30th Applications of Computer Algebra

Various Bialgebras of Representative Functions on Free Monoids

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14-18 July 2025, Heraklion (Greece)

Outline

1. Introduction: Fuchsian linear differential equations and functions on free monoids
2. Graded bialgebras[1] $(A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$ $(A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi})$
   - 2.1 Factorization of the diagonal series in the graded bialgebra $(A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$
   - 2.2 φ-deformation of shuffle product
   - 2.3 Factorization of the diagonal series in the graded bialgebra $(A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi})$
3. Sweedler's dual of graded bialgebras $(K\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$ $(K\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi})$
   - 3.1 Bialgebras $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}})$ $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \sqcup\!\sqcup_\varphi, 1_{Y^*}, \Delta_{\mathrm{conc}})$
   - 3.2 Factorization and decomposition of rational series
   - 3.3 Kleene stars of the planes

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[1] $\mathcal{X}$ denotes a finite or infinite alphabet.
A (resp. K) denotes a ring containing $\mathbb{Q}$ (resp. an algebraic closed field)
$A\langle\mathcal{X}\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$) denotes the set of noncommutative polynomials (resp. series).

INTRODUCTION

Fuchsian linear differential equations and hyperlogarithms

$$(ED)\quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}z}q(z) &= \Big(\sum_{i\geq 0} u_i(z)M_i\Big)q(z), \quad \text{with } M_i\in\mathcal{M}_{n,n}(\mathbb{C}) \text{ and } u_i(z)=(z-s_i)^{-1},\\ q(z_0) &= \eta,\\ y(z) &= \lambda q(z). \end{cases}$$

Let $\sigma=\{s_i\}_{i\geq 0}$ with $s_0=0$ and $B=\mathbb{C}\setminus\sigma$. For any $i\geq 0$, let $\omega_i=u_i\mathrm{d}z$. For simplification, if $i\neq j$ then $s_i\neq s_j$ and $s_i=e^{\mathrm{i}\theta_i},\theta_i\in[0,2\pi[$.

Let $X=\{x_i\}_{i\geq 0}$ and $\mathcal{H}(\tilde{B})$ be the ring of holomorphic functions over $\tilde{B}$. Considering the following functions over the monoid $(X^*,1_{X^*})$

$$\alpha_{z_0}^z : X^* \longrightarrow \mathcal{H}(\tilde{B}) \quad \text{and} \quad \mu : X^* \longrightarrow \mathcal{M}_{n,n}(\mathbb{C}),$$

defined by $\quad \alpha_{z_0}^z(1_{X^*})=1_{\mathcal{H}(\tilde{B})}, \quad \mu(1_{X^*})=\mathrm{Id}_n,$

$$\forall w=x_iv\in XX^*,\quad \alpha_{z_0}^z(w)=\int_{z_0}^{z}\omega_i(s)\alpha_{z_0}^s(v),\quad \mu(w)=M_i\mu(v),$$

one obtains $U(z_0;z)=\sum_{w\in X^*}\mu(w)\alpha_{z_0}^z(w)$ and $y(z)=\lambda U(z_0;z)\eta$.

The iterated integrals (of $\{\omega_i\}_{i\geq 0}$ and along $z_0\rightsquigarrow z$) $\{\alpha_{z_0}^z(w)\}_{w\in X^*}$, as being functions on the free monoid $(X^*,1_{X^*})$, are hyperlogarithms and their algebra is isomorphic to the shuffle $(\mathbb{C}\langle X\rangle,\sqcup\!\sqcup,1_{X^*})$.

The case of hypergeometric equation ($m = 1$)

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}y(z) + [t_2 - (t_0+t_1+1)z]\frac{\mathrm{d}}{\mathrm{d}z}y(z) - t_0t_1y(z) = 0.$$

Introducing $q_1(z) = y(z)$ and $q_2(z) = (1-z)\frac{\mathrm{d}}{\mathrm{d}z}y(z)$ and letting $\omega_0(z) = z^{-1}\mathrm{d}z$ and $\omega_1(z) = (1-z)^{-1}\mathrm{d}z$, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} 0 & \dfrac{1}{1-z} \\ \dfrac{ab}{z} & \dfrac{a+b-c}{1-z} - \dfrac{c}{z} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\
&= \left[\begin{pmatrix} 0 & 0 \\ -t_0t_1 & -t_2 \end{pmatrix}\omega_0(z) - \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}\omega_1(z)\right]\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.
\end{aligned}$$

For any $w = x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_1 \in X^*x_1$, where $X = \{x_0, x_1\}$, one has

$$\begin{aligned}
\alpha_0^z(w) = \mathrm{Li}_{s_1,\cdots,s_r}(z) &= \sum_{n_1>\cdots>n_r>0} n_1^{-s_1}\cdots n_r^{-s_r}z^{n_1}, \\
(1-z)^{-1}\,\mathrm{Li}_{s_1,\cdots,s_r}(z) &= \sum_{n\geq 0} \mathrm{H}_{s_1,\cdots,s_r}(n)z^n, \\
\mathrm{H}_{s_1,\cdots,s_r}(n) &= \sum_{n\geq n_1>\cdots>n_r>0} n_1^{-s_1}\cdots n_r^{-s_r}.
\end{aligned}$$

The map $\mathrm{Li}_\bullet$ (resp. $\mathrm{H}_\bullet$) is a function on $(\mathbb{N}_{\geq 1})^*$ to the rings of holomorphic (resp. arithmetical) functions $\{\mathrm{Li}_{s_1,\cdots,s_r}\}_{\substack{s_1,\cdots,s_r\geq 1 \\ r\geq 0}}$ (resp. $\{\mathrm{H}_{s_1,\cdots,s_r}\}_{\substack{s_1,\cdots,s_r\geq 1 \\ r\geq 0}}$).

For $s_1 > 1$, $\lim_{z\to 1}\mathrm{Li}_{s_1,\cdots,s_r}(z) = \lim_{n\to+\infty}\mathrm{H}_{s_1,\cdots,s_r}(n) = \zeta(s_1,\cdots,s_r)$.

The case of coulored polylogarithms ($m \geq 1$)

Let $X = \{x_0, \cdots, x_m\}$ and $\mathcal{O}_m = \{\rho_i\}_{1 \leq i \leq m}$, where $\rho_i = e^{\mathrm{i}\frac{2\pi}{m}i}$. Let

$$\omega_0(z) = \frac{\mathrm{d}z}{z}, \quad \omega_i(z) = \rho_i \frac{\mathrm{d}z}{1 - \rho_i z} = \frac{\mathrm{d}z}{\bar{\rho}_i - z}, \quad 1 \leq i \leq m.$$

For any $w = x_0^{s_1-1}x_{i_1} \cdots x_0^{s_r-1}x_{i_r} \in X^*X$, one has

$$\begin{aligned}
\alpha_0^z(w) = \mathrm{Li}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}(z) &= \sum_{n_1 > \cdots > n_r > 0} \frac{\rho_{i_1}^{n_1} \cdots \rho_{i_r}^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}} z^{n_1}, \\
(1-z)^{-1} \mathrm{Li}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}(z) &= \sum_{n \geq 0} \mathrm{H}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}(n) z^n, \\
\mathrm{H}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}(n) &= \sum_{n \geq n_1 > \cdots > n_r > 0} \frac{\rho_{i_1}^{n_1} \cdots \rho_{i_r}^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}}.
\end{aligned}$$

The map $\mathrm{Li}_\bullet$ (resp. $\mathrm{H}_\bullet$) is a function on $\binom{\mathcal{O}_m}{\mathbb{N}_{\geq 1}}^*$ to the rings of holomorphic (resp. arithmetical) functions $\{\mathrm{Li}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}\}_{\substack{\rho_{i_1}, \cdots, \rho_{i_r} \in \mathcal{O}_m \\ s_1, \cdots, s_r \geq 1, r \geq 0}}$ (resp. $\{\mathrm{H}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}\}_{\substack{\rho_{i_1}, \cdots, \rho_{i_r} \in \mathcal{O}_m \\ s_1, \cdots, s_r \geq 1, r \geq 0}}$). For $\binom{\rho_1}{s_1} \neq \binom{1}{1}$,

$$\lim_{z \to 1} \mathrm{Li}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}(z) = \lim_{n \to +\infty} \mathrm{H}_{\binom{\rho_{i_1}}{s_1}, \ldots, \binom{\rho_{i_r}}{s_r}}(n) = \zeta\binom{\rho_{i_1}}{s_1}, \cdots, \binom{\rho_{i_r}}{s_r}.$$

Functions on the free monoids $(X^*, 1_{X^*})$ and $(Y^*, 1_{Y^*})$

- Polylogarithms ($X = \{x_0, x_1\}, x_0 \prec x_1$) and hamonic sums ($Y = \{y_k\}_{k\geq 1}, y_1 \succ y_2 \succ \cdots$).
$$(s_1, \cdots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \cdots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_1 \in X^*x_1.$$
- Coulored polylogarithms ($X = \{x_0, \cdots, x_m\}, x_0 \prec \cdots \prec x_m$) and coulored hamonic sums ($Y = \{Y_k\}_{k\geq 1}$, $Y_1 \succ Y_2 \succ \cdots$, where $Y_k = \{y_{k,\rho_1}, \cdots, y_{k,\rho_m}\}, y_{k,\rho_1} \prec \cdots \prec y_{k,\rho_m}$).
$$\begin{pmatrix} \rho_{i_1} \\ s_1 \end{pmatrix}, \cdots, \begin{pmatrix} \rho_{i_r} \\ s_r \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_m \\ \mathbb{N}_{\geq 1} \end{pmatrix}^* \leftrightarrow y_{s_1,\rho_{i_1}} \cdots y_{s_r,\rho_{i_r}} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_{i_1} \cdots x_0^{s_r-1}x_{i_r} \in X^*(X \setminus \{x_0\}).$$
- Hyperlogarithms ($X = \{x_i\}_{i\geq 0}, x_0 \prec x_1 \prec \cdots$) and extended hamonic sums ($Y = \{Y_k\}_{k\geq 1}$, $Y_1 \succ Y_2 \succ \cdots$, where $Y_k = \{y_{k,\rho_i}\}_{i\geq 1}$, $y_{k,\rho_1} \prec y_{k,\rho_2} \prec \cdots$).
$$\begin{pmatrix} \rho_{i_1} \\ s_1 \end{pmatrix}, \cdots, \begin{pmatrix} \rho_{i_r} \\ s_r \end{pmatrix} \in \begin{pmatrix} \sigma \\ \mathbb{N}_{\geq 1} \end{pmatrix}^* \leftrightarrow y_{s_1,\rho_{i_1}} \cdots y_{s_r,\rho_{i_r}} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_{i_1} \cdots x_0^{s_r-1}x_{i_r} \in X^*(X \setminus \{x_0\}).$$

In any case, for $Y = \{y_k\}_{k\geq 1}$ or $Y = \{y_{k,\rho_i}\}_{k\geq 1, 1\leq i\leq m}$ or $Y = \{y_{k,\rho_i}\}_{k,i\geq 1}$,
$$l \in \mathcal{L}yn X - \{x_0\} \iff \pi_Y(l) \in \mathcal{L}yn Y.$$

Graph of representative function on free monoids

Let f be a function on the free monoid $(\mathcal{X}^*, 1_{\mathcal{X}^*})$ to A. It is said to be representative iff there is finitely many functions $\{f'_i, f''_i\}_{i \in I_{finite}}$ of $A^{\mathcal{X}^*}$, choosen to be representative functions s.t.

$$\forall u, v \in \mathcal{X}^*, \quad f(uv) = \sum_{i \in I_{finite}} f'_i(u) f''_i(v).$$

The coproduct of the representative function f is defined in duality with the concatenation (denoted by $\mathtt{conc}$) in $\mathcal{X}^*$ as follows

$$\forall u, v \in \mathcal{X}^*, \quad \Delta_{\mathtt{conc}}(f)(u \otimes v) = f(uv), \quad \Delta_{\mathtt{conc}}(f) = \sum_{i \in I_{finite}} f'_i \otimes f''_i.$$

The graph of f is given by the following noncommutative series[2]:

$$S = \sum_{w \in \mathcal{X}^*} \langle S|w \rangle w, \quad \text{where} \quad \langle S|w \rangle = f(w).$$

Using the following pairing

$$A\langle\langle \mathcal{X} \rangle\rangle \otimes A\langle \mathcal{X} \rangle \longrightarrow A, \quad T \otimes P \longmapsto \langle T|P \rangle := \sum_{w \in \mathcal{X}^*} \langle T|w \rangle \langle P|w \rangle,$$

there is a natural duality between $A^{\mathcal{X}^*} = A\langle\langle \mathcal{X} \rangle\rangle$ and $A[\mathcal{X}^*] \cong A\langle \mathcal{X} \rangle$:

$$A\langle\langle \mathcal{X} \rangle\rangle = A\langle \mathcal{X} \rangle^\vee.$$

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[2] Any series S is a function (on $\mathcal{X}^*$ to A) mapping $w \in \mathcal{X}^*$ to $\langle S|w \rangle \in A$. The sets of noncommutative series and of polynomials (over $\mathcal{X}$ and with coefficients in A) are denoted by $A\langle\langle \mathcal{X} \rangle\rangle$ and $A\langle \mathcal{X} \rangle$, respectively.

GRADED BIALGEBRAS $(A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{⧢})$ $(A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{⧢_\varphi})$

Dual bases in graded bialgebra $(A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup})$

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \sum_{w\in\mathcal{X}^*} P_w\otimes S_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} \exp(S_l\otimes P_l),$$

where (see **Reutenauer**, 1993)

- $\mathcal{L}yn\mathcal{X}$ is the set[3] of Lyndon words over $\mathcal{X}$.
- $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$: basis of the Lie algebra $\mathcal{L}ie_A\langle\mathcal{X}\rangle$ and P_l is defined by
  $P_l = l$ if $l\in\mathcal{X}$ and $P_l = [P_u, P_v]$ if $l\in\mathcal{L}yn\mathcal{X}$ and $\mathrm{st}(l) = (u, v)$.
- $\{P_w\}_{w\in\mathcal{X}^*}$: Lyndon-PBW basis of $\mathcal{U}(\mathcal{L}ie_A\langle\mathcal{X}\rangle)$, obtained by putting
  $P_w = P_{l_1}^{i_1}\cdots P_{l_k}^{i_k}$ for $w = l_1^{i_1}\cdots l_k^{i_k}, l_1,\cdots,l_k\in\mathcal{L}yn\mathcal{X}, l_1\succ\cdots\succ l_k$.
- The dual[4] basis $\{S_w\}_{w\in\mathcal{X}^*}$, containing $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$, is as follows

$$S_l = xS_u, \qquad \text{for} \quad l = xu\in\mathcal{L}yn\mathcal{X},$$

$$S_w = \frac{S_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup S_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1!\cdots i_k!}, \qquad \text{for} \quad w = l_1^{i_1}\cdots l_k^{i_k}, l_1\succ\cdots\succ l_k.$$

- $\begin{matrix}\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}\\ \{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}\end{matrix}$ are triangular and homogenous in length (b_v and $d_v\in A$):

$$S_l = l + \sum_{v\succ l, |v|=|l|} b_v v \quad \text{and} \quad P_l = l + \sum_{v\prec l, |v|=|l|} d_v v.$$

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[3] It forms a pure transcendence basis of $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$.

[4] *i.e.* $\langle P_u|S_v\rangle = \delta_{u,v}$, for u and $v\in\mathcal{X}^*$.

φ-deformed shuffle products

Let φ be an arbitrary mapping defined by its structure constants

$$\varphi : Y \times Y \longrightarrow AY, \quad (y_i, y_j) \longmapsto \sum_{k \in I} \gamma_{i,j}^k y_k.$$

Let $⧢_\varphi$ be the product $Y^* \times Y^* \longrightarrow A\langle Y \rangle$ satisfying

1. $\forall w \in Y^*$, $1_{Y^*} ⧢_\varphi w = w ⧢_\varphi 1_{Y^*} = w$,
2. $\forall a, b \in Y, \forall u, v \in Y^*$,
   $(R) \quad au ⧢_\varphi bv = a(u ⧢_\varphi bv) + b(au ⧢_\varphi v) + \varphi(a, b)(u ⧢_\varphi v)$,

defining a unique mapping $⧢_\varphi : Y^* \times Y^* \longrightarrow A\langle Y \rangle$ which is at once extended as a law $⧢_\varphi : A\langle Y \rangle \otimes A\langle Y \rangle \longrightarrow A\langle Y \rangle$ and $⧢_\varphi$ is said to be dualizable if there is $\Delta_{⧢_\varphi} : A\langle Y \rangle \longrightarrow A\langle Y \rangle \otimes A\langle Y \rangle$ s.t. the dual mapping $(A\langle Y \rangle \otimes A\langle Y \rangle)^\vee \longrightarrow A\langle\langle Y \rangle\rangle$ restricts to $⧢_\varphi$.

- $⧢_\varphi$ is commutative iff $\varphi : AY \times AY \longrightarrow AY$ is so.
- $⧢_\varphi$ is associative iff $\varphi : AY \times AY \longrightarrow AY$ is so.
- $⧢_\varphi$ is dualizable iff $(\gamma_{x,y}^z)_{x,y,z \in Y}$ satisfy

$$(\forall z \in Y)(\#\{(x, y) \in Y^2 | \gamma_{x,y}^z \neq 0\} < +\infty).$$

Example (q-shuffle product[5], $\varphi(y_i, y_j) = q^{i+j}$)

The q-shuffle is defined, for any $y_i, y_j \in Y^*$ and $u, v \in Y^*$, by

$$u \shuffle_q 1_{Y^*} = 1_{Y^*} \shuffle_q u = u,$$
$$(y_i u) \shuffle_q (y_j v) = y_i(u \shuffle_q y_j v) + y_j((y_i u) \shuffle_q v + q^{i+j} y_{i+j}(u \shuffle_q v).$$

Example (product of extended harmonic sums)

For $Y = \{y_{s,i}\}_{s,i\geq 1}$, the product, in particular, of coulored harmonic sums is defined, for any $y_{s,\rho_i}, y_{r,\rho_j} \in Y^*$ and $u, v \in Y^*$, by

$$u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u,$$
$$(y_{s,\rho_i} u) \uplus (y_{r,\rho_j} v) = y_{r,\rho_j}((y_{s,\rho_i} u) \uplus v) + y_{s,\rho_i}(u \uplus (y_{r,\rho_j} v)) + y_{s+r,\rho_i\rho_j}(u \uplus v).$$

Let $\mathcal{B}_\varphi = (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\shuffle_\varphi})$, for ass., comm. and dualizable $\shuffle_\varphi$. Then the following assertions are equivalent.

- $\mathcal{B}_\varphi$ is an enveloping algebra.
- $\mathcal{B}_\varphi \cong (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\shuffle})$, as bialgebras.

Now, we are in situation to consider the following Eulerian idempotent

$$\forall w \in Y^*, \quad \pi_1(w) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\cdots,u_k \in Y^+} \langle w | u_1 \shuffle_\varphi \cdots \shuffle_\varphi u_k \rangle u_1 \cdots u_k.$$

[5] It corresponds to the shuffle (resp. quasi-shuffle or minus-shuffle) for $q = 0$ (resp. $q = 1$ or $q = -1$).

Dual bases in graded bialgebra $(A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi})$

$$\mathcal{D}_Y := \sum_{w\in Y^*} w\otimes w = \mathcal{D}_Y = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l\otimes\Pi_l).$$

- Letting $Y' = \{y_k'\}_{k\geq 1}$, with $y_k' = \pi_1(y_k)$, $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ is a basis of the Lie algebra $\mathcal{L}ie_A\langle Y'\rangle$, where Π_l is defined by $\Pi_{y_k} = y_k'$ and $\Pi_l = [\Pi_u, \Pi_v]$ if $l\in\mathcal{L}ynY$ and $\mathrm{st}(l) = (u,v)$.
- $\{\Pi_w\}_{w\in Y^*}$, where for any $w = l_1^{i_1}\cdots l_k^{i_k}$ satisfying $l_1\succ\cdots\succ l_k$ and[6] $l_1,\cdots,l_k\in\mathcal{L}ynY$, $\Pi_w = \Pi_{l_1}^{i_1}\cdots\Pi_{l_k}^{i_k}$.
- The dual basis $\{\Sigma_w\}_{w\in Y^*}$, containing $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$, is as follows

$$\Sigma_l = \sum_{(*)} \frac{y_{s_{k_1}+\cdots+s_{k_i}}}{i!}\Sigma_{l_1\cdots l_n}, \qquad \text{for} \quad l = yl'\in\mathcal{L}ynY,$$

$$\Sigma_w = \frac{\Sigma_{l_1}^{\sqcup\!\sqcup_\varphi i_1} \sqcup\!\sqcup_\varphi \cdots \sqcup\!\sqcup_\varphi \Sigma_{l_k}^{\sqcup\!\sqcup_\varphi i_k}}{i_1!\cdots i_k!}, \quad \text{for} \quad w = l_1^{i_1}\cdots l_k^{i_k}, l_1\succ\cdots\succ l_k.$$

- $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ are triangular and homogenous in weight $(b_v, d_v\in A)$:

$$\Sigma_l = l + \sum_{v\succ l, (v)=(l)} b_v v \quad \text{and} \quad \Pi_l = l + \sum_{v\prec l,(v)=(l)} d_v v.$$

Hence, $(A\langle Y\rangle, \sqcup\!\sqcup_\varphi, 1_{Y^*})$ admits $\mathcal{L}ynY$ as pure trancendence basis.

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[6] The Lyndon-PBW basis of $\mathcal{U}(\mathcal{L}ie_A\langle Y'\rangle)$.

SWEEDLER'S DUAL OF THE GRADED BIALGEBRAS $(K\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$ $(K\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi})$

Ext. of products, coproducts over $A\langle\langle\mathcal{X}\rangle\rangle$ and Kleene stars

$$\begin{aligned}
\mathtt{conc}, ш : A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle &\longrightarrow A\langle\langle\mathcal{X}\rangle\rangle, \quad ш_{\varphi} : A\langle\langle Y\rangle\rangle \otimes A\langle\langle Y\rangle\rangle \longrightarrow A\langle\langle Y\rangle\rangle,\\
\forall S, R \in A\langle\langle\mathcal{X}\rangle\rangle, \quad SR &= \sum_{w\in\mathcal{X}^*}\Big(\sum_{\substack{u,v\in\mathcal{X}^*\\ uv=w}} \langle S|u\rangle\langle R|v\rangle\Big)w,\\
\forall S, R \in A\langle\langle\mathcal{X}\rangle\rangle, \quad S ш R &= \sum_{u,v\in\mathcal{X}^*} \langle S|u\rangle\langle R|v\rangle u ш v\\
\forall S, R \in A\langle\langle Y\rangle\rangle, \quad S ш_{\varphi} R &= \sum_{u,v\in Y^*} \langle S|u\rangle\langle R|v\rangle u ш_{\varphi} v.
\end{aligned}$$

If $\langle S|1_{\mathcal{X}^*}\rangle = 0$ then Kleene star of S is the sum $S^* = 1_{\mathcal{X}^*} + S + S^2 + \cdots$.
$A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the smallest algebra containing $\widehat{A\mathcal{X}}$, closed by $\{+, \mathtt{conc}, *\}$.
A series $S \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is said to be rational.

$$\begin{aligned}
\Delta_{\mathtt{conc}}, \Delta_{ш} : A\langle\langle\mathcal{X}\rangle\rangle &\longrightarrow A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle, \quad \Delta_{ш_{\varphi}} : A\langle\langle Y\rangle\rangle \longrightarrow A\langle\langle Y^* \otimes Y^*\rangle\rangle,\\
\forall S \in A\langle\langle\mathcal{X}\rangle\rangle, \quad \Delta_{\mathtt{conc}} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\mathtt{conc}} w \in A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,\\
\forall S \in A\langle\langle\mathcal{X}\rangle\rangle, \quad \Delta_{ш} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{ш} w \in A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,\\
\forall S \in A\langle\langle Y\rangle\rangle, \quad \Delta_{ш_{\varphi}} S &= \sum_{w\in Y^*} \langle S|w\rangle \Delta_{ш_{\varphi}} w \in A\langle\langle Y^* \otimes Y^*\rangle\rangle.
\end{aligned}$$

Characters, primitive and grouplike series

For $ш_\varphi$ (resp. $ш$ and conc), a series $S \in A\langle\langle \mathcal{X} \rangle\rangle$ is said to be

1. a character of $A\langle Y \rangle$ (resp. $A\langle \mathcal{X} \rangle$) iff $\langle S|1_{\mathcal{X}^*}\rangle = 1$ and, for any u and $v \in Y^*$ (resp. $\mathcal{X}^*$), one has
   $\langle S|u ш_\varphi v\rangle$ (resp. $\langle S|u ш v\rangle$ and $\langle S|uv\rangle$) $= \langle S|u\rangle\langle S|v\rangle$.
2. an infinitesimal character of $A\langle Y \rangle$ (resp. $A\langle \mathcal{X} \rangle$) iff, for any u and $v \in Y^*$ (resp. $\mathcal{X}^*$), one has
   $\langle S|u ш_\varphi v\rangle$ (resp. $\langle S|u ш v\rangle$ and $\langle S|uv\rangle$) $= \langle S|u\rangle\langle v|1_{Y^*}\rangle + \langle u|1_{Y^*}\rangle\langle S|v\rangle$.

For $\Delta_{ш_\varphi}$ (resp. $\Delta_ш$ and Δ_{conc}), a series S is

1. primitive iff $\Delta_{ш_\varphi} S$ (resp. $\Delta_ш S$ and $\Delta_{\mathrm{conc}} S$) $= 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$.
2. grouplike iff $\Delta_{ш_\varphi} S$ (resp. $\Delta_ш S$ and $\Delta_{\mathrm{conc}} S$) $= S \otimes S$ and $\langle S|1_{\mathcal{X}^*}\rangle = 1$.

Let $\mathcal{P}^Y_{ш_\varphi}$ (resp. $\mathcal{P}^{\mathcal{X}}_ш$ and $\mathcal{P}^{\mathcal{X}}_{\mathrm{conc}}$) and $\mathcal{G}^Y_{ш_\varphi}$ (resp. $\mathcal{G}^{\mathcal{X}}_ш$ and $\mathcal{G}^{\mathcal{X}}_{\mathrm{conc}}$) the sets of primitive and grouplike series, respectively. For $ш_\varphi$ (resp. $ш$ and conc),

1. $S \in \mathcal{G}^Y_{ш_\varphi}$ (resp. $\mathcal{G}^{\mathcal{X}}_ш$ and $\mathcal{G}^{\mathcal{X}}_{\mathrm{conc}}$) iff S is a character.
2. $S \in \mathcal{P}^Y_{ш_\varphi}$ (resp. $\mathcal{P}^{\mathcal{X}}_ш$ and $\mathcal{P}^{\mathcal{X}}_{\mathrm{conc}}$) iff S is an infinitesimal character.

$\mathcal{P}^Y_{ш_\varphi}$ is a Lie algebra and $\mathcal{G}^Y_{ш_\varphi}$ (resp. $\mathcal{G}^{\mathcal{X}}_ш$ and $\mathcal{G}^{\mathcal{X}}_{\mathrm{conc}}$) is a group.

Factorization and decomposition of rational series

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. $S \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ iff one of the following assertions holds

1. The shifts[7] $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module (**Jacob**, 1978).
2. There is $n \in \mathbb{N}$ and (ν, μ, η), where (**Schützenberger**, 1961) $\nu \in M_{1,n}(\mathbb{C})$ and $\eta \in M_{n,1}(\mathbb{C})$ and $\mu : \mathcal{X}^* \longrightarrow M_{n,n}(\mathbb{C})$ s.t.
$$S = \nu\Big(\sum_{w\in\mathcal{X}^*} \mu(w)w\eta\Big) = \nu\big((\mu \otimes \mathrm{Id})\mathcal{D}_{\mathcal{X}}\big)\eta.$$
The triplet (ν, μ, η) is so-called linear representation of rang n of S.

Letting $M : \mathcal{X}^* \longrightarrow M_{n,n}(A\langle\langle\mathcal{X}\rangle\rangle), x \longmapsto \mu(x)x$, one has $S = \nu M(\mathcal{X}^*)\eta$,

1. $M(\mathcal{X}^*) = \prod^{\searrow}_{l\in\mathcal{L}yn\mathcal{X}} e^{\mu(P_l)S_l}$ (resp. $M(Y^*) = \prod^{\searrow}_{l\in\mathcal{L}ynY} e^{\mu(\Pi_l)\Sigma_l}$).
2. If $\{M(x)\}_{x\in\mathcal{X}}$ are upper triangular then, using diagonal and strictly upper triangular matrices $D(x)$ and $N(x)$, $M(x) = D(x) + N(x)$. Moreover, since $D(\mathcal{X}^*)N(\mathcal{X})$ is nilpotent of order k then
$$M(\mathcal{X}^*) = \sum_{0\le i\le k} (D(\mathcal{X}^*)N(\mathcal{X}^*))^i D(\mathcal{X}^*).$$

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[7] The left (resp. right) shift of $S \in A\langle\langle\mathcal{X}\rangle\rangle$ by $P \in A\langle\mathcal{X}\rangle$ is defined by
$$\forall w \in \mathcal{X}^*, \quad \langle P \triangleright S|w\rangle = \langle S|wP\rangle, \quad \langle S \triangleleft P|w\rangle = \langle S|Pw\rangle).$$

Sweedler's dual of the graded bialgebras $(K\langle\mathcal{X}\rangle, \text{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$ $(K\langle Y\rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi})$

Let $S \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ admitting (ν, μ, η) as linear representation of rank n. For any $1 \le i \le n$, let $\{(\nu, \mu, e_i)\}_{1\le i\le n}$ (resp. $\{({}^t e_i, \mu, \eta)\}_{1\le i\le n}$) be a linear representation of rang n of the rational series G_i (resp. D_i), where

$$e_i \in \mathcal{M}_{1,n}(A) \quad \text{and} \quad {}^t e_i = (0 \quad \cdots \quad 0 \quad \underset{\uparrow i}{1} \quad 0 \quad \cdots \quad 0).$$

Theorem

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. With the above notations, one has

$$S \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \iff \Delta_{\text{conc}} S = \sum_{1\le i\le n} G_i \otimes D_i.$$

The Sweedler's dual $\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X})$ (resp. $\mathcal{H}^{\circ}_{\sqcup\!\sqcup_\varphi}(Y)$) of $(K\langle\mathcal{X}\rangle, \text{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$ (resp. $(K\langle Y\rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi})$) is defined, for any series S, as follows

$$S \in \mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X}) \text{ (resp. } \mathcal{H}^{\circ}_{\sqcup\!\sqcup_\varphi}(Y)) \iff \Delta_{\text{conc}} S = \sum_{i\in I} G_i \otimes D_i,$$

where I is finite, $\{G_i, D_i\}_{i\in I}$ are series, choosen in $\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X})$ (resp. $\mathcal{H}^{\circ}_{\sqcup\!\sqcup_\varphi}(Y)$).

Corollary

$$\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X}) \cong (K^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\text{conc}}),$$
$$\mathcal{H}^{\circ}_{\sqcup\!\sqcup_\varphi}(Y) \cong (K^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \sqcup\!\sqcup_\varphi, 1_{Y^*}, \Delta_{\text{conc}}).$$

Kleene stars of the plane (1/2)

Proposition

1. *The algebras* $(K[\{x^*\}_{x\in\mathcal{X}}],\sqcup\!\sqcup,1_{\mathcal{X}^*})$ *and* $(K\langle\mathcal{X}\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})$ *are K-algebraically disjoint and* $\{x^*,l\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}$ *generates freely*
$$\begin{aligned}(K[\{x^*\}_{x\in\mathcal{X}}]\langle\mathcal{X}\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*}) &\cong (K[\{x^*\}_{x\in\mathcal{X}}][\mathcal{L}yn\mathcal{X}],\sqcup\!\sqcup,1_{\mathcal{X}^*})\\ &\cong (K[\{x^*,l\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}],\sqcup\!\sqcup,1_{\mathcal{X}^*}).\end{aligned}$$
2. *Let* $\theta:(K[\{x^*\}_{x\in\mathcal{X}}]\langle\mathcal{X}\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})\longrightarrow(K,\times,1)$ *be a* $\sqcup\!\sqcup$*-morphism. Let* $E:=K[\{\theta(x^*)\}_{x\in\mathcal{X}}]$ *and* $F:=K[\{\theta(l)\}_{l\in\mathcal{L}yn\mathcal{X}}]$.

   Then the following assertions are equivalent

   2.1 *The morphism θ is injective.*

   2.2 *The K-algebras E and F s.t. $E\cap F=K.1$ are generated by the transcendent bases* $\{\theta(x^*)\}_{x\in\mathcal{X}}$ *and* $\{\theta(l)\}_{l\in\mathcal{L}yn\mathcal{X}}$*, respectively.*

If 1. (or 2.) holds then E and F are K-algebraically disjoint and $\{\theta(x^*),\theta(l)\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}$ *generates freely*
$$K[\{\theta(x^*)\}_{x\in\mathcal{X}}][\{\theta(l)\}_{l\in\mathcal{L}yn\mathcal{X}}]\cong K[\{\theta(x^*),\theta(l)\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}].$$

Kleene stars of the plane (2/2)

Corollary

Let $R, L \in A^{\mathrm{rat}}\langle\langle \mathcal{X} \rangle\rangle$ *such that* $\langle R|1_{\mathcal{X}^*}\rangle = 1, \langle L|1_{\mathcal{X}^*}\rangle = 0$ *and* $L^* = R$. *The following assertions are equivalent*

1. R *is a* conc-*character of* $(A\langle \mathcal{X} \rangle, \mathtt{conc}, 1_{\mathcal{X}^*})$.
2. *There is a family of coefficients* $(c_x)_{x \in \mathcal{X}}$ *such that* $R = (\sum_{x \in \mathcal{X}} c_x x)^*$.
3. *The series* R *admits a linear representation of rank* 1.
4. L *belongs to the plane* $A\mathcal{X}$.
5. L *is an infinitesimal* conc-*character of* $(A\langle \mathcal{X} \rangle, \mathtt{conc}, 1_{\mathcal{X}^*})$.

Corollary

Let $\{\alpha_x\}_{x \in \mathcal{X}}, \{\beta_x\}_{x \in \mathcal{X}}, \{a_s\}_{s \geq 1}, \{b_s\}_{s \geq 1}$ *be complex numbers. Then*

$$\Big(\sum_{x \in \mathcal{X}} \alpha_x x\Big)^* ⧢ \Big(\sum_{x \in \mathcal{X}} \beta_x x\Big)^* = \Big(\sum_{x \in \mathcal{X}} (\alpha_x + \beta_x) x\Big)^*,$$

$$\Big(\sum_{s \geq 1} a_s y_s\Big)^* ⧢_\varphi \Big(\sum_{s \geq 1} b_s y_s\Big)^* = \Big(\sum_{s \geq 1} (a_s + b_s) y_s + \sum_{r,s \geq 1} a_s b_r \varphi(y_s, y_r) y_{s+r}\Big)^*.$$

THANK YOU FOR YOUR ATTENTION