldots + 0.51979382819610620\ldots i$
$(10+\sqrt{34})/11$	136	$710.60045194400248945\ldots - 0.51979382819610620\ldots i$
$(33+\sqrt{205})/34$	205	$714.16034018225715592\ldots + 0.75363913959038068\ldots i$
$(25+\sqrt{205})/30$	205	$714.16034018225715592\ldots - 0.75363913959038068\ldots i$
$(21+\sqrt{221})/22$	221	$708.90991972070874730\ldots + 0.26703973546028996\ldots i$
$(23+\sqrt{221})/22$	221	$708.90991972070874730\ldots - 0.26703973546028996\ldots i$
$(47+\sqrt{305})/56$	305	$716.13898693848579303\ldots + 0.82184193359696810\ldots i$
$(35+\sqrt{305})/46$	305	$716.13898693848579303\ldots - 0.82184193359696810\ldots i$
$(23+\sqrt{79})/25$	316	$712.65948582687702503\ldots + 0.32545553768732463\ldots i$
$(13+\sqrt{79})/15$	316	$712.65948582687702503\ldots - 0.32545553768732463\ldots i$
$(17+\sqrt{79})/15$	316	$712.65948582687702503\ldots + 0.32545553768732463\ldots i$
$(17+\sqrt{79})/21$	316	$712.65948582687702503\ldots - 0.32545553768732463\ldots i$

First several non-real values of val(w)

Observation 2 (Vague) As the rational approximation of w improves, val(w) increases (i.e., approaches 744).

? A kind of "Diophantine continuity" ?

w	D	$\mathrm{val}(w)$	$\log \varepsilon$
[1]	5	706.3248135408125820559603...	0.9624236501192...
[2]	8	709.8928909199123368059253...	1.7627471740390...
[3]	13	713.2227192129106375260272...	2.3895264345742...
[4]	20	715.8658310509644567882877...	2.8872709503576...
[5]	29	717.9165510885627097946754...	3.2944622927421...
[6]	40	719.5292195149241565812037...	3.6368929184641...
[7]	53	720.8247553829016929089184...	3.9314409432993...
[8]	68	721.8878326202869588905005...	4.1894250945222...
[9]	85	722.7768914565219262830724...	4.4186954172306...
[10]	104	723.5327700907464960378584...	4.6248766825455...
[20]	404	727.6296000047325464824629...	5.9964459005959...
[30]	904	729.4314438625732480951697...	6.8046132909611...
[50]	2504	731.2426027524741005593885...	7.8248455312825...
[100]	10004	733.1113065597372736130899...	9.2105403419828...

Values of $\mathrm{val}(w)$ for $w = [n] = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\ddots}}}$

Experiments related to Markoff numbers

Recall Markoff's theory:

Theorem (Hurwitz) $\forall\alpha \in \mathbf{R} \setminus \mathbf{Q}$, $\exists\infty\ p/q \in \mathbf{Q}$ s.t.

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}\,.$$

$1/\sqrt{5}$: best possible, but

if $\alpha \not\sim (1+\sqrt{5})/2$ (under $\mathrm{GL}_2(\mathbf{Z})$), then $1/\sqrt{5} \Longrightarrow 1/\sqrt{8}$.

If further $\alpha \not\sim \sqrt{2}$, then $1/\sqrt{8} \Longrightarrow 5/\sqrt{221}$.

And so on...

In general, $\exists$ a sequence of "Markoff numbers",

$$\{m_i\}_{i=1}^{\infty} = \{1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, \ldots\},$$

and associated quadratic irrationalities θ_i and monotonically increasing L_i whose limit is 3, with the following property:

For any i, if $\alpha \not\sim \theta_1, \theta_2, \ldots, \theta_{i-1}$ (under $\mathrm{GL}_2(\mathbf{Z})$), then

$\exists \infty\, p/q \in \mathbf{Q}$ that satisfy

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{L_i q^2}.$$

Explicitly, the Markoff numbers m_i appear as solutions of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz, \tag{1}$$

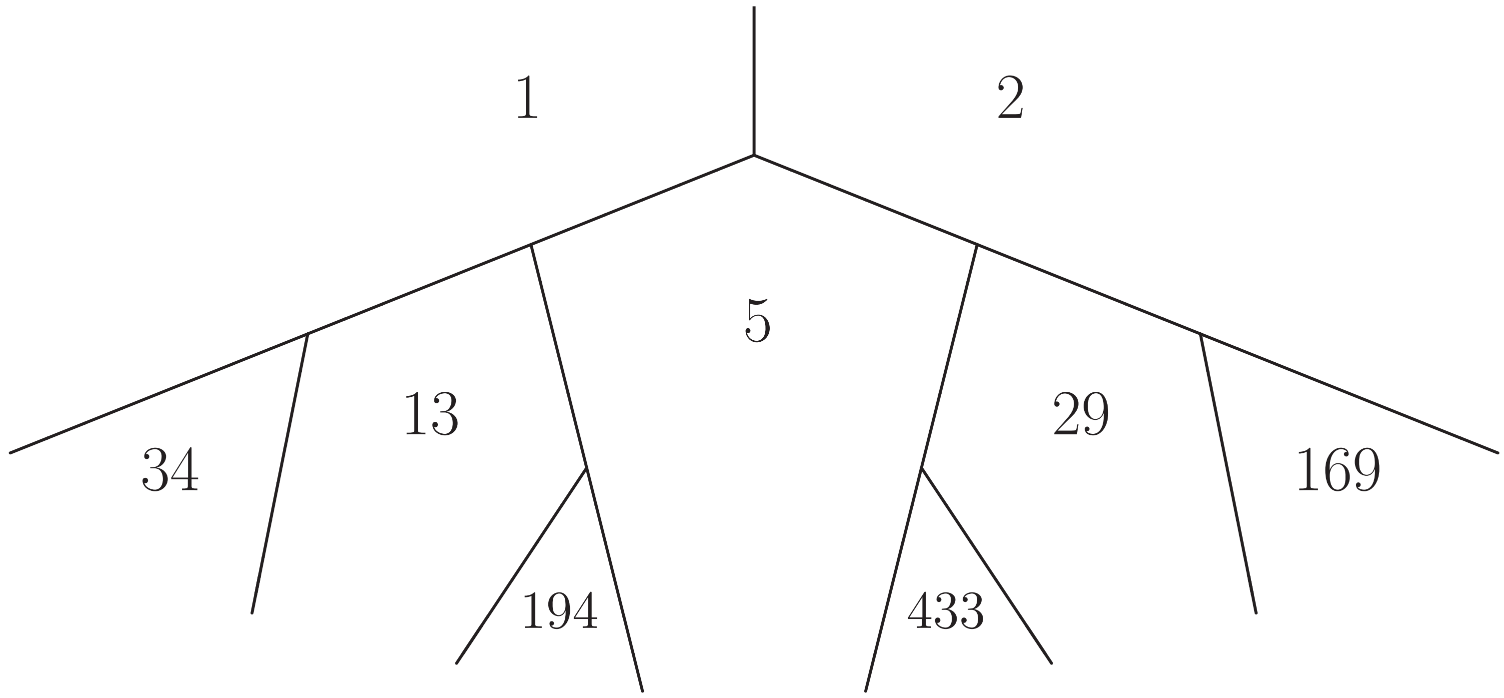
and

$$L_i = \sqrt{9 - 4/m_i^2}, \quad \theta_i = \frac{-3m_i + 2k_i + \sqrt{9m_i^2 - 4}}{2m_i},$$

where $k_i \in \mathbf{Z}$ s.t. $a_i k_i \equiv b_i \pmod{m_i}$, and (a_i, b_i, m_i) a solution of (1) with m_i maximal.

If (p, q, r) is a solution of (1), then $(p, q, 3pq - r)$ and $(p, r, 3pr - q)$ are too. $\Rightarrow$ a *tree* structure of Markoff numbers.

$$\Downarrow$$



Markoff numbers

Numerical computations of val(θ_i)

$\Rightarrow$ the following observations.

Observation 4 Only real values are

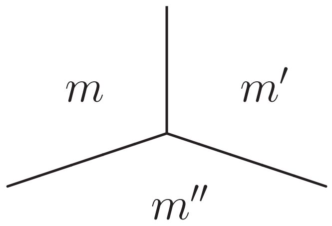
$$\mathrm{val}(\theta_1) = \mathrm{val}(\frac{-1+\sqrt{5}}{2}) = 706.32481354\ldots$$

and

$$\mathrm{val}(\theta_2) = \mathrm{val}(-1+\sqrt{2}) = 709.89289091\ldots.$$

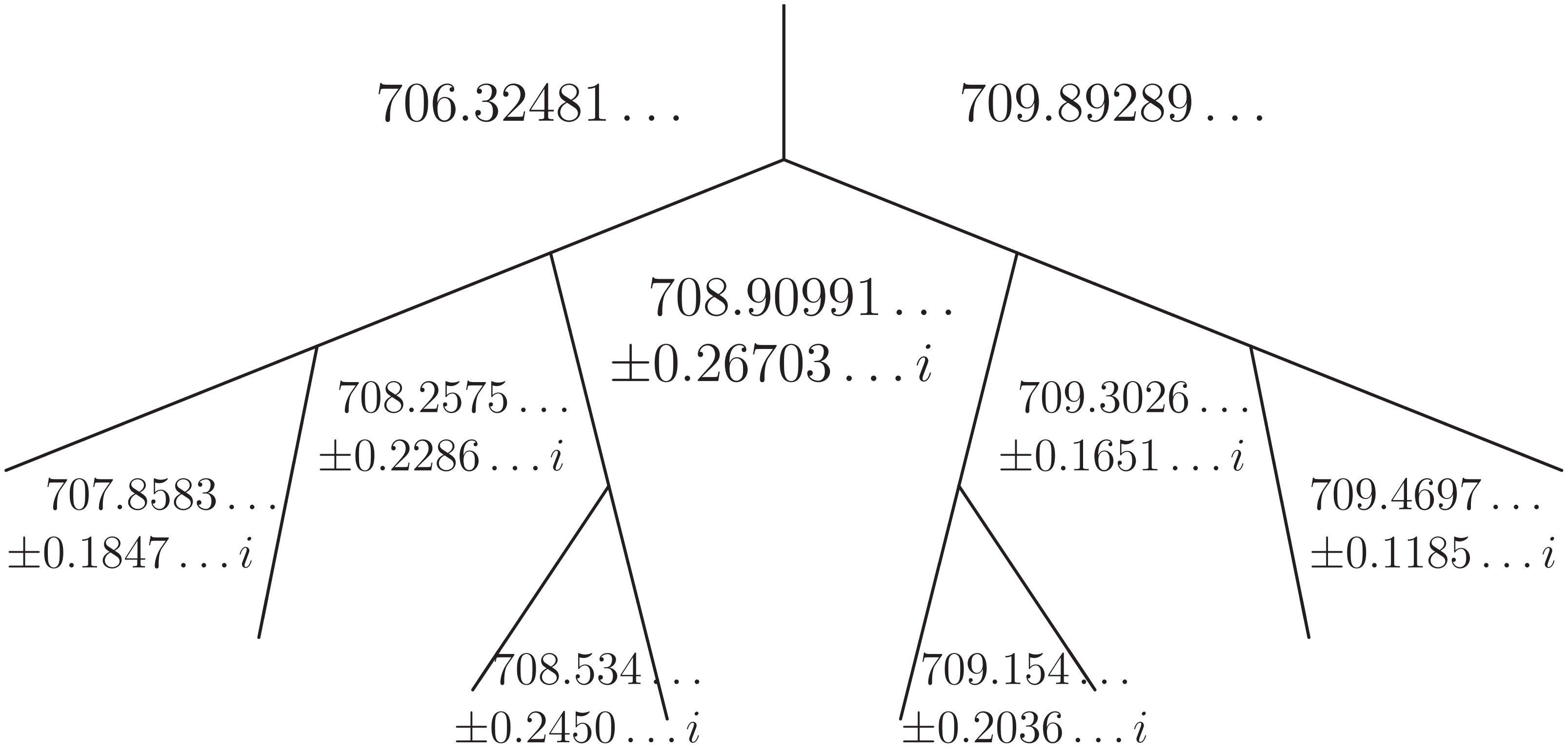
No other values val(θ_i) ($i \geq 3$) seem to be real.

Observation 5 Suppose three Markoff numbers m, m', m'' are

in the position like 

in the Markoff tree, and let θ, θ', θ'' be the associated quadratic numbers.

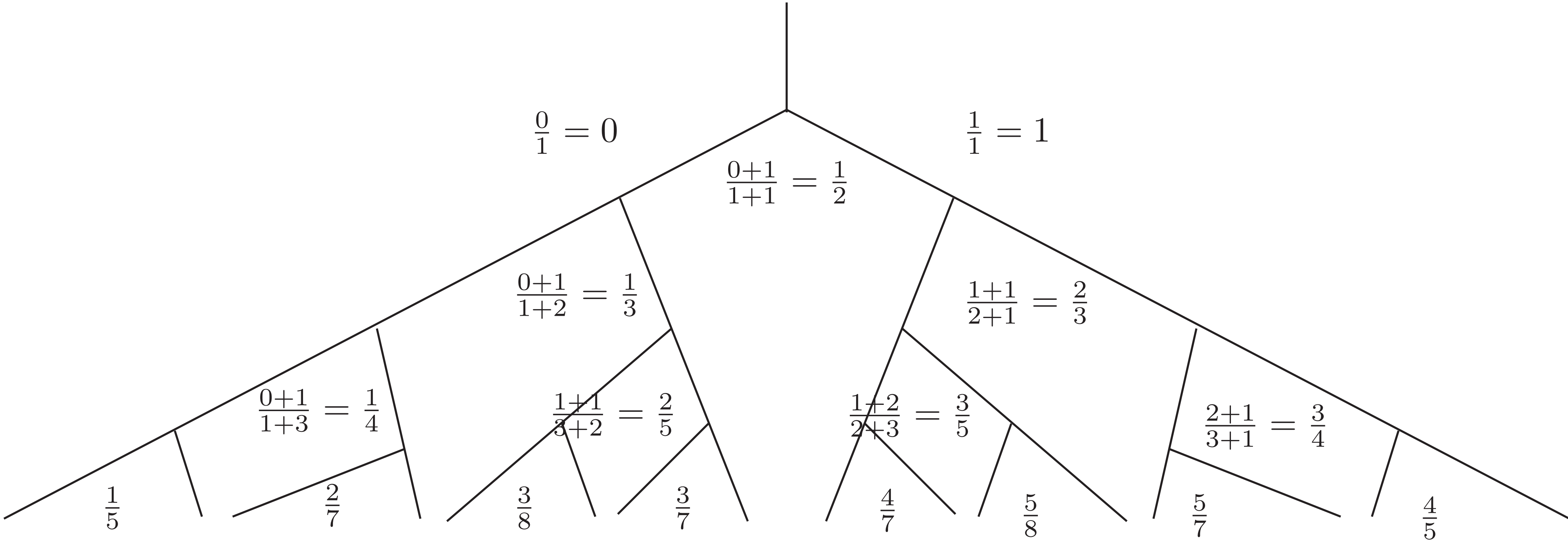
Then, the real and the imaginary parts of θ'' lie between those of θ and θ'.



Values at Markoff irrationalities

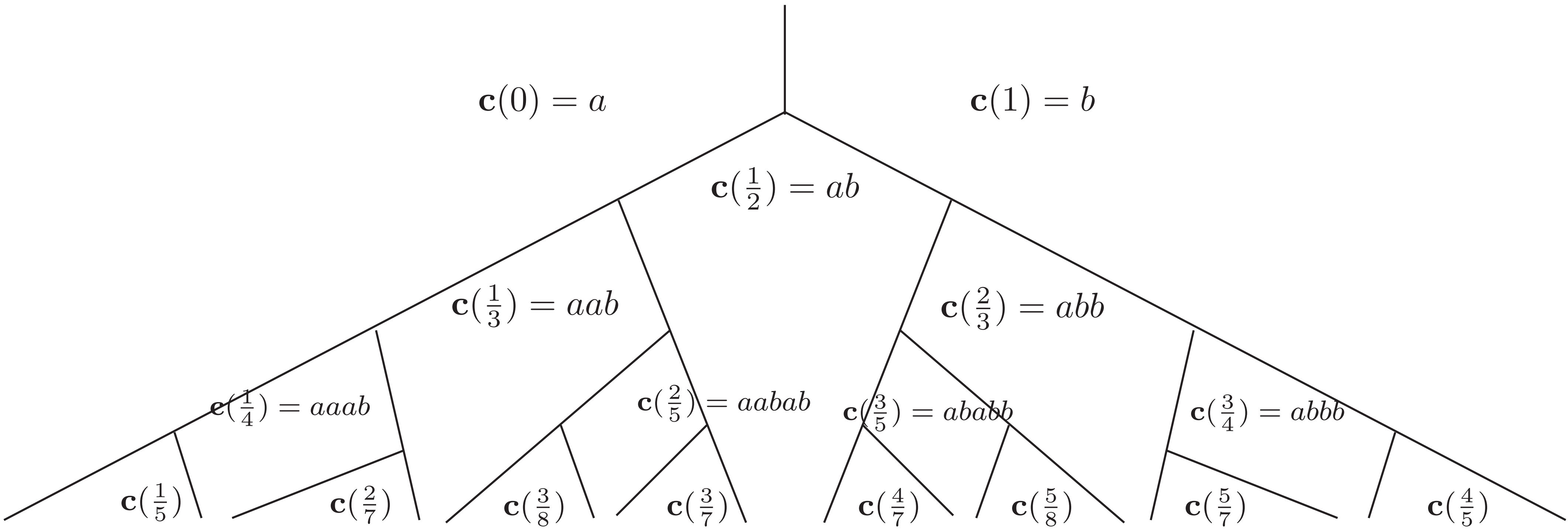
Another way: Farey parametrization of Markoff numbers

(Idea essentially due to H.Cohn)



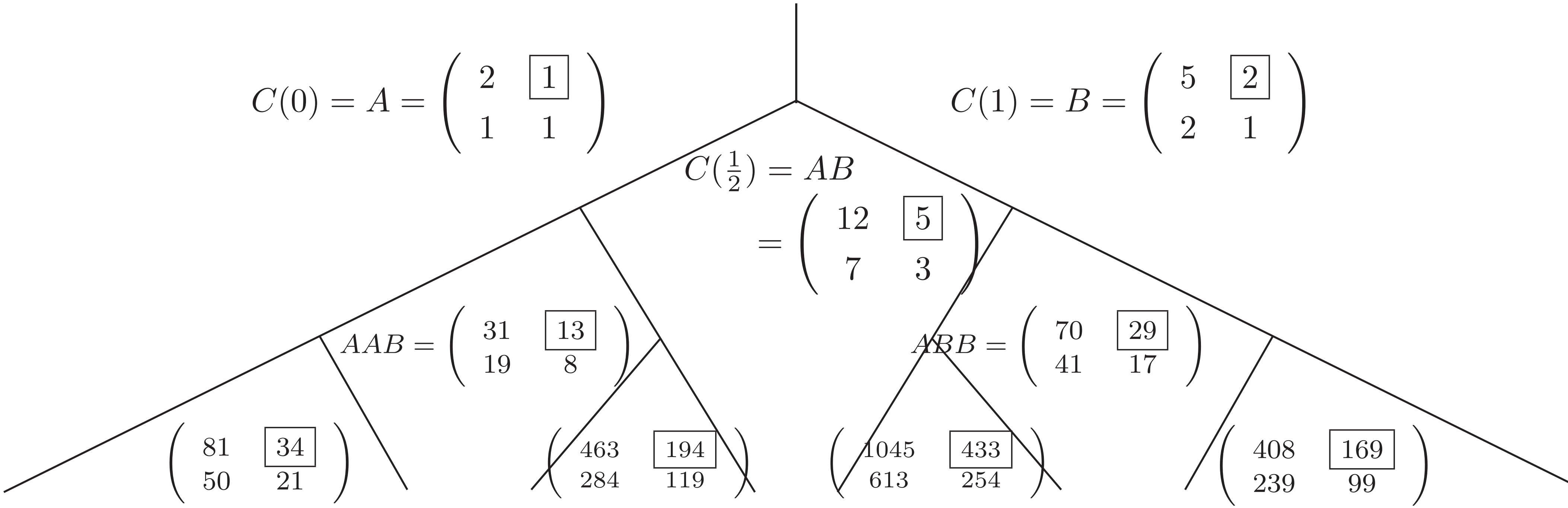
Farey tree

Associate a free word to each Farey fraction.



Farey word (Christoffel word) tree

$$a \mapsto A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad b \mapsto B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$



Cohn matrix tree

Markoff irrationalities

$=$ fixed points of Cohn matrices

$=$ periodic continued fractions with units $1,1$ and $2,2$

Example:

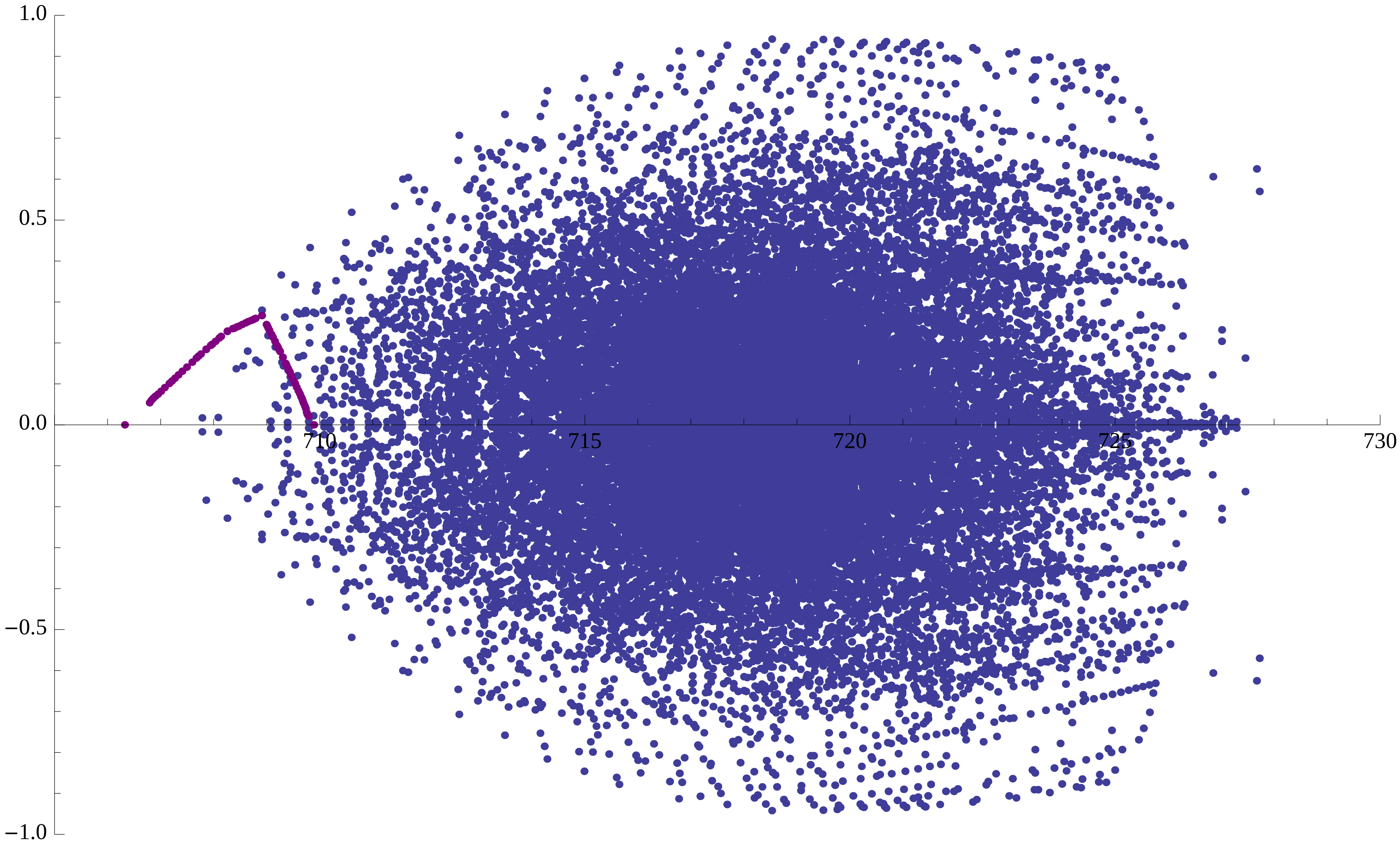
$$m_9 = 194 = M(\tfrac{2}{5}) \longleftrightarrow \mathbf{c}(\tfrac{2}{5}) = aabab \longleftrightarrow \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix},$$

$$\theta_9 = \theta(\tfrac{2}{5})$$

$$\theta_9 = \frac{463\,\theta_9 + 194}{284\,\theta_9 + 119} = [\overline{1,1,1,1,2,2,1,1,2,2}].$$

Fareylabel r	ab-word	$\mathrm{val}(\theta(r))$	
$^0/_1$	a	$706.3248135408125821\cdots$	
$^1/_7$	$aaaaaab$	$707.2692561396631393\cdots$	$+\ 0.1141395581120447229\cdots\ i$
$^1/_6$	$aaaaab$	$707.4080288468731756\cdots$	$+\ 0.1309034208870322692\cdots\ i$
$^1/_5$	$aaaab$	$707.5945659988763180\cdots$	$+\ 0.1533867749061698028\cdots\ i$
$^1/_4$	$aaab$	$707.8583723826967448\cdots$	$+\ 0.1847653353838999688\cdots\ i$
$^2/_7$	$aaabaab$	$708.0348125012714204\cdots$	$+\ 0.2041556894313982860\cdots\ i$
$^1/_3$	aab	$708.2575882428467797\cdots$	$+\ 0.2286358266649360649\cdots\ i$
$^2/_5$	$aabab$	$708.5346656664794211\cdots$	$+\ 0.2450132134683238542\cdots\ i$
$^3/_7$	$aababab$	$708.6465206449353398\cdots$	$+\ 0.2515790406999367673\cdots\ i$
$^1/_2$	ab	$708.9099197207087473\cdots$	$+\ 0.2670397354602899677\cdots\ i$
$^4/_7$	$abababb$	$709.0875535649402100\cdots$	$+\ 0.2209805094289926201\cdots\ i$
$^3/_5$	$ababb$	$709.1545395813431970\cdots$	$+\ 0.2036114408517187755\cdots\ i$
$^2/_3$	abb	$709.3026116673876565\cdots$	$+\ 0.1651964739421995694\cdots\ i$
$^5/_7$	$abbabbb$	$709.3999659857307715\cdots$	$+\ 0.1380102986089410898\cdots\ i$
$^3/_4$	$abbb$	$709.4697680246572326\cdots$	$+\ 0.1185180790830461506\cdots\ i$
$^4/_5$	$abbbb$	$709.5631046996452132\cdots$	$+\ 0.0923765232488907946\cdots\ i$
$^5/_6$	$abbbbb$	$709.6227038601741827\cdots$	$+\ 0.0756822399800070116\cdots\ i$
$^6/_7$	$abbbbbb$	$709.6640585451801062\cdots$	$+\ 0.0640983587355711778\cdots\ i$
$^1/_1$	b	$709.8928909199123368\cdots$	

Values of val at Markoff irrationalities



val(Markoff's) (purple) among imaginary values

What is the arithmetic meaning of val(w),

or "trace" of $\displaystyle\int_{\tau_0}^{\gamma_0\tau_0} j(\tau)\frac{\sqrt{D}\,d\tau}{Q(\tau,1)}$ **?**

(Can we sometime recognize the true trinity, if any, of elliptic, hyperbolic, and parabolic world?)

Expansion?

1) Two variable version

$$\boxed{j(\tau)} = 2^7 \left(\vartheta^8 + \vartheta_2^8 + \vartheta_3^8\right) \left(\frac{1}{\vartheta^8} + \frac{1}{\vartheta_2^8} + \frac{1}{\vartheta_3^8}\right),$$

where

$$\begin{aligned}
\vartheta &= \sum_{n\in\mathbf{Z}} (-1)^n q^{n^2/2}, \\
\vartheta_2 &= \sum_{n\in\mathbf{Z}} q^{(n+1/2)^2/2}, \\
\vartheta_3 &= \sum_{n\in\mathbf{Z}} q^{n^2/2}
\end{aligned}$$

are Jacobi's ‘Theta Nullwerte’.

Originally, theta's are two variable: Consider

$$\widetilde{j}(\tau, z)$$

$$:= \; 2^7 \left(\vartheta(z)^8 + \vartheta_2(z)^8 + \vartheta_3(z)^8\right) \left(\frac{1}{\vartheta(z)^8} + \frac{1}{\vartheta_2(z)^8} + \frac{1}{\vartheta_3(z)^8}\right)$$

$$\begin{aligned}
\vartheta(z) &= \sum_{n \in \mathbf{Z}} (-1)^n q^{n^2/2} w^n \quad (w = e^{2\pi i z}), \\
\vartheta_2(z) &= \sum_{n \in \mathbf{Z}} q^{(n+1/2)^2/2} w^{n+1/2}, \\
\vartheta_3(z) &= \sum_{n \in \mathbf{Z}} q^{n^2/2} w^n.
\end{aligned}$$

$$\begin{aligned}\widetilde{j}(\tau,0) &= \boxed{j(\tau)},\\ \widetilde{j}(\tau,1/3) &= \widetilde{j}(\tau,2/3) = 256(T_{3A}+93/2),\\ \widetilde{j}(\tau,1/4) &= \widetilde{j}(\tau,3/4) = 16(T_{2B}+48).\end{aligned}$$

(T_{3A}, T_{2B} are Mckay-Thompson series.)

$$\left(\frac{d}{dz}\right)^2 \widetilde{j}(\tau,z)\,|_{z=0} = 2(2\pi i)\frac{d}{d\tau}\widetilde{j}(\tau,z)\,|_{z=0} = 2(2\pi i)\frac{d}{d\tau}j(\tau).$$

2) $SL(2,\mathbf{R})$ version?

Fact:

$$SL(2,\mathbf{R})/SL(2,\mathbf{Z}) \approx S^3 \setminus \{\text{trefoil knot}\}$$