Multiplicative structure of some multivariate functions

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1 Introduction

It is now well known that the famous Riemann zeta function admits a multivariate extension given by

$$\zeta(s_1;\ldots;s_n)=\sum_{n_1>\ldots>n_r}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}}$$

with, initially, $s_1,\ldots,s_r$ being natural numbers. This family of functions is studied for both its arithmetic [1] and computational interest, particularly in quantum physics. To obtain this, an efficient coding approach has been proposed using two-letter or n-letter alphabets. We will generalize this process to other families of functions: colored polyzetas, reals Hurwitz polyzetas [5], multifunctional Polylerch, etc. For each of these families, we explain how their multiplicative structures relate via their coding to the φ-shuffle, defined recursively by:

$$\forall(a,b)\in X^2,\forall(u,v)\in(X^*)^2,\quad au\sqcup\!\sqcup_\varphi vb=a(u\sqcup\!\sqcup_\varphi bv)+b(au\sqcup\!\sqcup_\varphi v)+\varphi(a,b)(u\sqcup\!\sqcup_\varphi v)\ ,\qquad(1)$$

thus providing an implementable computational path. We will then begin a systematic study of φ-shuffles.

2 II- different type of φ-shuffle

In this study, we will be led to distinguish five types of φ-shuffle :

1. Type I : factor φ comes from a product (possibly with zero) between letters (i.e. $X\cup\{0\}$ is a semigroup).
2. Type II : factor φ comes from the deformation of a semigroup product by a bicharacter.
3. Type III : factor φ comes from the deformation of a semigroup product by a colour factor.
4. Type IV : factor φ is the commutative law of an associative algebra (CAA) on $A.X$
5. Type V : factor φ is the law of an associative algebra (AA) on $A.X$

These classes are ordered by the following (strict) inclusion diagram:

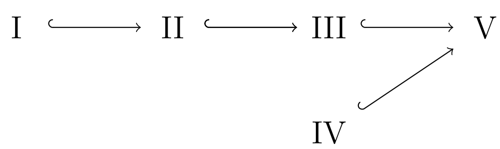

Figure 1: Hasse diagram of the inclusions between classes.

We have collected examples from the literature, with the corresponding formulas, and give a classication table.

3 III - Radford's theorem for AC-shuffle

Once the multiplicative laws are transferred via coding, it becomes easier to work on the alphabet X. This alphabet has a total order relation $<$, we then have the family $\mathcal{L}yn(X)$ of Lyndon words on X [3]. We will note the decompositions based on $\mathcal{L}yn(X)$ as follows:

Definition 1 *Let $\star$ be an associative law, with unit, over $A\langle X\rangle$. For any $\alpha \in \mathbb{N}^{(Lyn(X))}$ and $\{l_1, \cdots, l_r\} \supset supp(\alpha)$ in strict decreasing order (i.e. $l_1 > \cdots > l_r$), we set*

$$\mathbb{X}^{\star\alpha} = l_1^{\star\alpha_1} \star \cdots \star l_r^{\star\alpha_r}, \tag{2}$$

where $\alpha_i = \alpha(l_i)$ for all i and, for short, $\mathbb{X} = Lyn(X)$.

We will establish the following theorem :

Theorem 3.1 *Let A be a commutative ring (with unit) such that[1] $\mathbb{Q} \subset A$ and $\sqcup\!\sqcup_\varphi : A\langle X\rangle \otimes A\langle X\rangle \to A\langle X\rangle$ is associative.*
If X is totally ordered by $<$, then $(\mathbb{X}^{\sqcup\!\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}}$ is a linear basis of $A\langle X\rangle$.

So, $\mathcal{L}yn(X)$ is a transcendence basis of $\mathcal{A} = (A\langle X\rangle, \sqcup\!\sqcup_\varphi, 1_{X^*})$.

4 IV -Bialgebra structure ... in the way of Hofp algebra

It turns out to be very efficient to dualize, if possible, the algebra $\mathcal{A} = (A\langle X\rangle, \sqcup\!\sqcup_\varphi, 1_{X^*})$ in the following way :

Definition 2 *A law $\star$ defined over $A\langle X\rangle$ is a dual law (or dualizable) if there exists a linear mapping $\Delta_\star : A\langle X\rangle \to A\langle X\rangle \otimes A\langle X\rangle$ such*

$$\forall (u, v, w) \in X^* \times X^* \times X^*, \quad \langle u \star v | w\rangle = \langle u \otimes v | \Delta_\star(w)\rangle^{\otimes 2} \ . \tag{3}$$

In this case, Δ_* will be called the comultiplication dual to $\star$.

Here again, exactly the same conditions allow dualization :

[1] This condition amounts to ask that $\mathbb{N}^+.1_A \subset A^\times$

Theorem 4.1 *Let A be a commutative ring (with unit). We suppose that the product $\sqcup\!\sqcup_\varphi : A\langle X\rangle \otimes A\langle X\rangle \to A\langle X\rangle$ is an associative and commutative law on $A\langle X\rangle$, then the algebra $(A\langle X\rangle, \sqcup\!\sqcup_\varphi, 1_{X^*})$ can be endowed with the comultiplication Δ_{conc} dual to the concatenation*

$$\Delta_{\text{conc}}(w) = \sum_{uv=w} u \otimes v \tag{4}$$

and the "constant term" character $\epsilon(P) = \langle P|1_{X^*}\rangle$.

(i) With this setting

$$\mathcal{B}_\varphi = (A\langle X\rangle, \sqcup\!\sqcup_\varphi, 1_{X^*}, \Delta_{\text{conc}}, \epsilon) \tag{5}$$

is a bialgebra [2].

(ii) The bialgebra (5) is, in fact, a Hopf Algebra.

So we need to find the condition over $\sqcup\!\sqcup_\varphi$ to be commutative and associative. We will given :

Theorem 4.2 *(i) The law $\sqcup\!\sqcup_\varphi$ is commutative if and only if the extension $\varphi : AX \otimes AX \to AX$ is commutative.*

(ii) The law $\sqcup\!\sqcup_\varphi$ is associative if and only if the extension $\varphi : AX \otimes AX \to AX$ is associative.

Proposition 1 *We call $\gamma_{x,y}^z := \langle \varphi(x,y)|z\rangle$ the structure constants of φ (w.r.t. the basis X). The product $\sqcup\!\sqcup_\varphi$ is a dual law if and only if $(\gamma_{x,y}^z)_{x,y,z\in X}$ is dualizable in the following sense*

$$(\forall z \in X)(\#\{(x,y) \in X^2 | \gamma_{x,y}^z \neq 0\} < +\infty) \ . \tag{6}$$

4.1 The Hopf-Hurwitz algebra

In the end we explicitly give the product $\sqcup\!\sqcup\!\!\!\bullet$ of reals Hurwitz polyzetas, and check its commutativity and associativity, making $(A\langle N\rangle, \sqcup\!\sqcup\!\!\!\bullet, 1_N)$ into a A-CAAU : the Radford's theorem can be generalised here.

References

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[2] Commutative and, when $|X| \geq 2$, noncocommutative.