RECONSTRUCTION OF FIELDS FROM

ABSOLUTE GALOIS GROUPS

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Part I: Local Fields

A joint work with Ivan Fesenko

Math. Res. Letters **6** (1999), 345–356

Part II: Global Aspects

Problem: K, K_0 fields, K_0 local

$$G_K \cong G_{K_0} \quad \Longrightarrow \quad K = ??$$

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(0) $K_0 = \mathbb{C}$

$G_K \cong G_{\mathbb{C}} \quad \Longleftrightarrow \quad K$ separably closed

(the fundamental theorem of algebra)

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(1) $K_0 = \mathbb{R}$

$G_K \cong G_{\mathbb{R}} \quad \Longleftrightarrow \quad K$ real closed

E. Artin – O. Schreier 1927

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(2) $G_K \cong G_{K_0}$, $[K_0 : \mathbb{Q}_p] < \infty$

$\Longleftrightarrow$ K p-adically closed

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ henselian valuation v on K such that

$|\bar{K}_v| < \infty$, char $\bar{K}_v = p$,

$0 \ \to \ \mathbb{Z} \ \to \ ,_v \ \to \ \Delta \ \to \ 0$ exact

$v(p)$ divisible

Neukirch '69, Pop '88,'95, E. ('95; $p \neq 2$), Koenigsmann ('95; all p)

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(3) $G_K \cong G_{\mathbb{F}_q((t))}$, $q = p^m$ $\iff$ $K = ??$

Expect: $\exists$ henselian valuation v on K with

- value group $,_v$ "close" to $\mathbb{Z}$ and
- residue field $\bar{K}_v$ "close" to $\mathbb{F}_q$

Example (*Koch '67*):

$$G_{\mathbb{F}_q((t))} \;\cong\; \hat{F}_{\aleph_0}(p) \rtimes \langle \sigma, \tau \mid \sigma\tau\sigma^{-1} = \tau^q \rangle_{\text{profinite}}$$

↑

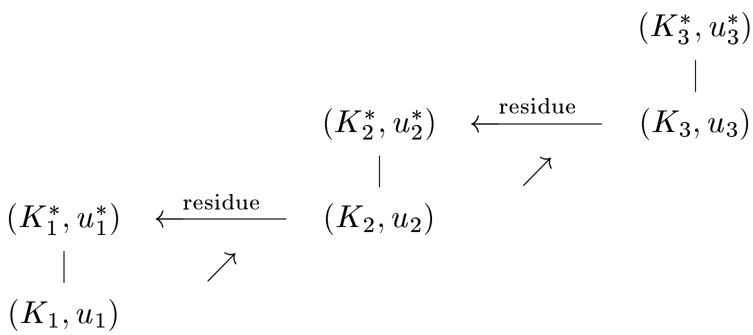
universal action

Example (A): Construct inductively (K_r, u_r) as follows:

(K_1, u_1) = henselization of $\mathbb{F}_q(t_0)$ at (t_0)

(K_r^*, u_r^*) = maximal totally tamely ramified extension of (K_r, u_r)

(K_{r+1}, u_{r+1}) = henselization of $K_r^*(t_r)$ at (t_r)



$\Longrightarrow \quad G_{K_r} \cong G_{\mathbb{F}_q((t))}$

u_r henselian , , ${}_{u_r} \cong \mathbb{Z}$

$(\overline{K_{r+1}})_{u_{r+1}} = K_r^*$ *non-perfect!*

$G_{K_r^*} \cong \hat{F}_{\aleph_0}(p) \times \hat{\mathbb{Z}} \not\cong \hat{\mathbb{Z}}$

Example (B):

Take (K_r, u_r) as in Example (A)

Set $\quad w_r^* = u_1^* \circ u_2^* \circ \cdots \circ u_{r-1}^* \circ u_r^*$

$w_r = \mathrm{Res}_{K_r}(w_r^*)$.

$\Longrightarrow \quad (K_r, w_r)$ henselian

$(\overline{K_r})_{w_r} = \mathbb{F}_q$,

$$, \; {}_{w_r}/l \cong \begin{cases} \mathbb{Z}/l & \text{for } l \neq p \text{ prime} \\ (\mathbb{Z}/p)^r & \text{for } l = p \text{ prime} \end{cases}$$

Example (C): *Examples in characteristic* 0 *[E., '95]*

F = arbitrary field of characteristic p

$E = (W(F_{\text{ins}}))$

$\exists$ *split* epimorphism $G_E \to G_{F_{\text{ins}}} \cong G_F$

(Kuhlmann–Pank–Roquette)

K = fixed field of the image of a section

$\Longrightarrow \qquad \operatorname{char} K = 0, \quad G_K \cong G_F$

Example (D):

Fields of Norms (Fontaine – Winterberger)

E = finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$

K = arithmetically profinite extension of E

$\Longrightarrow \qquad G_K \cong G_{\mathbb{F}_q((t))}.$

THEOREM 1:

Suppose: $G_K \cong G_{\mathbb{F}_q((t))}$.

Then there exists a henselian valuation v on K s.t.:

(1) $\forall l \neq p$ prime: $,_{v}/l \cong \mathbb{Z}/l$

(2) $\operatorname{char} \bar{K}_v = p$

(3) $G_{\bar{K}_v}(p') \cong \hat{\mathbb{Z}}(p')$ $(= \prod_{l \neq p} \mathbb{Z}_l)$

(4) $\mathrm{Syl}_p(G_{\bar{K}_v})$ is a *non-trivial* free pro-p group of rank $\leq |\bar{K}_v|$

(5) $\operatorname{char} K = 0 \implies ,_{v}/p = 0$ and $\bar{K}_v$ is perfect

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Construction of valuations from K-theory

Jacob '81, Ware '81, Arason–Elman–Jacob '87, Hwang–Jacob '95, E. '99

Alternative approaches: *Bogomolov '92, Koenigsmann '95*

Theorem: Suppose: E field , $l \neq \operatorname{char} E$ prime , $\langle -1, (E^\times)^l \rangle \ \leq \ T \ \leq \ E^\times$;

(a) $\forall x \in E \setminus T \ \forall y \in T \setminus (E^\times)^l$: $\{x, y\} \neq 0$ in $K_2^M(E)$

(b) $\forall x, y \in E^\times$: x, y $\mathbb{F}_l$-linearly independent mod T

$\Longrightarrow \quad \{x, y\} \neq 0$ in $K_2^M(E)$.

Then: $\exists$ valuation v on E such that:

- $\operatorname{char} \bar{E}_v \neq l$
- $\dim_{\mathbb{F}_p}(\,_v/l) \geq \dim_{\mathbb{F}_l}(E^\times/T) - 1$
- either $\dim_{\mathbb{F}_l}(\,_v/l) = \dim_{\mathbb{F}_l}(E^\times/T)$ or $\bar{E}_v \neq \bar{E}_v^l$.

Corollary: Suppose :

E a field, $l \neq \operatorname{char} E$ prime, $-1 \in (E^\times)^l$, and

$\bigwedge^2(E^\times/l) \xrightarrow{\sim} K_2^M(E)/l$ naturally.

Then $\exists$ valuation v on E such that:

- $\operatorname{char} \bar{E}_v \neq l$
- $\dim_{\mathbb{F}_l}(,\ _v/l) \geq \dim_{\mathbb{F}_l}(E^\times/l) - 1$
- either $\dim_{\mathbb{F}_l}(,\ _v/l) = \dim_{\mathbb{F}_l}(E^\times/l)$ or $\bar{E}_v \neq \bar{E}_v^l$

The construction of v:

One chooses $T \leq H \leq E^\times$ appropriately

(in the Corollary: $T = (E^\times)^l$)

$O^- = \{x \notin H \mid 1 - x \in T\}$

$O^+ = \{x \in H \mid xO^- \subseteq O^-\}$

$O = O^- \cup O^+$ is a valuation ring with the desired properties!

In our case:

Suppose $\sigma\colon G_K \xrightarrow{\sim} G_{\mathbb{F}_q((t))}$, $q = p^m$.

Take $l \neq p$ prime and E_l/K finite and separable with $\mu_l \subseteq E_l, E_l'$ ($\mu_4 \subseteq E_l, E_l'$ if $l = 2$).

Then: $\quad G_{E_l}(l) \cong \langle \sigma, \tau \mid \sigma\tau\sigma^{-1} = \tau^q \rangle_{\text{pro}-l}$

$$\Longrightarrow H^1(G_{E_l}(l), \mathbb{Z}/l) \cong (\mathbb{Z}/l)^2$$
$$H^2(G_{E_l}(l), \mathbb{Z}/l) \cong \textstyle\bigwedge^2 H^1(G_{E_l}(l), \mathbb{Z}/l) \text{ (via } \cup)$$

$$\Longrightarrow E_l^\times/l \cong (\mathbb{Z}/l)^2 \qquad \textit{(Kummer theory)}$$
$$K_2^M(E_l)/l \cong \textstyle\bigwedge^2(E_l^\times/l) \qquad \textit{(Merkur'ev–Suslin)}$$

$\Longrightarrow$ $\exists$ valuation u_l on E_l such that $\operatorname{char}(\overline{E_l})_{u_l} \neq l$,

$\dim_{\mathbb{F}_l}(\,{}_{u_l}/l) = 1$, $(\overline{E_l})_{u_l} \neq (\overline{E_l})_{u_l}^l$.

$\Longrightarrow u_l$ is henselian

$\Longrightarrow v_l = \operatorname{Res}_K u_l$ is henselian

$\Longrightarrow O_v = \bigcap_{l \neq p} O_{v_l}$ is henselian and

$\forall l \neq p :$ $\operatorname{char} \bar{K}_v \neq l$ and $\dim_{\mathbb{F}_l}(\,{}_v/l) = 1$

Proposition (*E., '95*):

Suppose: (E, u) valued field

$l \neq \operatorname{char} E, 2$ prime (or $l = 2$, $\sqrt{-1} \in E$)

$\bar{E}_u \neq \bar{E}_u^l$

$\sup_{[F:E]<\infty} \operatorname{rank} G_F(l) < \infty$.

Then u is henselian .

Proposition (*Endler–Engler '77*):

Suppose: v, v' valuations on a field K

v henselian

$\bar{K}_{v'}$ not algebraically closed .

Then either $O_v \subseteq O_{v'}$ or $O_v \subseteq O_{v'}$.

Claim: $\operatorname{char} \bar{K}_v = p$

Key Fact:

The (first) ramification group V of $G_{\mathbb{F}_q((t))}$ intersects every non-trivial normal closed subgroup of $G_{\mathbb{F}_q((t))}$.

Let $T = G_{K_{v,\mathrm{ur}}}$ and take $l \neq p, \operatorname{char} \bar{K}_v$ prime.

Then: $\quad \mathrm{Syl}_l(T) \cong \mathbb{Z}_l$

$\Longrightarrow \quad T \neq 1$ and is normal in G_K

$\Longrightarrow \quad \sigma(T) \neq 1$ and is normal in $G_{\mathbb{F}_q((t))}$

$\Longrightarrow \quad \sigma(T) \cap V \neq 1$ and is normal in V $(\cong \hat{F}_{\aleph_0}(p))$

$\Longrightarrow \quad \sigma(T) \cap V$ non-abelian, pro-p

$\Longrightarrow \quad \mathrm{Syl}_p(T)$ non-abelian

$\Longrightarrow \quad \operatorname{char} \bar{K}_v = p \qquad \square$

II. FINITELY GENERATED FIELDS

Grothendieck's anabelian conjecture - 0-dim case:

Theorem *(Pop)*:

Let K, K' be finitely generated infinite fields.

Let $\sigma: G_K \xrightarrow{\sim} G_{K'}$.

Then there is a unique $\varphi: \tilde{K}' \xrightarrow{\sim} \tilde{K}$ inducing σ.

- K, K' global – *Neukirch '69, Ikeda '77, Iwasawa Uchida '77+*
- K, K' of transcendence degree 1 over $\mathbb{Q}$ – *Pop '90, Spiess '96*
- K, K' arbitrary – *Pop '95 +*

Definition:

$$\dim(K) = \begin{cases} \mathrm{tr.deg}(K/\mathbb{F}_p) & \text{if } \mathrm{char}\, K = p > 0 \\ \mathrm{tr.deg}(K/\mathbb{Q}) + 1 & \text{if } \mathrm{char}\, K = 0 \end{cases}$$

A valuation v on K is 1-*defectless* if

$$\dim(K) = \dim(\bar{K}_v) + 1 \quad .$$

The Local Correspondence:

Let L, L' be separable extensions of K, K', respectively, with $\sigma(G_L) = G_{L'}$.

Then:

L is a henselization of K with

respect to a 1-defectless valuation

$\Updownarrow$

L' is a henselization of K' with

respect to a 1-defectless valuation

Earlier Approaches:

Hasse Principles + Model Theory

(Brauer–Hasse–Noether, Tate–Lichtenbaum–Saito,

Kato – Jannsen)

An "algebraic" proof ??

Definition: Let L be a field of dimension d and $p \neq \operatorname{char} L$ a prime number. L is p**-divisorial** if there exist $L \subseteq E \subseteq M \subseteq L_{\text{sep}}$ such that:

(1) M/L is Galois

(2) $\operatorname{Syl}_p(G_M) \cong \mathbb{Z}_p$

(3) ${}_p(G_L) = d + 1$

(4) Either $d = 1$ or $\operatorname{Gal}(M/L)$ has no normal pro-solvable closed subgroups $\neq 1$

(5) $\forall F/E$ finite separable:

$H^1(G_F, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^{d+1}$

$H^2(G_F, \mathbb{Z}/p) \cong \bigwedge^2 H^1(G_F, \mathbb{Z}/p)$ via $\cup$.

Theorem 2:

Suppose :

K finitely generated field,
L/K separable algebraic extension,
$p \neq \operatorname{char} K$.

TCAE :

(a) L is a henselization of K with respect to a 1-defectless valuation

(b) L is a minimal p-divisorial separable algebraic extension of K.

Condition (b) is Galois–theoretic !

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