Joint MSJ–RIMS Conference

The 3rd Seasonal Institute of the Mathematical Society of Japan
and
RIMS Workshop 2010

Development of Galois–Teichmüller Theory and Anabelian Geometry

Dates:
October 25–30, 2010

Venue:
RIMS, Kyoto University
Kyoto, Japan

Organizing committee:
Hiroaki Nakamura (Chair)
Florian Pop
Leila Schneps
Akio Tamagawa

Joint MSJ-RIMS Conference

"Development of Galois -Teichmüller Theory and Anabelian Geometry"

Date: October 25—30, 2010／Place: RIMS, Kyoto University, Room 420
Organizers: H.Nakamura (Chair), F.Pop, L.Schneps, A.Tamagawa

October 25 Monday

9:30--9:45	Takashi Tsuboi (President of the Mathematical Society of Japan): Opening remarks
9:45--9:50	General information from the organizing committee
10:00--11:00	P. Cartier (IHÉS) : Towards Grothendieck's "Dessins d'Enfants"
11:30--12:20	Y. André (CNRS) : Introduction to tempered anabelian geometry
-- Lunch --	
14:30--15:20	P. Lochak (CNRS): Grothendieck-Teichmüller theory from a topological viewpoint
-- Tea --	
16:00--16:50	M. Asada (Kyoto Institute of Technology), H. Nakamura (Okayama University), N. Takao (RIMS, Kyoto University), H. Tsunogai (Sophia University): Easy walking in GT theory and anabelian geometry, I
17:10--18:00	M. Asada(Kyoto Institute of Technology), H. Nakamura (Okayama University), N. Takao (RIMS, Kyoto University), H. Tsunogai (Sophia University): Easy walking in GT theory and anabelian geometry, II

October 26 Tuesday

10:00--11:00	L. Schneps (CNRS): Survey of the theory of multiple zeta values
11:30--12:20	I. Marin (Université Paris Diderot): Rigidity characters for the Grothendieck-Teichmüller group
-- Lunch --	
14:30--15:20	A. Schmidt (Heidelberg University): Motivic aspects of anabelian geometry
-- Tea --	
16:00--16:50	J. Stix (Heidelberg University) : On the passage from local to global in Grothendieck's section conjecture
17:10--18:00	J. Ellenberg (University of Wisconsin) : Ihara's braid group and fundamental groups of random curves

October 27 Wednesday

10:00--11:00	K. Wickelgren (Harvard University) : Etale π_1 obstructions to rational points
11:30--12:20	F. Pop (University of Pennsylvania): On the p-adic section conjecture
-- Lunch --	

Free discussions

October 28 Thursday

10:00--11:00 M. Saïdi (Exeter University), A. Tamagawa (RIMS, Kyoto University):
Survey on anabelian geometry in positive characteristic

11:30--12:20 R. Sharifi (University of Arizona) :
Investigations in the Galois cohomology of number fields

-- Lunch --

14:30--15:20 Y. Ihara (RIMS, Kyoto University):
Arithmetic questions on $\pi_1(\mathrm{P}^1 - \{0,1,\infty\})$ at p

-- Tea --

16:00--16:50 D. Harbater (University of Pennsylvania) :
Local-global principles over arithmetic curves

18:30-- Banquet

October 29 Friday

10:00--11:00 S. Mochizuki (RIMS, Kyoto University):
Inter-universal Teichmüller Theory: A Progress Report

11:30--12:20 M. Matsumoto (University of Tokyo) :
Universal mixed elliptic motive and derivation algebra of the fundamental group of one-punctured elliptic curve (joint work with Richard Hain)

-- Lunch --

14:30--15:20 F. Brown (CNRS) :
On the coalgebra structure of motivic multiple zeta values

-- Tea --

16:00--16:50 H. Furusho (Nagoya University) :
Geometric interpretation of double shuffle relations of multiple polylogarithms at roots of unity

October 30 Saturday

10:00--11:00 M. A. Garuti (Università degli Studi di Padova):
Galois closures and the fundamental group scheme

11:30--12:20 Y. Hoshi (RIMS, Kyoto University):
Survey on the combinatorial anabelian geometry of hyperbolic curves

-- Lunch --

Free discussions

October 25 Monday

9:30–9:45 Takashi Tsuboi (President of the Mathematical Society of Japan):
Opening remarks

9:45–9:50 General information from the organizing committee

10:00–11:00 P. Cartier:
Towards Grothendieck's "Dessins d'Enfants"

Abstract: During his Montpellier period, Grothendieck changed his style and the focus of his mathematical research. Perhaps motivated by the need to give "elementary" lectures, he became interested in a kind of more explicit mathematics. Already in a Cartan/Grothendieck seminar of 1961, he got interest in the construction of the Teichmüller space, to be used in the construction of the moduli space of compact Riemann surfaces. The new central tool was the Belyi theorem creating a connection between algebraic curves, number fields and certain combinatorial dissections ("dessins d'enfants"). This enabled him to formulate a very fruitful program in the well-known paper "Sketch of a program". We shall mention some possible connections with Maass automorphic forms.

11:30–12:20 Y. André:
Introduction to tempered anabelian geometry

Abstract: We will present an outline of tempered fundamental groups of p-adic curves, with emphasis on their applications and on their role in the anabelian context.

– Lunch –

14:30–15:20 P. Lochak:
Grothendieck–Teichmüller theory from a topological viewpoint

Abstract: I will survey part of what could be called geometric (as contrasted with 'motivic') Grothendieck–Teichmüller theory, which started around twenty years ago, partly following (with delay) Grothendieck's 'Sketch of a program'. I will explain in particular how one can topologically understand and prove a version of the 'two level principle', which lies at the root of the very existence of the Grothendieck–Teichmüller group and its ubiquity. Time permitting I will then delineate a program centered around completed versions of the so-called curve complexes which should help unify (and hopefully attack) certain conjectures on moduli stacks of curves and their fundamental groups.

– Tea –

16:00–16:50 M. Asada, H. Nakamura, N. Takao, H. Tsunogai:
Easy walking in GT theory and anabelian geometry, I

Abstract: In this talk we shall introduce some basic notions to understand profinite aspects of the title of this conference for a wider public of mathematicians including graduate students. We introduce the fundamental exact sequence associated with arithmetic fundamental groups, and discuss typical basic examples: hyperbolic curves, their configuration spaces, and moduli spaces. After Belyi's Theorem, Grothendieck raised a series of questions that encourages closely looking at the extention structures of arithmetic fundamental groups, equivalently, understanding outer Galois representations (or more generally, universal monodromy representation arising from the moduli space of curves). We discuss generalization of Belyi's injectivity theorem. If time allows, the definition of Grothendieck–Teichmüller group and its characterization as the automorphism group of certain towers will be discussed.

17:10–18:00 M. Asada, H. Nakamura, N. Takao, H. Tsunogai:
Easy walking in GT theory and anabelian geometry, II
Abstract: In this talk we shall introduce some basic notions to understand pro-l (pro-unipotent) aspects of the title of this conference for a wider public of mathematicians including graduate students. The Galois actions on the pro-l fundamental group of algebraic curves have been an important subject to find arithmetic nature of anabelian curves since Ihara's works on $P^1 - \{0, 1, \infty\}$ in 1980's. We explain weight filtration, associated Lie algebras and derivation algebras in the case of hyperbolic curves, and generalization to configuration spaces of curves. A fundamental result concerned here is injectivity of a sequence of derivation algebras and its stability, that leads to settlement of Oda's conjecture on the common Galois factor of the universal pro-l monodromy representation. If time allows, we mention relationships of Grothendieck–Teichmüller Lie algebra, Zagier's conjecture on multiple zeta values and Ihara's stable derivation algebra.

October 26 Tuesday

10:00–11:00 L. Schneps:
Survey of the theory of multiple zeta values
Abstract: The theory of multiple zeta values consists in the algebraic and geometric study of the values at positive integers $\zeta(k_1, \ldots, k_r)$ of many-variabled ζ-functions. These numbers satisfy a double family of fundamental algebraic relations called "double shuffle relations". In this lecture, we will pose some of the main questions facing the theory at present, and give some of the major results. Then we will cover the astonishing connections between the double shuffle algebra and many other parts of mathematics: moduli spaces of curves and mixed Tate motives, modular forms and the Eichler-Shimura correspondence, and Grothendieck-Teichmüller theory.

11:30–12:20 I. Marin:
Rigidity characters for the Grothendieck–Teichmüller group
Abstract: Drawing a parallel with the theory of rigid local systems and the possibility they provide to construct representations of the braid group, this talk will present a notion of 'GT-rigid' representations of the braid groups: these are representations whose isomorphism class is unchanged under the natural action of the Grothendieck–Teichmüller group. We will show how to recover the extensions of some natural arithmetic character in this way.

– Lunch –

14:30–15:20 A. Schmidt:
Motivic aspects of anabelian geometry
Abstract: TBA

– Tea –

16:00–16:50 J. Stix:

On the passage from local to global in Grothendieck's section conjecture

Abstract: The passage from local to global has a long tradition in number theory. The talk will introduce Grothendieck's section conjecture and discuss it regarding the passage from local to global. We will present results on Brauer–Manin obstructions for sections and the relation of the descent obstruction to sections of the fundamental exact sequence. The latter is joint work with David Harari.

17:10–18:00 J. Ellenberg:

Ihara's braid group and fundamental groups of random curves

Abstract: Let p be a prime and S a finite set of primes not including p. Let $G_S(p)$ be the Galois group of the maximal pro-p extension of Q unramified away from S. What does $G_S(p)$ look like when S is a “random" set of primes of fixed size? Questions of this kind pertaining to _abelian_ unramified pro-p extensions of number fields (i.e. p-parts of ideal class groups) are the subject of the Cohen–Lenstra conjectures. But the non-abelian case has been studied much less. We discuss two routes to a heuristic for the distribution of $G_S(p)$; one along the lines of the original Cohen–Lenstra argument, and another via the analogy with function fields, in which we model the action of Frobenius on the arithmetic fundamental group of a curve by a random element of Ihara's pro-p braid group. It turns out that both routes lead to the same heuristic, which agrees with the few results one can prove, and is reasonably consistent with the experimental data we can gather. This is join! Work with Nigel Boston – a preprint can be seen at http://www.math.wisc.edu/˜ellenber /randombraid.pdf

October 27 Wednesday

10:00–11:00 K. Wickelgren:

Etale π_1 obstructions to rational points

Abstract: Grothendieck's anabelian conjectures say that hyperbolic algebraic curves over number fields should behave like $K(\pi, 1)$'s in algebraic geometry. For instance, conjecturally the rational points on such a curve are the sections of etale π_1 of the structure map. We use cohomological obstructions of Jordan Ellenberg coming from the lower central series of the etale fundamental group to study such sections. We give a complete calculation of the two and three nilpotent local mod 2 obstructions for $\mathbb{P}^1 - \{0, 1, \infty\}$. Globally, we give a characterization in terms of splitting varieties. This is tantamount to computing the splitting variety of a Massey product in Galois cohomology, which was done jointly with M. Hopkins. Over $\mathbb{R}$, we show a 2-nilpotent section conjecture.

11:30–12:20 F. Pop:

On the p-adic section conjecture

Abstract: I plan to report on joint work with Jakob Stix on the p-adic section conjecture. If not already done by speakers before me, I will make a short introduction to the section conjecture as it evolved from Grothendieck's program in the Eighties and its [birational] (p-adic) variants. I will present a new result which reduces the p-adic section conjecture to a local problem. I will comment on possible strategies / difficulties about how to proceed.

– Lunch –

Free discussions

October 28 Thursday

10:00–11:00 M. Saïdi, A. Tamagawa:
Survey on anabelian geometry in positive characteristic
Abstract: We will review the anabelian geometry of hyperbolic curves over finite fields, and discuss the anabelian geometry of hyperbolic curves over algebraically closed fields of positive characteristics, which is beyond the original anabelian programme of Grothendieck.

11:30–12:20 R. Sharifi:
Investigations in the Galois cohomology of number fields
Abstract: We will survey a number of different results and conjectures relating to the cohomology of the Galois group of the maximal extension of a number field unramified outside of a finite set of primes. Of particular interest are the cyclotomic fields, for which there are connections between cup products of cyclotomic units and p-adic L-values of cusp forms.

– Lunch –

14:30–15:20 Y. Ihara:
Arithmetic questions on $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$ at p
Abstract: For an odd prime p, let Π_p (resp. $\bar{\Pi}_p$) denote the quotient of the algebraic fundamental group of $X = \mathbf{P}^1 - \{0, 1, \infty\}$ over $\bar{\mathbf{Q}}_p$ (resp. $\bar{\mathbf{F}}_p$) defined by the condition: the ramification indices above $0, 1, \infty$ are not divisible by p. Let $\Pi_p^0 = \bar{\Pi}^0{}_p$ denote the Galois group of the tower of modular curves $\{X(2N)/X\}_{N \not\equiv 0(p)}$ of level $2N$ over these fields under the identification $X = X(2)$. Look at the canonical surjective homomorphisms $f : \Pi_p \to \bar{\Pi}_p, \quad g : \bar{\Pi}_p \to \Pi_p^0$. Then we see that (i) the kernel of $g \circ f$ is generated by p conjugacy classes, (ii) that of g is generated by $(p+1)/2$ conjugacy classes essentially coming from $(p-1)/2$ supersingular Frobenius elements. These follow easily from our old work on the connections between modular curves over $\mathbf{F}_{p^2}$ and the modular groups over $\mathbf{Z}[1/p]$, which we shall first briefly review. We then go on to discuss some basic (mostly open) questions related to these conjugacy classes and the kernels.

– Tea –

16:00–16:50 D. Harbater:
Local-global principles over arithmetic curves
Abstract: (Joint work with Julia Hartmann and Daniel Krashen.) The classical Tate–Shafarevich group Ш considers torsors for an abelian variety over a global field, and classifies those that become trivial at each completion. More generally, one may consider other fields F and other algebraic groups G (though Ш becomes just a pointed set if G is not commutative). This talk concerns the case in which G is a linear algebraic group that is rational (though possibly disconnected) over the function field F of a curve defined over a complete discretely valued field. In this situation, we show that Ш is finite, and we explicitly give its order in terms of the fundamental group of the reduction graph of a regular model of the curve and the maximal finite quotient of G. In particular, for such G, we show that a local-global principle holds if and only if either G is connected or the reduction graph is a tree. This has applications to the study of quadratic forms and central simple algebras.

18:30– Banquet

October 29 Friday

10:00–11:00 S. Mochizuki:

Inter-universal Teichmüller Theory: A Progress Report

Abstract: The analogy between *number fields* and *function fields of curves* (e.g., hyperbolic curves) *over finite fields* is quite classical. In the present talk, we survey work in progress concerning a theory developed by the lecturer during the last decade — in the spirit of this analogy — whose goal is to construct an analogue for *number fields "equipped with an elliptic curve"* of the ***p*-adic Teichmüller theory** developed by the lecturer during the early 1990's for *hyperbolic curves over a finite field "equipped with a nilpotent ordinary indigenous bundle"*. From an even more classical point of view, one may think of this theory as a sort of analogue for number fields of classical complex Teichmüller theory, in which canonical deformations of the holomorphic structure of a hyperbolic Riemann surface of finite type are constructed by dilating one of the two underlying real dimensions of the Riemann surface, while leaving the other dimension fixed (i.e., "undeformed").

In the case of number fields equipped with an elliptic curve, one thinks of the ring structure of the number field as a sort of "arithmetic holomorphic structure". One then constructs *canonical deformations of this arithmetic holomorphic structure* — i.e., analogues of the canonical liftings of p-adic Teichmüller theory — by applying the general theory of Frobenioids, as well as the theory of the *Frobenioid-theoretic theta function* (developed in earlier papers by the lecturer). At a more concrete level, if one thinks of the ring structure (i.e., "arithmetic holomorphic structure") of the given number field as consisting of *"two underlying combinatorial dimensions"* corresponding to **addition** and **multiplication**, then working with Frobenioids corresponds, roughly speaking, to performing operations with the *multiplicative monoids* involved (i.e., multiplicative portions of the rings involved) — in a fashion motivated by the theory of log structures; in particular, such operations are *not necessarily compatible* with the additive portions of the ring structures involved. Alternatively, if one thinks of the ring structure (i.e., "arithmetic holomorphic structure") of the various local fields that arise as localizations of the given number field as consisting of *"two underlying combinatorial dimensions"* corresponding to the group of units and the value group, then one may think of these canonical deformations of the arithmetic holomorphic structure as deformations in which the *value groups* are **(canonically!) dilated** — by means of the **theta function** — while the *units* are left undeformed. Since such "arithmetic Teichmüller dilations" are *manifestly incompatible with the ring structure* of the given number field, it follows that they are not compatible, in general, with various classical scheme-theoretic constructions performed over the number field which depend on this ring structure. In particular, these arithmetic Teichmüller dilations fail to be compatible with the various **basepoints** *of arithmetic fundamental groups* involved (e.g., Galois groups) which are defined by considering scheme-theoretic geometric points. The resulting incompatibility of (conventional scheme-theoretic) basepoints on either side of the "arithmetic Teichmüller dilation" gives rise to numerous indeterminacies; these indeterminacies lead naturally to the introduction of tools from anabelian geometry. It is this fundamental aspect of the theory that is referred to by the term **"inter-universal"**.

The (expected) **main theorem** *of inter-universal Teichmüller theory* consists of a fairly explicit computation, up to certain relatively mild indetermacies, of the "arithmetic Teichmüller deformations of a number field equipped with an elliptic curve" discussed above by applying various results obtained in previous papers by the lecturer concerning local and global absolute anabelian geometry, tempered anabelian geometry, and the étale theta function. This passage from the **Frobenioid-theoretic definition** of the arithmetic deformations involved to a more explicit **Galois-theoretic description** may be thought of as a sort of global arithmetic analogue of the classical computation of the Gaussian integral (i.e., $\int_{-\infty}^{\infty} e^{-x^2}\, dx$) by means of the passage from *cartesian* to *polar* coordinates. Inequalities of interest in diophantine geometry may then be obtained as (expected) corollaries of this (expected) main theorem.

11:30–12:20 M. Matsumoto:
Universal mixed elliptic motive and derivation algebra of
the fundamental group of one-punctured elliptic curve
(joint work with Richard Hain)

Abstract: Let $G_{\mathbb{Q}}$ denote the absolute Galois group over $\mathbb{Q}$. Let X be a three-point punctured projective line over $\mathbb{Q}$, and choose a tangential base point $\overrightarrow{01}$. Then we have a Galois representation on the $\mathbb{Q}_\ell$-prounipotent fundamental group

$$G_{\mathbb{Q}} \to \mathrm{Aut}\pi_1^{\mathrm{un}}(\overline{X}, \overrightarrow{01})/\mathbb{Q}_\ell.$$

We enumerate properties of (the Lie-algebrization of) this representation: (1) it has the weight filtration, (2) its weight graded quotients are Tate, (3) unramified outside ℓ and crystalline at ℓ. Conversely, if we consider the category of finite dimensional $\mathbb{Q}_\ell$-linear Galois representation with these properties, then we have a Tannakian category whose Tannakian fundamental group is an extension of G_m by a free prounipotent group generated by Soulé's elements. This coincides with that of the category of mixed Tate motives by Deligne-Goncharov, after extension of scaler to $\mathbb{Q}_\ell$.
One motivation of our research is to know what happens if we replace the family $X \to \mathrm{Spec}\mathbb{Q} = \mathcal{M}_{0,3}$ with the universal family of elliptic curves $\mathcal{E} \to \mathcal{M}_{1,1}$. We enumerate several properties of the corresponding monodromy representation

$$\pi_1(\mathcal{M}_{1,1}) \to \mathrm{Aut}\pi_1^{\mathrm{un}}(E_{\vec{01}}, \vec{01})/\mathbb{Q}_\ell.$$

Then the Tannakian fundamental group of the category of the representations $\pi_1(\mathcal{M}_{1,1})$ with these properties has a generating set consisting of Soulé's elements together with geometric generators corresponding to the Eisenstein series, and possible relations arising from the cusp forms. These possible relations turn out to be actual relations in the derivation algebra in the right hand side (due to Aaron Pollack's result).

– Lunch –

14:30–15:20 F. Brown:
On the coalgebra structure of motivic multiple zeta values

Abstract: In this talk I shall review Goncharov's coproduct formula for the motivic multiple zeta values in the most elementary possible terms, and deduce some simple consequences from it.

– Tea –

16:00–16:50 H. Furusho:
Geometric interpretation of double shuffle relations of multiple
polylogarithms at roots of unity

Abstract: I will give a geometric interpretation of the generalized (including the regularization relation) double shuffle relation for multiple L-values. I will explain that Enriquez' mixed pentagon equation implies the relations. As a corollary, an embedding from his cyclotomic analogue of Grothendieck–Teichmüller group into Racinet's cyclotomic double shuffle group is obtained, which extends my provious result.

October 30 Saturday

10:00–11:00 M. A. Garuti:
Galois closures and the fundamental group scheme
Abstract: The fundamental group scheme of a scheme X over a base B, when it exists, classifies torsors over X under finite flat B-group schemes. We will give a short introduction to the fundamental group scheme, and explain how its existence is related to the problem of finding Galois closures of finite flat morphisms.

11:30–12:20 Y. Hoshi:
Survey on the combinatorial anabelian geometry of hyperbolic curves
Abstract: In this talk, I will give a survey on the combinatorial anabelian geometry of hyperbolic curves. First, I will review briefly the notion of a semi-graph of anabelioids of PSC-type, which is one of the main objects of interest in combinatorial anabelian geometry, and discuss Grothendieck conjecture-type results for outer isomorphisms between the fundamental groups of semi-graphs of anabelioids of PSC-type equipped with certain outer representations. Next, I will explain various consequences of these Grothendieck conjecture-type results: (1) the injectivity portion of combinatorial cuspidalization, (2) faithfulness of the outer Galois representations associated to hyperbolic curves, (3) a version of the Grothendieck conjecture for universal curves over moduli spaces of curves over algebraically closed fields. Finally, I will discuss a generalization of Yves Andre's result concerning the intersection of the outer Galois representation associated to a tripod over a number field and the group of outer automorphisms of the tempered fundamental group of the tripod.

– Lunch –

Free discussions