Arithmetic Milnor invariants and modular forms

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Acknowledgement & apology

- I would like to thank the organigers for giving me the chance to give a talk in this conference.
- I am sorry that I am poor at speaking English and so please allow me to speak Japanese

Contents

Legendre-Gauss symbol (1798)

Legendre (1752 - 1833)

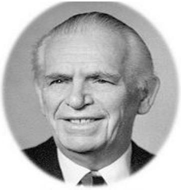
p_1, p_2 : odd primes

$$\left(\frac{p_1}{p_2}\right) = \begin{cases} 1 & \cdots \quad \exists x \in \mathbb{Z} \text{ s.t } x^2 \equiv p_1 \pmod{p_2}, \\ -1 & \cdots \quad \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 & \cdots \quad p_2 \text{ is decomposed in } \mathbb{Q}(\sqrt{p_1})/\mathbb{Q}, \\ -1 & \cdots \quad \text{otherwise.} \end{cases}$$

Gauss (1777 - 1855)

Rédei symbol (1939)



L. Rédei (1900-1980)

p_1, p_2, p_3 : odd primes $\equiv 1 \pmod 4$

$\left(\frac{p_i}{p_j}\right) = 1$ $(1 \leqq i \neq j \leqq 3)$.

$$[p_1, p_2, p_3] = \begin{cases} 1 & \cdots p_3 \text{ is completely decomposed} \\ & \quad \text{in the Rédei ext. } \mathfrak{R}/\mathbb{Q}, \\ -1 & \cdots \text{otherwise.} \end{cases}$$

Here $\mathfrak{R}$ is a D_8-extension determined by p_1, p_2.

Ex. $p_1 = 5, p_2 = 29 \Rightarrow \mathfrak{R} = \mathbb{Q}(\sqrt{5}, \sqrt{29}, \sqrt{7+2\sqrt{5}})$.

Problem and my results (rough form) :

- Are Rédei's ext. $\mathfrak{R}/\mathbb{Q}$ and triple symbol $[p_1, p_2, p_3]$ the right (natural) objects to generalize $\mathbb{Q}(\sqrt{p_1})/\mathbb{Q}$ and $\left(\frac{p_1}{p_2}\right)$?

  Yes, they are the right and natural objects !

- Can we generalize the Legendre and Rédei symbols to an n-th multiple residue symbol $[p_1, \ldots, p_n] \in \{\pm 1\}$?

  Yes, I can, for $n = 4$!

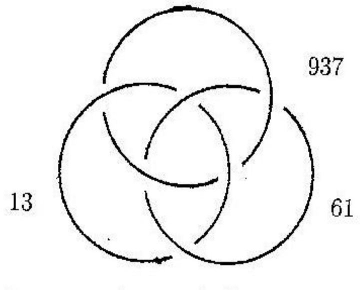
§0 Introduction

Idea: Arithmetic Milnor invariants (Morishita, 1999)

$$\mu_2(12\cdots n) \in \mathbb{Z}/2\mathbb{Z}$$

introduced by the analogy with link theory:

$$\left(\frac{p_1}{p_2}\right) = (-1)^{\mu_2(12)}, \ [p_1, p_2, p_3] = (-1)^{\mu_2(123)}.$$



$\mu_2(ij) = 0$
$(i \neq j)$
$\mu_2(ijk) = 1$
$(\{i, j, k\}$
$= \{1, 2, 3\})$

J. Milnor (1931-)

Thm. (Morishita). $\mu_2(12\cdots n)$ describes the decomposition law of p_n in the Galois extension $K_n/\mathbb{Q}$, unramified outside $p_1, \ldots, p_{n-1}, \infty$, and

$$\mathrm{Gal}(K_n/\mathbb{Q}) = \begin{pmatrix} 1 & \mathbb{F}_2 & \cdots & \mathbb{F}_2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{F}_2 \\ 0 & \cdots & 0 & 1 \end{pmatrix} =: N_n(\mathbb{F}_2)$$

Note $N_2(\mathbb{F}_2) = \mathbb{Z}/2\mathbb{Z}, \; N_3(\mathbb{F}_2) = D_8$.

Remark. Morishita's work is conceptual, and is NOT useful for practical computation !

So, the really number-theoretic problem is to construct the Galois ext. $K_n/\mathbb{Q}$ concretely !

My results (more precise):

- Give a characterization of Rédei's ext. $\mathfrak{R}/\mathbb{Q}$ which yields $K_3 = \mathfrak{R}$.
(Rédei ext. $\mathfrak{R}$ is right and natural.)

- Construct an $N_4(\mathbb{F}_2)$-extension K concretely and introduce the 4-th multiple residue symbol $[p_1, p_2, p_3, p_4]$ and prove

$$[p_1, p_2, p_3, p_4] = (-1)^{\mu_2(1234)}.$$

Assumption:

$p_1,\ p_2,\ p_3$: primes $\equiv 1$ mod 4 s.t. $\left(\frac{p_i}{p_j}\right) = 1\ (i \neq j)$.

Consider the equation

$$x^2 - p_1 y^2 - p_2 z^2 = 0$$

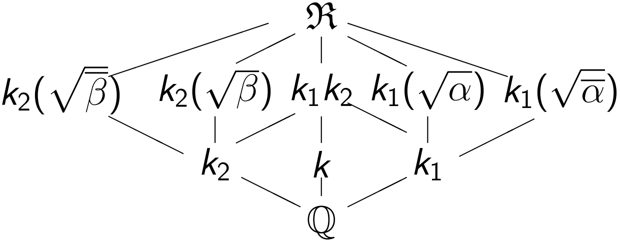
which has non-trivial integer solution (x, y, z) s.t. $(x, y, z) = 1, x - y \equiv 1 \pmod 4, y \equiv 0 \pmod 2$.

Def. Rédei's extension:

$$\mathfrak{R} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}), \quad \alpha = x + y\sqrt{p_1}$$

Note. $\mathfrak{R}$ is independent of the choice of (x, y, z), depends only on p_1, p_2: $\mathfrak{R} = \mathfrak{R}_{\{p_1, p_2\}}$

Thm. (Rédei). $\mathfrak{R}_{\{p_1,p_2\}}$ is a D_8-ext. of $\mathbb{Q}$ and unramified outside p_1, p_2, ∞, ram. index of $p_i = 2$.



$k_i = \mathbb{Q}(\sqrt{p_i})$ $(i = 1, 2)$, $k = \mathbb{Q}(\sqrt{p_1 p_2})$
$\alpha = x + y\sqrt{p_1}$, $\beta = 2(x + z\sqrt{p_2})$

Def. Rédei triple symbol:

$$[p_1, p_2, p_3] := \begin{cases} 1 & \cdots p_3 \text{ is completely decomposed} \\ & \quad \text{in } \mathfrak{R}_{\{p_1,p_2\}}/\mathbb{Q}, \\ -1 & \cdots \text{otherwise} \end{cases}$$

Thm. (F. A.). (1) $K/\mathbb{Q}$: D_8-extension unramified outside p_1, p_2, ∞, ram. index of $p_i = 2$

$$\Longrightarrow K = \mathfrak{R}_{\{p_1,p_2\}}$$

(2) Simple proof of Rédei reciprocity law

$[p_i, p_j, p_k] = [p_1, p_2, p_3]$ for $\{i, j, k\} = \{1, 2, 3\}$, i.e.

p_k is c.d. in $\mathfrak{R}_{\{p_i,p_j\}}/\mathbb{Q} \Leftrightarrow p_3$ is c.d. in $\mathfrak{R}_{\{p_1,p_2\}}/\mathbb{Q}$

Koch type pro-2 group

$S = \{p_1, \ldots, p_n\}$ (p_i : odd prime),

$G_S = \pi_1^{\text{pro-2}}(\mathrm{Spec}(\mathbb{Z}) \setminus S)$

pro-2 fundamental (Poincaré) group

Thm. (Koch).

$G_S = \langle x_1, \ldots, x_n \mid x_1^{p_1 - 1}[x_1, y_1] = \cdots = x_n^{p_n - 1}[x_n, y_n] = 1\rangle,$

x_i : monodromy over p_i ,

y_i : Frobenius auto. over p_i (pro-2 word of x_i's),

$[x, y] := xyx^{-1}y^{-1}$.

$F = \langle x_1, \ldots, x_n \rangle$: free pro-2 group
Magnus-Fox non-comm. power series expansion

$$\begin{array}{ccc} F & \longrightarrow & \mathbb{Z}_2\langle\langle X_1, \ldots, x_n \rangle\rangle \\ x_i & \mapsto & 1 + X_i \\ y_j & \mapsto & 1 + \displaystyle\sum_{1 \le i_1, \ldots, i_r \le n} \epsilon\left(\frac{\partial^r y_j}{\partial x_{i_1} \cdots \partial x_{i_r}}\right) X_{i_1} \cdots X_{i_r} \end{array}$$

Here $\partial/\partial x_i$:= pro-2 Fox free derivative (T.Oda - Ihara)
ϵ := augmentation map.

Def. Arithmetic Milnor invariant:

$$\mu_2(i_1 \cdots i_r j) := \epsilon\left(\frac{\partial^r y_j}{\partial x_{i_1} \cdots \partial x_{i_r}}\right) \bmod 2.$$

Thm. (Morishita).
There is a canonical Galois ext. $K_n/\mathbb{Q}$ unramified outside $p_1, \cdots, p_{n-1}, \infty$ and $\mathrm{Gal}(K_n/\mathbb{Q}) = N_n(\mathbb{F}_2)$ s.t. $\mu_2(12\cdots n) = 0$
$\Leftrightarrow$ p_n is completely decomposed in $K_n/\mathbb{Q}$.

Ex. $K_2 = \mathbb{Q}(\sqrt{p_1})$, $\left(\dfrac{p_1}{p_2}\right) = (-1)^{\mu_2(12)}$

$K_3 =$ Rédei's ext. $\mathfrak{R}$, $[p_1, p_2, p_3] = (-1)^{\mu_2(123)}$.

Assumption:

$p_1,\ p_2,\ p_3,\ p_4$: odd primes $\equiv 1 \pmod 4$,

$$\left(\frac{p_i}{p_j}\right) = 1,\ \ (1 \leqq i \neq j \leqq 3), \quad [p_1, p_2, p_3] = 1.$$

$\alpha = x + y\sqrt{p_1}$, $\beta = x' + y'\sqrt{p_3}$ (Rédei's for p_3, p_2).

Consider the equation

$$X^2 - \alpha Y^2 - p_3 Z^2 = 0$$

which has a non-trivial integral solution in $\mathbb{Q}(\sqrt{p_1})$.

Def. $K := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\theta})$,
where $\theta = X + Y\sqrt{\alpha}$ (good choice).

Thm. (F.A.).
$K/\mathbb{Q}$ is a Galois extension of degree 64 s.t.
(1) unramified outside p_1, p_2, p_3, ∞
(2) $\mathrm{Gal}(K/\mathbb{Q}) = N_4(\mathbb{F}_2)$

Proof) It's very long.
The key is to control the ramification over 2 !

Def. 4-th multiple residue symbol

$$[p_1, p_2, p_3, p_4] = \begin{cases} 1 & \cdots p_4 \text{ is completely decomposed in } K/\mathbb{Q}, \\ -1 & \cdots \text{otherwise} \end{cases}$$

Thm. (F.A.).

$$(-1)^{\mu_2(1234)} = [p_1, p_2, p_3, p_4].$$

Proof) $S = \{p_1, p_2, p_3, p_4\}$

$$\begin{array}{ccc} G_S & \longrightarrow & \mathrm{Gal}(K/\mathbb{Q}) \\ \mu_2(1234) & & [p_1, p_2, p_3, p_4] \end{array}$$

Use the structure of

$$N_4(\mathbb{F}_2) = \left\langle g_1, g_2, g_3 \;\middle|\; \begin{array}{l} g_1^2 = g_2^2 = g_3^2 = (g_1g_3)^2 = 1 \\ (g_2g_3)^4 = (g_1g_2g_3)^4 = 1 \\ ((g_1g_2g_3g_2)^2g_3)^2 = 1 \end{array} \right\rangle,$$

(informed by Y. Mizusawa)

Ex. $(p_1, p_2, p_3, p_4) := (5, 8081, 101, 449)$

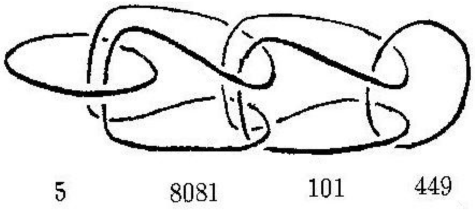
$\alpha = 241 + 100\sqrt{5}$, $\beta = 1009 + 100\sqrt{101}$,

$\theta = 50 - 4\sqrt{5} - 2\sqrt{\alpha}$

$K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\theta})$

$\left(\frac{p_i}{p_j}\right) = 1$, $[p_i, p_j, p_k] = 1$, $(\{i, j, k\} = \{1, 2, 3\})$.

$[p_1, p_2, p_3, p_4] = -1$



p_1, p_2, p_3 primes $\equiv 1 \pmod 4$

$\left(\dfrac{p_i}{p_j}\right) = 1$ $(i \neq j)$.

$$\Longrightarrow \begin{cases} \text{Rédei ext. } \mathfrak{R} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha})/\mathbb{Q}, \\ \text{Rédei symbol } [p_1, p_2, p_3] \end{cases}$$

Our aim: Express $[p_1, p_2, p_3]$ in terms of the p_3-th Fourier coefficient of a modular form $f_{\{p_1, p_2\}}$.

"Analytic realization of non-abelian reciprocity".

$$\mathrm{Gal}(\mathfrak{R}/\mathbb{Q}) = \langle a, b \mid a^2 = b^4 = 1, aba = b^{-1} \rangle,$$
$$a : (\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}) \mapsto (\sqrt{p_1}, -\sqrt{p_2}, \sqrt{\alpha}),$$
$$b : (\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}) \mapsto (-\sqrt{p_1}, -\sqrt{p_2}, -\sqrt{\alpha}).$$

2-dim. representation & Artin L-function:

$$\rho : \ \mathrm{Gal}(\mathfrak{R}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$$
$$\rho(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{cases} L(\rho, s) & = \prod_p L_p(\rho, s) \\ L_p(\rho, s) & = \det(\mathrm{I} - \rho(\mathrm{Fr}_p)p^{-s}|V^{I_p})^{-1}, \ V = \mathbb{C}^2 \end{cases}$$

$\Rightarrow$

$$\begin{cases} \frac{1}{2}\mathrm{Tr}(\rho(\mathrm{Fr}_{p_3})) & = [p_1, p_2, p_3], \\ L_{p_3}(\rho, s) & = (1 - 2[p_1, p_2, p_3]p_3^{-s} + p_3^{-2s})^{-1} \end{cases}$$

Hecke character & Hecke L-function:

$k := \mathbb{Q}(\sqrt{p_1 p_2})$

- $\mathfrak{R}/k$ is unramified in the narrow sense.

  Artin map H_k^+ (narrow class gr.) $\twoheadrightarrow$ $\mathrm{Gal}(\mathfrak{R}/k)$

$\chi :\ \mathrm{Gal}(\mathfrak{R}/k) = \langle b \mid b^4 = 1 \rangle \to \mathbb{C}^\times;\ b \mapsto \sqrt{-1}$

$$\Rightarrow \boldsymbol{\chi} := \chi \circ (\mathrm{Artin}) : H_k^+ \longrightarrow \mathbb{C}^\times$$

- $\rho = \mathrm{Ind}(\chi)$

$$\Rightarrow L(\rho, s) = L(\boldsymbol{\chi}, s) =: \sum_{n=1}^{\infty} a_{\boldsymbol{\chi}}(n) n^{-s}.$$

$\boldsymbol{\chi}$ is determined by p_1, p_2: $a_{\{p_1,p_2\}}(n) := a_{\boldsymbol{\chi}}(n)$.

Maass wave form (1941)

$$\Theta_{\{p_1,p_2\}}(z) := \sum_{n\neq 0} a_\chi(n)\sqrt{y}K(2\pi|n|y)e^{2\pi inx},$$
$$(z = x + iy, y > 0)$$
$$K(t) := \frac{1}{2}\int_0^\infty e^{-t(u+u^{-1})}\frac{du}{u}.$$

Maass (1911 - 1992)

Thm. $\Theta_{\{p_1,p_2\}}(z)$ is a real analytic modular (cusp) form of weight 0, character $\left(\frac{p_1p_2}{\cdot}\right)$ w.r.t. $\Gamma_0(p_1p_2)$ s.t.

$$a_{\{p_1,p_2\}}(p_3) = 2[p_1, p_2, p_3].$$

Cor. (Reciprocity of Fourier coefficients).
$a_{\{p_i,p_j\}}(p_k) = a_{\{p_1,p_2\}}(p_3)$ for $\{i, j, k\} = \{1, 2, 3\}$.

Imaginary variant

$p_1 \equiv 3 \bmod 4,\ p_2,\ p_3 \equiv 1 \bmod 4 \quad \left(\frac{p_i}{p_j}\right) = 1 \ (i \neq j)$

$$\Rightarrow \begin{cases} \text{Rédei } D_8\text{-ext. } \mathfrak{K} := \mathbb{Q}(\sqrt{-p_1}, \sqrt{p_2}, \sqrt{\alpha}), \\ \qquad (\alpha = x + y\sqrt{-p_1},\ x^2 + p_1 y^2 = p_2 z^2) \\ \text{Rédei symbol } [-p_1, p_2, p_3] \end{cases}$$

2-dim. repr. ρ and Hecke char. χ are defined similarly.

$k := \mathbb{Q}(\sqrt{-p_1 p_2})$.

$\mathfrak{K}/k$ is (strictly) unramified, hence

$\chi : H_k \twoheadrightarrow \mathrm{Gal}(\mathfrak{K}/k) \to \mathbb{C}^\times$

$$L(\rho, s) = L(\chi, s) =: \sum_{n=1}^{\infty} a_\chi(n) n^{-s} =: \sum_{n=1}^{\infty} a_{\{-p_1, p_2\}}(n) n^{-s}.$$

$H_k = \{C_0, C_1, \cdots, C_{h-1}\}$, $h :=$ class no. of k.
$C_i \leftrightarrow Q_i =$ binary quadratic form mod $\mathrm{SL}_2(\mathbb{Z})$-equiv.
(cf. T. Ono's Rikkyo Lect. Note)

Prop.

$$2a_{\{-p_1,p_2\}}(n) = \sum_{i=0}^{h-1} \chi(C_i) \cdot \#\{(x,y) \in \mathbb{Z}^2 | Q_i(x,y) = n\}.$$

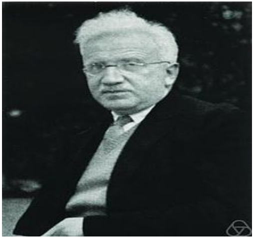
Ex. $(p_1, p_2) = (3, 73)$, $p_1 p_2 = 219$
$H_k = \{C_0, C_1, C_2 = C_1^2, C_3 = C_1^3\} \simeq \mathbb{Z}/4\mathbb{Z}$.
$C_0 = [\mathcal{O}_k]$, $C_1 = [5, \frac{1+\sqrt{-219}}{2}]$, $C_2 = [3, \frac{3+\sqrt{-219}}{2}]$, $C_3 = [1, \frac{9+\sqrt{-219}}{2}]$
$Q_0 = X^2 + XY + 3Y^2$, $Q_1 = 5X^2 + XY + 11Y^2$,
$Q_2 = 3X^2 + 3XY + 19Y^2$, $Q_3 = 5X^2 + 9XY + 15Y^2$.

n	1	2	3	4	5	6	7	8	9	$\cdots$
$a_{\{-p_1,p_2\}}(n)$	1	0	-1	1	0	0	0	0	1	$\cdots$

Hecke theta function. (1927)

$$\Theta_{\{-p_1,p_2\}}(z) = \sum_{n=0}^{\infty} a_{\{-p_1,p_2\}}(n) e^{2\pi i n z}$$

Thm. $\Theta_{\{-p_1,p_2\}}(z)$ is a holomorphic Hecke eigen cusp form of weight 1, character $\left(\frac{-p_1p_2}{\cdot}\right)$ w.r.t. $\Gamma_0(p_1p_2)$ s.t.



E. Hecke (1887 - 1947)

$$a_{\{-p_1,p_2\}}(p_3) = 2[-p_1, p_2, p_3].$$

Cor. (Reciprocity of Fourier coefficients).

$$a_{\{-p_1,p_2\}}(p_3) = a_{\{-p_1,p_3\}}(p_2).$$

Ex. Assume the class no. of $\mathbb{Q}(\sqrt{-p_1p_2})$ is 4.
$\Leftrightarrow \mathfrak{R}$ is the Hilbert class field of $\mathbb{Q}(\sqrt{-p_1p_2})$.

Note M. Kida told us that the following 21 pairs of (p_1, p_2) from H. Wada's list are complete:

$$(p_1, p_2) = (3, 13),\ (11, 5),\ (31, 5),\ (7, 29),\ (3, 73),\ (7, 37),\ (3, 97),\ (19, 17),\ (71, 5),\ (23, 29),\ (3, 241),\ (7, 109),\ (191, 5),\ (59, 17),\ (79, 13),\ (3, 409),\ (11, 113),\ (19, 73),\ (83, 17),\ (11, 137),\ (311, 5)$$

Thm. (Ogasawara). For such 21 pairs of (p_1, p_2),

$$\Theta_{\{-p_1,p_2\}}(z) = \frac{1}{2}(\vartheta(z)\vartheta(p_1p_2z) - \vartheta(p_1z)\vartheta(p_2z))|T(4)$$

and

$$a_{\{-p_1,p_2\}}(n) = \frac{1}{2}\#\{(X,Y)\in\mathbb{Z}^2 \mid X^2 + p_1p_2Y^2 = 4n\} - \frac{1}{2}\#\{(X,Y)\in\mathbb{Z}^2 \mid p_1X^2 + p_2Y^2 = 4n\}$$

where

$$\vartheta(z) := \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z} \text{ (Jacobi theta function)},$$

$$T(4) := \text{Hecke operator.}$$

Cor. (Reciprocity). Assume class no's of $\mathbb{Q}(\sqrt{-p_1p_2})$ and $\mathbb{Q}(\sqrt{-p_1p_3})$ are 4, i.e., $(p_1, p_2, p_3) = (3, 73, 97), (3, 97, 241), (7, 29, 109)$
$\Rightarrow$

$$\begin{aligned}
&\#\{(X, Y) \in \mathbb{Z}^2 \mid X^2 + p_1p_2Y^2 = p_3\} \\
&\quad - \#\{(X, Y) \in \mathbb{Z}^2 \mid p_1X^2 + p_2Y^2 = p_3\} \\
&\qquad = \#\{(X, Y) \in \mathbb{Z}^2 \mid X^2 + p_1p_3Y^2 = p_2\} \\
&\qquad\quad - \#\{(X, Y) \in \mathbb{Z}^2 \mid p_1X^2 + p_3Y^2 = p_2\}
\end{aligned}$$

Analytic expression for topological Milnor invariants:

Number Theory	Physics
Zeta function	Partition function
L-function	Correlation function

Koyama-Kurokawa's book "Riemann Hypotheis in Math. Physics"

Zeta function of modular form	Partition function of gauge-inv. Lagrangian
Fourier expansion at cusps	Perturbative expansion at classical solutions

§4 Analytic expression for Rédei symbol

SU(2) Chern-Simons QFT with link $\mathcal{K}_1 \cup \cdots \cup \mathcal{K}_n$

Lagrangian: $\mathrm{CS}(A) = \dfrac{1}{8\pi^2} \displaystyle\int_{\mathbb{R}^3} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$

Correlation function (Witten invariant):

$$Z(\mathrm{CS}, \hbar) = \int_{\Omega^1(\mathbb{R}^3)\otimes \mathfrak{su}(2)} \mathcal{D}A \exp(\frac{i}{\hbar}\mathrm{CS}(A)) \prod_{j=1}^{n} \mathrm{Tr} \exp(i \int_{K_j} A)$$

$$\sim \sum a_n \hbar^n \quad (\hbar \to 0)$$

Result (L. Rozansky, H. Kodani). Formulas on the relation between top. Milnor invariants $\mu(1 \cdots n)$ and perturbative coefficients a_n of $Z(\mathrm{CS}, \hbar)$ corr. to tree Feynman diagrams.

The Witten invariant and the above Result goes back to Gauss' integral formula for the linking number in electro-magnetic theory (1833):

$$\mathrm{lk}(\mathcal{K}_1, \mathcal{K}_2) = \int_{x \in \mathcal{K}_1} \int_{y \in \mathcal{K}_2} \omega(x - y)$$

$$\omega(x) := \frac{x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2}{4\pi ||x||^3}$$

I started with Legendre-Gauss symbol and now end up with Gauss again !

Gauss (1777 - 1855)

References

F. Amano, On Rédei's dihedral extension and triple reciprocity law, Proc. Japan Acad., 90, Ser. A, (2014).

F. Amano, On a certain nilpotent extension over $\mathbb{Q}$ of degree 64 and the 4-th multiple residue symbol, arxiv. math. 1311.1289 (Nov. 2013), to appear in Tohoku Math. J.

F. Amano, H. Kodani, M. Morishita, T. Sakamoto, T. Yoshida, Rédei's tripe symbols and modular forms, with Appendix by T. Ogasawara, Tokyo J. Math., 36, No 2, (**2013**).

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