Preface

This volume is an outgrowth of the research project "The Inverse Galois Problem and its Application to Number Theory" which was carried out in three academic years from 1999 to 2001 with the support of the Grant-in-Aid for Scientific Research (B) (1) No.11440013. In September, 2001, an international conference "Galois Theory and Modular Forms" was held at Tokyo Metropolitan University after some preparatory workshops and symposia in previous years. The title of this book came from that of the conference, and the authors were participants of those meetings. All of the articles here were critically refereed by experts. Some of these articles give well prepared surveys on branches of research areas, and many articles aim to bear the latest research results accompanied with carefully written expository introductions.

When we started our research project, we picked up three areas to investigate under the key word "Galois groups"; namely, "generic polynomials" to be applied to number theory, "Galois coverings of algebraic curves" to study new type of representations of absolute Galois groups, and explicitly described "Shimura varieties" to understand well the Galois structures of some interesting polynomials including Brumer's sextic for the alternating group of degree 5. The topics of the articles in this volume are widely spread as a result. At a first glance, some readers may think this book somewhat unfocussed. The editors, however, believe it is of interest to present this collection of articles because they discuss those topics each of which could trace its source back to a spring of advanced ideas on Galois theory.

On the Inverse Galois Problem, Hilbert gave a remarkable result on the symmetric groups over the rational number field at the beginning of the last century. Then E. Noether proposed the so called "Noether's Problem" which became one of basic problems of modern algebra. If we go back to the 19th century, it was Abel who found new Abelian equations other than cyclotomic ones with moduli and values at division points of periods of elliptic functions. He was also interested in characterizing Abelian polynomials which were algebraically solvable. Then, for example, Kronecker formulated the Kronecker-Weber Theorem, and proposed 'the fondest dream in his youth' to determine all of the Abelian extensions of the rational number field, and those of imaginary quadratic fields by special values of the exponential function, and elliptic, and elliptic modular functions. The theory of complex multiplications of elliptic curves was certainly a starting point towards the later vast studies of automorphic forms and of Shimura varieties. The reader will find several articles in this book which concern with the beautiful theory and examples explaining/indicating mysterious phenomena arising in various arithmetic functions and varieties related to modular forms.

In the 1920's, algebraic number theory succeeded in establishing class field theory, and, as a result, opened the door to non-Abelian worlds. These were there behind the work of Hilbert and the abstractions of E. Noether stated above. Later in the latter half of the previous century, when the classification of finite simple groups was realized, we see a quite new development of the Inverse Galois Problem. For example, Thompson gave a stimulating plan for finite simple groups and their automorphism groups. Matzat is certainly one of the most active mathematicians who took this course. The editors are very happy to include his substantial article in the book, which establishes a foundation for the recent active studies of the differential Galois theory.

Investigation of structures of Galois groups is also an important stream in the above history of the Inverse Galois Problem. In 1953, Iwasawa showed that the commutator subgroup of the Galois group of the maximal solvable extension over the rational number field is a free pro-finite-solvable group with a countable number of generators. Then Shafarevich's conjecture claims that the commutator subgroup of the Galois group of the algebraic closure of the rational number field should be a free profinite group with a countable number of generators. Recently, our knowledge on structures of profinite fundamental groups has been vastly progressed under the name of "anabelian geometry". The reader will find an excellent paper by S. Mochizuki, where a new frontier has been pioneered for our understanding the arithmetic fundamental groups of curves over p-adic fields.

Classification of Galois extensions is another important branch of the Inverse Galois Problem. Here let us mention a problem which is concerned with "generic polynomials" for finite cyclic groups. When the local class field theory was established by the global class field theory, E. Noether immediately pointed out that the converse should be the right way; namely, the global class field theory should be constructed on the basis of the much simpler local class field theory. As a result of both theories, we know that every Abelian extension of a local completion field can be obtained by an Abelian extension of the base global field. This has not, however, been proved yet in any simple and direct manners. One way may be to utilize "generic polynomials" for finite cyclic groups over some small algebraic number fields. It should be noted that there exist no generic polynomials for cyclic group of order 8 over the rational number field.

The editors would like to express their hearty gratitude to all the participants and the speakers at our meetings, to the authors of articles here and the referees behind them, for their intimate cooperation at various stages of our modest project, all of which have been indispensable for collecting resulted fruits successfully in the present proceedings volume. Special thanks should also go to Dr. Yasuhiro Kishi and Mr. Masafumi Imaoka for their efforts in arranging the articles of the volume in the publisher's format.

March, 2003
Katsuya Miyake,
Ki-ichiro Hashimoto,
Hiroaki Nakamura