j-LINE COVERS AND θ-CHARACTERISTICS

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Consider covers $\phi : X \to \mathbb{P}^1_z$ with r branch with this equivalence: $\phi : X \to \mathbb{P}^1_z$ is *D-equivalent* to $\phi' : X' \to \mathbb{P}^1_z$ if there is a smooth deformation from ϕ to ϕ' through r branch point covers.

Deformation means there is a total family $\Phi : \mathcal{T} \to S \times \mathbb{P}^1_z$ with $\text{pr}_S : S \times \mathbb{P}^1_z \to S$ and $\text{pr}_z : S \times \mathbb{P}^1_z \to \mathbb{P}^1_z$ projection maps. Further:

- $\mathcal{T}$ and $\text{pr}_S \circ \Phi$ are smooth; Φ is finite.
- For $s \in S$, a fiber $\text{pr} \circ \Phi_s : \mathcal{T}_s \to s \times \mathbb{P}^1_z$ is an r branch point cover.
- For $s, s' \in S$, $\mathcal{T}_s$ (resp. $\mathcal{T}_{s'}$) is inner (or absolute, reduced) equivalent to ϕ (resp. ϕ').

MOTIVATION

Take all curves, and covers over $\mathbb{C}$. If $G_{\mathbb{Q}}$ action occurs, assume everything defined over $\bar{\mathbb{Q}}$. Let $\phi : X \to \mathbb{P}^1_z$ be a cover. Call it a $(G, \mathbf{C})$ cover (or in the *Nielsen class* $\mathrm{Ni}(G, \mathbf{C})$) if G is the group of its Galois closure $\widehat{\phi} : \hat{X} \to \mathbb{P}^1_z$ and its local monodromy groups are in the set of *conjugacy classes* $\mathbf{C}$.

Problem! Given $(G, \mathbf{C})$:

- Explain the geometric significance of D-equivalence classes of $(G, \mathbf{C})$ covers.
- Find fields of moduli and $G_{\mathbb{Q}}$ orbits of such classes.

MAIN DEFINITION

Up to D-equivalence may assume two covers have the same branch points. For

$$\boldsymbol{g} = (g_1, \dots, g_r) \in G^r,\ \Pi(\boldsymbol{g}) \stackrel{\text{def}}{=} g_1 \cdots g_r.$$

Then, $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C})$ means:

$$\boldsymbol{g} \in \mathbf{C},\ \langle \boldsymbol{g} \rangle = G \text{ and } \Pi(\boldsymbol{g}) = 1.$$

Two D-equivalent covers are in the same Nielsen class $\text{Ni}(G, \mathbf{C})$ computed from classical generators on the r-punctured sphere.

Given $(G, \mathbf{C})$, consider $\psi_H : H \twoheadrightarrow G$, and a choice of conjugacy classes $\mathbf{C}_H$: r conjugacy classes lifting to H those of $\mathbf{C}$.

For $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C})$, consider the $(H, \mathbf{C}_H)$*-lifting invariant*: $s_{H,\mathbf{C}_H}(\boldsymbol{g})=$

$$\{\Pi(\boldsymbol{g}^*) \mid \boldsymbol{g}^* = \boldsymbol{g} \bmod \ker(\psi_H),\ \boldsymbol{g}^* \in \mathbf{C}_H,\ \langle \boldsymbol{g}^* \rangle = H\}.$$

MAIN TOOL

Use $\mathcal{Y}_{\boldsymbol{g}}$ for the collection of $s_{H,\mathbf{C}_H}(\boldsymbol{g})$ running over all $(H,\mathbf{C}_H)$ to separate D-inequivalent elements in Ni$(G,\mathbf{C})$. Recall the *Hurwitz monodromy group* H_r from [Fri99, Talk 2].

Lemma 1. *$\mathcal{Y}_{\boldsymbol{g}}$ is an H_r invariant:*

$$s_{H,\mathbf{C}_H}(\boldsymbol{g}) = s_{H,\mathbf{C}_H}((\boldsymbol{g})Q) \text{ for } Q \in H_r,$$

any $(H,\mathbf{C}_H)$. It is a D-equivalence invariant.

Definition 2. Call $\boldsymbol{g}$ and $\boldsymbol{g}'$ in Ni$(G,\mathbf{C})$ *Nielsen separated* if the collection $\mathcal{Y}_{\boldsymbol{g}}$ differs from $\mathcal{Y}_{\boldsymbol{g}'}$.

Question 3. Does this *reasonably (practically, significantly)* separate D-equivalence classes?

GROUP THEORY THAT WORKS

Here is a situation that produces several D-equivalence classes (H_r orbits) in a Nielsen class. Let $\psi : H \to\!\!\!\to G$ be a Frattini central extension with $\ker(\psi)$ of order prime to the orders of elements in $\mathbf{C}$. Choose $\mathbf{C}_H$ to be the lifted conjugacy classes of $\mathbf{C}$ *with the same orders*.

Lemma 4. *For* $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$*,* $s_{H,\mathbf{C}_H}(\boldsymbol{g}) = s(\boldsymbol{g})$ *is a single element. Assume* $\mathrm{Ni}(G, \mathbf{C}) \neq \emptyset$ *and*

- *conjugacy classes in* $\mathbf{C}$ *appear many times.*

Then $\{s(\boldsymbol{g}) \mid \boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})\} = \ker(\psi)$*. So, there are at least* $|\ker(\psi)|$ *orbits for* H_r *on* $\mathrm{Ni}(G, \mathbf{C})$*.*

SMALL $H \to\!\!\to G$ EXAMPLE

Example 5. Let $n \geq 4$, $A_n = G$, and $\mathbf{C} = \mathbf{C}_{3^r}$: $r \geq n-1$ repetitions of conjugacy class of 3-cycles. Take $\mathbb{Z}/2 \to \hat{A}_n \to A_n$ the Spin cover of A_n. With $H = \hat{A}_n$, hypotheses of Lem. 4 hold. For $\boldsymbol{g} \in \text{Ni}(A_n, \mathbf{C}_{3^r})$, $s(\boldsymbol{g}) \in \{\pm 1\}$.

Nielsen separation components, Ex. 5 [Fri96]

Genus at (n,r) of degree n cover: $g_{n,r} = r - n + 1$
of the Galois cover: $\hat{g}_{n,r} = \frac{(r-3)n!}{6}$

$\overset{g \geq 1}{\longrightarrow}$	$\otimes \oplus$	$\otimes \oplus$	$\overset{1 \leq g}{\longleftarrow}$
$\overset{g = 0}{\longrightarrow}$	$\otimes$	$\oplus$	$\overset{0 = g}{\longleftarrow}$
$n \geq 4$	n even	n odd	$4 \leq n$

Locations containing symbols $\otimes$ or $\oplus$ attach to pair (n,r). Labels for rows are by genuses of degree n covers of $\mathbb{P}^1_z$. *Each symbol is one component:* $\otimes$ means lifting invariant is value -1; $\oplus$ that it is $+1$.

REMAINING TOPICS

Nielsen Separation uses complicated witnessing pairs $(H, \mathbf{C}_H)$. In the last example, $(H, \mathbf{C}_H)$ witnessing separation is a *small* cover of $(G, \mathbf{C})$. Remaining topics relate to *Frattini covers*.

- Relation of $H = \hat{A}_n$, $G = A_n$ to θ functions.
- Using $\hat{A}_n$ when $G \le A_n$: *Serre's Theorem*.
- Complete D-equivalence separation for $\mathbf{C}$ *large*: *Conway-Parker Theorem* ([CP87], [FV91, App]).
- How the universal p-Frattini cover of G shows Conway-Parker isn't enough: Infinitely many groups with $\mathbf{C}$ fixed.

I cover these topics until I run out of time!

θ-CHARACTERISTICS

Definition 6. Let X be a compact Riemann surface, and ω a meromorphic differential on X. Suppose $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$ is a coordinate chart for X and $\phi_\alpha^*(\omega)(z_\alpha) = (f_\alpha(z_\alpha))^2 dz_\alpha$. Then, $f_\alpha(z_\alpha)\sqrt{dz_\alpha}$ is a *half-canonical differential*. It defines a (meromorphic) section of some sheaf, attached to a divisor $(\sqrt{d\omega})$: $2(\sqrt{d\omega})$ is in the *canonical class*.

Example 7. Let $\phi : X \to \mathbb{P}^1_z$ be a cover with *odd order ramification*. Then, $\sqrt{d\,\phi}$ is a half-canonical differential. If $\gamma \in \mathrm{PSL}_2(\mathbb{C})$, $(\sqrt{d(\gamma \circ \phi)})$ is *linearly equivalent* to $(\sqrt{d\phi})$ on X.

APPEARANCE OF θ-FUNCTIONS

Let $[D]$ be a θ-characteristic. Denote divisor classes of degree d on X by $\text{Pic}^d(X)$. There is a natural map $X^{g-1} \to \text{Pic}^{g-1}(X)$. Denote its image by Θ_X. Then, $\Theta_X - [D]$ defines a divisor in $\text{Pic}^0(X)$: *The Θ divisor* attached to $[D]$. Let $L(D) = \{f \in \mathbb{C}(X) \mid (f) + D \geq 0\}$.

According to $\dim_{\mathbb{C}} L(D) \bmod 2$, $[D]$ is called *even* or *odd*. If $[D]$ is odd, the corresponding θ function is odd. So it is zero at the origin. The following is a special case of [Ser90b]

Let $\hat{\phi}_{\boldsymbol{p}} : \hat{X}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ run over equivalence classes of covers parametrized by $\boldsymbol{p} \in \mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{in,rd}}$.

Theorem 8. *$(\sqrt{d\hat{\phi}_{\boldsymbol{p}}})$ is even if and only if $\boldsymbol{p}$ is on the $\oplus$ component of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{rd}}$.*

A THETA NULL

Example: If H_r orbit on $\text{Ni}(A_n, \mathbf{C}_{3^r})$ contains *H-M reps.* ([Fri99, Talk 4]), it gives $\oplus$ component of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{rd}}$. *Moduli map*:

$$\mu_{n,r} : \boldsymbol{p} \in \mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{in}} \mapsto [\hat{X}_{\boldsymbol{p}}] \in \mathcal{M}_{\hat{g}_{n,r}}.$$

Let $\theta_{\boldsymbol{p}}$ be the *theta function* (almost canonical) on the *universal cover* of $\text{Pic}^0(\hat{X}_{\boldsymbol{p}})$ ($\boldsymbol{p}$ on the $\oplus$ component). Let $\mathbf{0}$ be the origin of $\text{Pic}^0(\hat{X}_{\boldsymbol{p}})$.

Theorem 9. *If $\mu_{n,r}$ is generically surjective, then $\Theta_X - [D_{\boldsymbol{p}}]$ doesn't contain the origin of* $\text{Pic}^0(\hat{X}_{\boldsymbol{p}})$ *for most $\boldsymbol{p}$ in $\oplus$ component. Also, $\theta_{\boldsymbol{p}}(\mathbf{0})$ is locally nonconstant as a function of $\boldsymbol{p}$.*

Yet, $\mu_{n,r}$ isn't often generically surjective. Let $\theta_{\boldsymbol{p}}(\mathbf{0})$ be its value at the origin.

A VARYING THETA NULL

Theorem 10 (Tentative). *$\theta_{\boldsymbol{p}}(\mathbf{0})$ is locally non-constant.*

Idea! So, $\theta_{\boldsymbol{p}}(\mathbf{0})$ is an *automorphic function* on $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{rd}}$. (Case: $r = 4$, $n = 5$, $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{rd}}$ a j-line cover.)

There is a $Y \to \hat{X}_{\boldsymbol{p}}$ unramified cover with $Y \to \mathbb{P}^1_z$ having group $\tilde{A}_n$: from an A_n-equivarient 2-division point on $\text{Pic}^0(\hat{X}_{\boldsymbol{p}}) \setminus \{\boldsymbol{p}\}$. As $\boldsymbol{p}$ approaches an H-M cusp, this cover degenerates. Gives a pole of $\theta_{\boldsymbol{p}}(\mathbf{0})$ (inspired by [IN97]).

SERRE'S LIFTING INVARIANT THEOREM

For g with disjoint cycle lengths $s_1, \ldots, s_t$, let $\omega(g) = \sum(s_i^2 - 1)/8$.

Proposition 11 ([Ser90a]). *Assume* $G \le A_n$ *and entries of* $\boldsymbol{g} = (g_1, \ldots, g_r) \in \text{Ni}(G, \mathbf{C})$ *have odd order. Let* $\hat{\boldsymbol{g}} \in (\hat{A}_n, \mathbf{C})$ *lift* $\boldsymbol{g}$*. Suppose:*

- *Transitivity:* $\langle \boldsymbol{g} \rangle$ *is transitive in* A_n.
- *Genus 0 condition:* $\sum_{i=1}^r \text{ind}(g_i) = 2(n-1)$.

Then, $s(\boldsymbol{g}) \stackrel{\text{def}}{=} \prod_{i=1}^r g_i^* = (-1)^{\sum_{i=1}^r \omega(g_i)}$.

Exercise Lifting Computation:

- $\boldsymbol{g}_1 = ((1\,2\,3), (1\,3\,4), (1\,4\,3))$: $s(\boldsymbol{g}_1) = -1$.
- $\boldsymbol{g}_2 = ((1\,2\,3\,4\,5), (1\,3\,5\,2\,4), (3\,5\,1), (2\,4\,1))$: $s(\boldsymbol{g}) = -1$ [BFr99, §5].

D-EQUIVALENCE CLASSES: **C** LARGE

Let $M_G = H^2(G, \mathbf{C}^*)$ be the *Schur-Multiplier* of G. Suppose $\mathrm{C}'_1, \dots, \mathrm{C}'_t$ are conjugacy classes of G. Let $R \to G$ be a representation cover of G: *Central Frattini cover* having M_G as kernel. Choose $\hat{\mathrm{C}}'_j$ to be a lift of C'_i to R. For $g \in G$ let $\hat{g}$ be any lift of g to R. Denote the set

$$\{(\hat{g}_i, \hat{g}_j) \mid g_i \in \hat{C}'_i, g_j \in \hat{C}'_j,\ 1 \le i, j \le t\} \cap M_G$$

by $R_{\mathbf{C}'}$. The following is in [FV91, App] based on a preprint of Conway-Parker.

Theorem 12. *Given* $(G, \mathbf{C})$*, assume:*

- *Each conjugacy class in* **C** *is from the list* $\mathrm{C}'_1, \dots, \mathrm{C}'_t$.
- *Each* C'_i *appears in* **C** *many times (how many?).*

The number of components of M_r *acting on* $\mathrm{Ni}(G, \mathbf{C})$ *is exactly* $|M/\langle R_{\mathbf{C}'} \cap M\rangle|$. *Branch cycle lemma computes their fields of definition.*

UNIVERSAL p-FRATTINI ${}_p\tilde{G}$

Recall the *universal p-Frattini cover* ${}_p\tilde{G}$ of G and its characteristic quotients $\{G_k\}_{k=0}^{\infty}$ [Fri99, Talk 3,4]. Let Sch_k be the exponent p quotient of the Schur multiplier of G_k.

Lemma 13. *Let* $\ker_k$ *be the kernel of* ${}_p\tilde{G} \to G_k$*. A* G_k *quotient of* $\ker_k / \ker_{k+1}$ *is isomorphic to* Sch_k*. Also,* $|\text{Sch}_{k+1}| \ge |\text{Sch}_k|$*,* $k \ge 0$*.*

Example 14 (Use of [CP87]). Assume

- C is a fixed p' conjugacy class of G.
- G has nontrivial p Schur multiplier.

Let $\mathbf{C}_r$ be r repetitions of C. Then, there is r_k with $\text{Ni}(G_j, \mathbf{C}_{r_k})$ having *exactly* $|\text{Sch}_j|$ orbits under H_{r_k}, $j \le k$.

COMPONENTS FROM FIXED $\mathbf{C}$

Use assumptions of Example 14.

- What do M_r orbits of $\text{Ni}(G_k, \mathbf{C})$ look like if r is fixed, and k is large?
- Let $G_k = {}_2\tilde{A}_5/\ker_k$ be the k-th characteristic quotient of ${}_2\tilde{A}_5$. What do M_4 orbits of $\text{Ni}(G_k, \mathbf{C}_{34})$ look like if k is large.

Recall: When $r = 4$, each M_4 orbit is a curve whose closure covers the j-line. Denote the M_4 orbits on $\text{Ni}(G_k, \mathbf{C}_{34})$ by $\text{Ni}(G_k, \mathbf{C}_{34})/M_4$. The following is from [Fri99, Talk 4].

Theorem 15. $|\text{Ni}(G_0, \mathbf{C}_{34})/M_4| = 1$ *and the curve component has genus 0.*

Also, $|\text{Ni}(G_0, \mathbf{C}_{34})/M_4| = 2$ *and the curve components have genus 12 and genus 9. Only the genus 12 component has real points.*

Web: http://www.math.uci.edu/~mfried/#ret, Overview of RET.

[BFr99] P. Bailey and M. Fried, *The Hurwitz monodromy group H_4 and components of an A_5 Modular Tower*, 60 page preprint, Oct. 1999.

[CP87] J.H. Conway and R. Parker, *Braid group orbits and Schur multipliers*, preprint, Spring 1987.

[FrV89] M.D. Fried and H. Völklein, *Unramified abelian extensions of Galois covers*, *Proceedings of Symposia in Pure Mathematics, Part 1* **49** (1989), 675–693.

[Fri95] M.D. Fried, *Introduction to Modular Towers: Generalizing the relation between dihedral groups and modular curves*, vol. 186, 1995, Cont. Math series, Recent Dev. in IGP, pp. 111–171.

[Fri96] M.D. Fried, *Alternating groups and lifting invariants*, Preprint as of 07/01/96 (1996), 1–34.

[Fri99] M.D. Fried, *Talks 1-4 of this MSRI Semester on website.*

[FV91] M.D. Fried and H. Völklein, *The inverse Galois problem and rational points on moduli spaces*, Math. Annalen **290** (1991), 771–800.

[IN97] , Y. Ihara and H. Nakamura, *On deformation of maximally degenerate stable marked curves and Oda's problem*, J. Reine. Angew. Math **487** (1997), 125–151.

[Ser90a] J.-P. Serre, *Relêvements dans* $\tilde{A}_n$, C. R. Acad. Sci. Paris **311** (1990), 477–482.

[Ser90b] J.-P. Serre, *Revêtements a ramification impaire et thêta-caractéristiques*, C. R. Acad. Sci. Paris **311** (1990), 547–552.