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Weighted Completion of Galois Groups and the Deligne-Ihara Conjecture*

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1. Deligne-Ihara Conjecture

Ihara stated a conjecture in

" Galois group over $\mathbb{Q}$" Publ. MSRI 16 (1989),
attributed to Deligne.

Conjecture (Deligne-Ihara)

The Lie algebra

$$\mathrm{Gr}^{Ih}\, G_{\mathbb{Q}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

has the following properties.

(G) as $\mathbb{Q}_\ell$-Lie algebra, it has a generating set consisting of one element in each odd grade ≥ 3.

(F) these are free generators. $\square$

We shall prove (G) by weighted completion of Galois groups. The proof suggests the reason of this conjecture.

Notation:

- $\mathbb{P}^1_{01\infty}$: the projective line over $\mathbb{Q}$ minus three points.
- $\pi_1 =< x, y >$: its topological fundamental group (free with two generators),
- $\pi_1\hat{}$: its profinite completion,
- π_1^ℓ: its pro-ℓ completion, (ℓ: a prime)
- $G_\mathbb{Q} :=$ Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$.

We have the pro-l nonabelian Galois representation

$$\rho^\ell : G_{\mathbb{Q}} \to \operatorname{Out} \pi_1\hat{} \to \operatorname{Out} \pi_1^\ell.$$

Let $L^m \pi_1^\ell$ be the lower central series

$$\begin{array}{ll} & L^1\pi_1^\ell := \pi_1^\ell \\ \supset & L^2\pi_1^\ell := [\pi_1^\ell, \pi_1^\ell] \\ \supset & L^3\pi_1^\ell := [\pi_1^\ell, [\pi_1^\ell, \pi_1^\ell]] \\ & \cdots . \end{array}$$

Then, induce a descending filtration on $G_{\mathbb{Q}}$ by

$$Ih^m(G_{\mathbb{Q}}) := \ker[G_{\mathbb{Q}} \to \mathrm{Out}(\pi_1^\ell / L^{m+1}\pi_1^\ell)].$$

By group theory, $Ih^m (m \geq 1)$ is a central filtration, and

$$\mathrm{Gr}_m^{Ih} G_{\mathbb{Q}} := Ih^m G_{\mathbb{Q}} / Ih^{m+1} G_{\mathbb{Q}}$$

is abelian and a free $\mathbb{Z}_\ell$-module of finite rank.

The direct sum of $\mathbb{Z}_\ell$-modules

$$\mathrm{Gr}^{Ih} G_{\mathbb{Q}} := \bigoplus_{m \geq 1} \mathrm{Gr}_m^{Ih} G_{\mathbb{Q}}$$

has a graded Lie algebra structure:

$$s_m \in Ih^m(G_{\mathbb{Q}}), s_n \in Ih^n(G_{\mathbb{Q}}) \\ \mapsto s_m s_n s_m^{-1} s_n^{-1} \in Ih^{n+m}(G_{\mathbb{Q}})$$

gives Lie product

$$\mathrm{Gr}_m^{Ih} G_{\mathbb{Q}} \otimes_{\mathbb{Z}_\ell} \mathrm{Gr}_n^{Ih} G_{\mathbb{Q}} \ \to \ \mathrm{Gr}_{m+n}^{Ih} G_{\mathbb{Q}}.$$

Known:

- $G_{\mathbb{Q}} \overset{\exists}{\to} \mathrm{Aut}\,\pi_1^\ell \to \mathrm{Aut}(\pi_1^\ell / L^{m+1}\pi_1^\ell)$ which defines the same Ih.

  $Ih^1 G_{\mathbb{Q}} = Ih^2 G_{\mathbb{Q}} = Ih^3 G_{\mathbb{Q}}$ is the kernel of $\chi : G_{\mathbb{Q}} \to \mathbb{Z}_\ell{}^\times$.

- Soulé's elements: $\exists \sigma_m \neq 0 \in \mathrm{Gr}_m^{Ih} G_{\mathbb{Q}}$ ($m \geq 3$, odd). (Ihara, Soulé).

- Some nonvanishing of their products.

- $\mathrm{Gr}^{Ih} G_{\mathbb{Q}} \hookrightarrow \mathfrak{g}t$,
  Lie version of Grothendieck-Teichmüller group. (Deligne, Drinfeld, Ihara)

- $\dim \mathrm{Gr}^{Ih}_m G_{\mathbb{Q}} = \dim \mathfrak{g}t_m \ (m \leq 12)$
  fits to Conjecture (Tsunogai)

- (F) implies (G). (Ihara)

- The Lie algebra appears in $\mathrm{Gr}\, \pi_1^{\text{arith}}$ of the moduli stack $\mathcal{M}_{g,n}$
  (H. Nakamura, T. Oda).

Outline of Proof of (G)

- It factors through $G_\ell := \pi_1(\mathbb{Z}[\frac{1}{\ell}])$.

- Construct the weighted completion $\mathcal{A}_\ell$ of G_ℓ with Zar.dense $G_\ell \to \mathcal{A}_\ell(\mathbb{Q}_\ell)$ which is "relative-unipotent group closure $/\mathbb{Q}_\ell$" (cf. Deligne, same book). $\mathcal{A}_\ell$ has a natural weight filtration.

- The kernel $\mathcal{K}_\ell$ of $\mathcal{A}_\ell \to \mathbb{G}_m$ is free unipotent group generated by Soulé's elements. This follows from Soulé's computation of $H^i(G_\ell, \mathbb{Q}_\ell(m))$.
  So generated is $\mathfrak{k}_\ell := Lie(\mathcal{K}_\ell) \cong \mathrm{Gr}\,\check{}\,\mathcal{K}_\ell$.

- Let $\mathcal{P}$ be the "unipotent group closure" of π_1^l. Then, we have

$$\rho^{unip} : G_\ell \to \mathcal{A}_\ell \to \mathrm{Aut}\,\mathcal{P}.$$

- The image $Im\mathcal{A}_\ell \subset \mathrm{Aut}\,\mathcal{P}$ has two filtrations:

  1. image of the weight filtration $\mathcal{A}_\ell$
  2. one from $L^{m+1}\mathcal{P}$ (induces $Ih^m G_\ell$).

  They coincide.

- $\mathrm{Gr}^{Ih} G_\ell \otimes \mathbb{Q}_\ell \cong \mathrm{Gr}(Im\mathcal{K}_\ell)$ is generated by Soulé's elements.

2. Weighted Completion of $G_\ell := \pi_1(\mathrm{Spec}\,\mathbb{Z}[\frac{1}{l}])$

Known: ρ^ℓ factors through

$$\rho^\ell : G_{\mathbb{Q}} \overset{f}{\twoheadrightarrow} G_\ell := \mathrm{Gal}(\mathbb{Q}^{ur,l}/\mathbb{Q}) \to \mathrm{Out}\,\pi_1^\ell,$$

where $\mathbb{Q}^{ur,l}$ is a maximal algebraic extension of $\mathbb{Q}$ unramified outside ℓ.

Easy to see:

$$\mathrm{Gr}^{Ih}\,G_{\mathbb{Q}} = \mathrm{Gr}^{Ih}\,G_\ell,$$

where $Ih^m G_\ell := \ker G_\ell \to \mathrm{Out}(\pi_1^\ell / L^{m+1}\pi_1^\ell)$, since $Ih^m G_{\mathbb{Q}} = f^{-1}(Ih^m G_\ell)$.

Representation Theory

V: finite dimensional $\mathbb{Q}_\ell$-vector space.

$\mathbb{G}_m \to \mathrm{Aut}\, V$: a rational action of $\mathbb{G}_m$.

decomposition of $\mathbb{G}_m$-representations

$$\Downarrow$$

$$V = \oplus_{m \in \mathbb{Z}} V_m \text{ (unique)},$$

where $\mathbb{G}_m$ acts on V_m by m-th power multiplication, i.e.
$c \in \mathbb{G}_m$ acts by $c^m \times (-) : V_m \to V_m$.

By Levi-decomposition Theorem, any

$$(*) \qquad 1 \to U \to G \to \mathbb{G}_m \to 1$$

(U unipotent, G algebraic $/\mathbb{Q}_\ell$) has a section $G \leftarrow \mathbb{G}_m$ unique upto the inner automorphisms of U. Thus

$$\mathbb{G}_m \to G \stackrel{conjugation}{\to} \mathrm{Aut}\, U \cong \mathrm{Aut}\, \mathfrak{u}$$

($\mathfrak{u}$: the Lie algebra of U) gives

$$\mathfrak{u} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{u}_m.$$

The sequence (*) is *negatively weighted* if, for some (hence any) section,

$$\mathfrak{u}_m = 0 \quad \text{if } m \leq 0.$$

We equip the *weight filtration* on $\mathfrak{u}$ by

$$W_{-N}\mathfrak{u} := \bigoplus_{m \geq \frac{N}{2}} \mathfrak{u}_m.$$

Facts:

- Is independent of choice of section.

- $\mathfrak{u} = W_0\mathfrak{u} = W_{-1}\mathfrak{u} \supset W_{-2}\mathfrak{u} = W_{-3}\mathfrak{u} \supset \cdots$
  is a central filtration of Lie ideals.
  $\therefore Gr^W\mathfrak{u}$: a graded $\mathbb{Q}_\ell$-Lie algebra.

- $\exists$ A bijection $\exp : \mathfrak{u} \to U$,
  gives a central filtration on group
  $U = W_0\mathfrak{u} = W_{-1}U \supset W_{-2}U = W_{-3}U \supset \cdots$
  with $\mathrm{Gr}^W U \cong \mathrm{Gr}^W \mathfrak{u}$.

- If we have two negative sequences:

$$\begin{array}{ccccccccc} 1 & \to & U & \to & G & \to & \mathbb{G}_m & \to & 1 \\ & & f\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & & \\ 1 & \to & U' & \to & G' & \to & \mathbb{G}_m & \to & 1, \end{array}$$

Then $f : U \to U'$ is strictly weight preserving, i.e.:

$$f(W_N(U)) = f(U) \cap W_N \mathcal{U}' \quad (N \leq -1)$$

($\because$ Decompose $\mathfrak{u}, \mathfrak{u}'$ by $\mathbb{G}_m$-action.
Then f restricts to $f : \mathfrak{u}_m \to \mathfrak{u}'_m$.)
In particular, f: surjective $\Rightarrow$
$\mathrm{Gr}(f) : \mathrm{Gr}\,\mathcal{U} \to \mathrm{Gr}\,\mathcal{U}'$: surjective.

For a profinite group $,$ with continuous homomorphism $\chi : , \to \mathbb{G}_m(\mathbb{Q}_\ell)$, the weighted completion of $,$ relative to χ is the projective limit of the following objects.

$$1 \to U \to G \to \mathbb{G}_m \to 1$$

is negative, with a $\phi : , \to G(\mathbb{Q}_\ell)$ is specified so that:

$$\begin{array}{ccccccccc} & & & & , & = & , & & \\ & & & & \phi \downarrow & \circlearrowleft & \chi \downarrow & & \\ 1 & \to & U & \to & G & \to & \mathbb{G}_m & \to & 1. \end{array}$$

Apply this to , $= G_\ell$ and the cyclotomic character $\chi : G_\ell \to {\mathbb{Z}_\ell}^\times \subset \mathbb{G}_m(\mathbb{Q}_\ell)$.

We get the weighted completion of G_ℓ:

$$\begin{array}{ccccccccc} & & & & G_\ell & = & G_\ell & & \\ & & & & \Phi \downarrow & & \chi \downarrow & & \\ 1 & \to & \mathcal{K}_\ell & \to & \mathcal{A}_\ell & \to & \mathbb{G}_m & \to & 1 \end{array}$$

with $\Phi : G_\ell \to \mathcal{A}_\ell(\mathbb{Q}_\ell)$ Zariski dense in the proalgebraic group $\mathcal{A}_\ell$.

It has the following universality:

For any $G \to \mathbb{G}_m$ with unipotent kernel U and for any $\phi : , \; \to G(\mathbb{Q}_\ell)$ with

$$\begin{array}{ccc} G_\ell & = & G_\ell \\ \phi \downarrow & \circlearrowleft & \chi \downarrow \\ G(\mathbb{Q}_\ell) & \to & \mathbb{G}_m(\mathbb{Q}_\ell), \end{array}$$

we have unique $\varphi : \mathcal{A}_\ell \to G$ such that

$$\begin{array}{ccccccccc} & & & & G_\ell & = & G_\ell & & \\ & & & & \Phi \downarrow & & \chi \downarrow & & \\ 1 & \to & \mathcal{K}_\ell & \to & \mathcal{A}_\ell & \to & \mathbb{G}_m & \to & 1 \\ & & \downarrow & & \varphi \downarrow & \circlearrowleft & \| & & \\ 1 & \to & U & \to & G & \to & \mathbb{G}_m & \to & 1. \end{array}$$

$\mathfrak{k}_\ell$: Lie algebra of $\mathcal{K}_\ell$.

Theorem 1 *$\mathfrak{k}_\ell$ is a pronilpotent free $\mathbb{Q}_\ell$-Lie algebra. We can choose free generators $\sigma_m \in W_{-2m}\mathfrak{k}_\ell$ $(m = 1, 3, 5, \ldots)$, such that their emages freely generate $\mathrm{Gr}\,\mathfrak{k}_\ell$.*

$\because$ (Wt. compl.) ${\mathfrak{k}_\ell}^{ab}$ is the product of

$$H^1_{cts}(G_\ell, \mathbb{Q}_\ell(m))^* \otimes \mathbb{Q}_\ell(m) \qquad (m \geq 1).$$

Soulé: $= \mathbb{Q}_\ell, 0$ (m:odd, even).

$\mathfrak{k}_\ell$: pronilpotent $\Rightarrow$ generated by these.
Free $\Leftarrow H^2(\mathfrak{k}_\ell) = 0$ since

$$H^2_{cts}(G_\ell, \mathbb{Q}_\ell(m)) = 0 \ (m \geq 2).$$

$\because$ Fix $\mathbb{G}_m$-section to $\mathcal{A}_\ell$. Then $\mathfrak{k}_\ell = \prod_{m>0} \mathfrak{k}_{\ell m}$. We can show for $m \geq 1$,

$$(\mathfrak{k}_\ell{}^{ab})_m = H^1_{cts}(G_\ell, \mathbb{Q}_\ell(m))^* = \begin{cases} \mathbb{Q}_\ell & (m : \text{odd}) \\ 0 & (m : \text{even}) \end{cases}$$

(the last step is Soulé's result + some).

$$H^2(\mathfrak{k}_\ell) \hookrightarrow \bigoplus_{m \geq 2} H^2_{cts}(G_\ell, \mathbb{Q}_\ell(m)) = 0$$

(the right 0 is by Soulé + some).

$\mathfrak{k}_\ell$, acted by $\mathbb{G}_m$, is automatically graded:

$$\mathfrak{k}_\ell = \prod_{m>0} (\mathfrak{k}_\ell)_m \cong (\mathrm{Gr}\,\mathfrak{k}_\ell)\hat{}$$

gives a (completed) graded Lie algebra structure on $\mathfrak{k}_\ell$. General theory on nilpotent Lie algebra says that if we take $\sigma_m \in \mathfrak{k}_{\ell m} \twoheadrightarrow \mathfrak{k}^{ab}_{\ell m}$ whose image generates $\mathfrak{k}^{ab}_{\ell m}$ for odd m, then $\sigma_1, \sigma_3, \ldots$ generate $\mathfrak{k}_\ell$ as pronilpotent Lie algebra, and hence also $\mathrm{Gr}\,\mathfrak{k}_\ell$.

Malcev completion of $\pi_1(\mathbb{P}^1_{01\infty})$

Recall $\pi_1 = < x, y >$.

- Consider $A := \lim_{\leftarrow m} \mathbb{Q}_\ell[\pi_1]/J^m$,

  $J = (x-1, y-1)$ is the augmentation ideal.

  A is a complete Hopf-algebra.

- $\mathcal{P} \subset 1 + J$: the set of group-like elements.

  Is a subgroup of $A^\times$.

  $\mathfrak{p} \subset J$: the set of Lie-like elements.

  Is a $\mathbb{Q}_\ell$-Lie algebra, & $\exists$ bijection:

$$\log : \mathcal{P} \subset 1 + J \to \mathfrak{p} \subset J.$$

Proposition

1. $\pi_1 \to A^\times$ continuously extends to $\pi_1^l \to A^\times$, and gives
$$\mathrm{Aut}\,\pi_1^l \to \mathrm{Aut}_{Hopf}\,A \cong \mathrm{Aut}_{Lie}\,\mathfrak{p}.$$

2. $$\begin{array}{ccc} \pi_1^l & \to & A^\times \\ \cup & \square & \cup \\ L^m\pi_1^l & \to & 1+J^m. \end{array}$$

3. $W'_{-2m}\,\mathrm{Aut}\,\mathfrak{p}$
$:= \ker[\mathrm{Aut}\,\mathfrak{p} \to \mathrm{Aut}(\mathfrak{p}/\mathfrak{p}\cap J^{m+1})]$.
Then
$$\begin{array}{ccccc} G_\ell & \to \mathrm{Aut}\,\pi_1^l \to & & \mathrm{Aut}\,\mathfrak{p} \\ \cup & \square & & \cup \\ Ih^m G_\ell & \to & & W'_{-2m}\,\mathrm{Aut}\,\mathfrak{p}. \end{array}$$

Proposition

1. There is a proalgebraic group $\mathcal{A}\mathrm{ut}\,\mathfrak{p}$ over $\mathbb{Q}_\ell$ such that $\mathrm{Aut}\,\mathfrak{p} \cong (\mathcal{A}\mathrm{ut}\,\mathfrak{p})(\mathbb{Q}_\ell)$.

2. $\mathrm{Aut}\,\mathfrak{p}^{ab} \cong GL(2, \mathbb{Q}_\ell)$.
By scaler embedding $\mathbb{G}_m \to GL(2)$, we pull back

$$\begin{array}{ccccccccc} 1 & \to & K & \to & \mathcal{A}\mathrm{ut}\,\mathfrak{p} & \to & \mathcal{A}\mathrm{ut}\,\mathfrak{p}^{ab} & \to & 1 \\ & & \uparrow & \circlearrowleft & \uparrow & \circlearrowleft & \uparrow & & \\ 1 & \to & K' & \to & \mathcal{A}\mathrm{ut}^*\,\mathfrak{p} & \to & \mathbb{G}_m & \to & 1. \end{array}$$

Then, the lower exact sequence is a negative sequence.

3. The weight filtration on K' coincides with W' $(m \geq 1)$:

$$W_{-2m}K' = \ker[\mathrm{Aut}\,\mathfrak{p} \to \mathrm{Aut}\,\mathfrak{p}/(\mathfrak{p} \cap J^m)].$$

Lemma 2

$$G_\ell \to \mathrm{Aut}\,\pi_1^l \to \mathrm{Aut}\,\mathfrak{p} \to \mathrm{Aut}\,\mathfrak{p}^{ab}$$

is $G_\ell \overset{\chi}{\to} \mathbb{Z}_\ell{}^\times \subset \mathbb{G}_m(\mathbb{Q}_\ell) \overset{diag}{\to} GL(2, \mathbb{Q}_\ell)$.

$\because$ $G_\ell \to \mathrm{Aut}\,\pi_1^{ab}$ is χ-multiplication.

Now, by the universality of $G_\ell \to \mathcal{A}_\ell(\mathbb{Q}_\ell)$, we have

$$\begin{array}{ccccccccc}
 & & & & G_\ell & = & G_\ell & & \\
 & & & & \phi\downarrow & & \chi\downarrow & & \\
1 & \to & \mathcal{K}_\ell & \to & \mathcal{A}_\ell & \to & \mathbb{G}_m & \to & 1 \\
 & & \downarrow & & \varphi\downarrow & \circlearrowleft & \| & & \\
1 & \to & K' & \to & \mathcal{A}\mathrm{ut}^*\,\mathfrak{p} & \to & \mathbb{G}_m & \to & 1.
\end{array}$$

Denote the image of $\mathcal{K}_\ell \to K'$ by $Im(\mathcal{K}_\ell)$.

Strictness of filtration in $\mathcal{K}_\ell \twoheadrightarrow Im(\mathcal{K}_\ell)$ says

$$\mathrm{Gr}\,\mathcal{K}_\ell \twoheadrightarrow \mathrm{Gr}\,Im(\mathcal{K}_\ell),$$

so

$\mathrm{Gr}\,Im(\mathcal{K}_\ell)$ is generated by the images

$\sigma_1, \sigma_3, \sigma_5, \ldots$ as Lie algebra.

Only two steps left for proving (G):
Step 1. $\mathrm{Gr}\,Im\mathcal{K}_\ell \cong \mathrm{Gr}^{Ih}\,G_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.
Step 2. (Already commented:)
$\mathrm{Gr}^{Ih}_1\,G_\ell = 0$, hence σ_1 vanishes there.

Lemma 3

$\mathcal{U}$: a prounipotent group over $\mathbb{Q}_\ell$,

$W_{-m}\mathcal{U}$: filtration such that every $\mathrm{Gr}_{-m}\mathcal{U}$ is finite dimensional abel.

, : a profinite group,

$\rho : , \to \mathcal{U}(\mathbb{Q}_\ell)$: a continuous map.

$I_{-m}, := \rho^{-1}(W_{-m}\mathcal{U}(\mathbb{Q}_\ell))$.

If the image of ρ is Zariski dense, then

$$Gr^{I}, \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong (\mathrm{Gr}^W \mathcal{U})(\mathbb{Q}_\ell).$$

($\because$ induction).

For , $= Ih^1G_\ell$ and

$U := Im\mathcal{K}_\ell \supset W_{-1}Im\mathcal{K}_\ell \supset \cdots$, we saw that $I_{-2m}, = Ih^m, .$ Thus, for Step 1, we need only:

$$Ih^1G_\ell \to Im\mathcal{K}_\ell$$

is Zariski dense.

Lemma 4 *Let $\mathcal{S}$ be the weighted completion of $\mathbb{Z}_\ell \to \mathbb{G}_m(\mathbb{Q}_\ell)$. Then (*) in the below is Zariski dense.*

$$\begin{array}{ccccccccc} 1 & \to & Ih^1 G_\ell & \to & G_\ell & \to & {\mathbb{Z}_\ell}^\times & \to & 1 \\ & & (*)\downarrow & & \Phi \downarrow & & \downarrow & & \\ 1 & \to & \mathcal{K} & \to & \mathcal{A}_\ell & \to & \mathcal{S} & \to & 1 \\ & & \downarrow & & \| & & \downarrow & & \\ 1 & \to & \mathcal{K}_\ell & \to & \mathcal{A}_\ell & \to & \mathbb{G}_m & \to & 1 \end{array}$$

Moreover, by

$H^1_{cts}({\mathbb{Z}_\ell}^\times, \mathbb{Q}_\ell(m)) = 0 \quad (m \neq 0)$,

$\mathcal{S} = \mathbb{G}_m$ *and consequently*

$Ih^1 G_\ell \to \mathcal{K} \cong \mathcal{K}_\ell$ *is dense.*

Concluding Remarks

Our proof of (G) says nothing on (F).

(F) says the faithfullness of the "motivic Galois group" action

$$\mathcal{A}_\ell / < \exp(\sigma_1) > \to \mathcal{A}\mathrm{ut}\,\mathcal{P}$$

($\mathcal{P}$: completion of π_1^l. c.f. Belyı's injectivity of $G_{\mathbb{Q}} \to \mathrm{Aut}\,\pi_1\hat{}$).