A combinatorial property of multiple polylogarithms at non-positive indices

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Outline

1. Introduction — Multiple polylogarithms
2. Motivation and Background — Various relations in “non-positive settings”
3. Main results and examples of calculation

Notation

- $\mathbb{N}, \mathbb{Q}, \mathbb{C}$: a set of non-negative integers, rational numbers, complex numbers, resp.,
- $\mathbb{N}^\infty := \bigsqcup_{r \in \mathbb{N}} \mathbb{N}^r$, where $\mathbb{N}^0$ is a point,
- If R is a commutative ring with a unit and $x_1, \ldots, x_n$ are letters (or symbols), then $R\langle x_1, \ldots, x_n \rangle$ is a free associative R-algebra generated by $x_1, \ldots, x_n$,
- If $y_1, \ldots, y_n$ are symbols, then $R[y_1, \ldots, y_n]$ is a commutative R-algebra generated by $y_1, \ldots, y_n$.

Introduction—MZV and MPL

Multiple zeta value (=MZV): for $k_1, \ldots, k_d \in \mathbb{N}_{>0}$,

$$\zeta_{(k_1,\ldots,k_d)} := \sum_{n_1>\cdots>n_d>0} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} (\in \mathbb{C} \text{ if } k_1 > 1).$$

Multiple polylogarithm (=MPL): for $k_1, \ldots, k_d$,

$$\mathrm{Li}_{(k_1,\ldots,k_d)}(z) := \sum_{n_1>\cdots>n_d>0} \frac{z^{n_1}}{n_1^{k_1} \cdots n_d^{k_d}}$$

This is a holomorphic function on $\Omega := \mathbb{C} - ((-\infty, 0] \cup [1, +\infty))$, and if $k_1 > 1$, then $\lim_{z\to 1} \mathrm{Li}_{(k_1,\ldots,k_d)} = \zeta_{(k_1,\ldots,k_d)}$.

Introduction—iterated integral presentation

Iterated integral presentation of MPL

Let $\omega_0 := dz/z$, $\omega_1 := dz/(1-z)$ be 1-forms on $\Omega \subset \mathbb{C}$.
Then MPL can be expressed as an iterated integral: for $k_1, \dots, k_d \in \mathbb{N}_{>0}$,

$$\mathrm{Li}_{(k_1,\dots,k_d)}(z) = \int_0^z \omega_0^{k_1-1}\omega_1 \cdots \omega_0^{k_d-1}\omega_1.$$

Let x_0, x_1 be letters and let X^* be a free monoid generated by x_0, x_1.
For $k_1, \dots, k_d \in \mathbb{N}_{>0}$:

$$X^* x_1 \ni x_0^{k_1-1}x_1 \cdots x_0^{k_d-1}x_1 \mapsto \int_0^z \omega_0^{k_1-1}\omega_1 \cdots \omega_0^{k_d-1}\omega_1 \in \mathcal{O}(\Omega)$$

Here, $\mathcal{O}(\Omega)$ is a commutative ring of holomorphic functions on Ω.
Then $\mathbb{C}$-linear map extended from this correspondence is injective and we can write $\mathrm{Li}_w := \mathrm{Li}_{(k_1,\dots,k_d)}$ for a word $w = x_0^{k_1-1}x_1 \cdots x_0^{k_d-1}x_1 \in X^* x_1$.

Introduction—shuffle relation

Shuffle product

A *shuffle product* $\sqcup\!\sqcup$ on $R\langle x_0, x_1\rangle$ is defined as follows:
for any $u, v \in R\langle x_0, x_1\rangle$,

- $u \sqcup\!\sqcup 1 = 1 \sqcup\!\sqcup u = u$,
- $x_i u \sqcup\!\sqcup x_j v = x_i(u \sqcup\!\sqcup x_j v) + x_j(x_i u \sqcup\!\sqcup v)$ $(i, j = 0$ or $1.)$

Consider the $\mathbb{C}$-linearly extended map $\mathrm{Li}_\bullet : \mathbb{C}\langle x_0, x_1\rangle \to \mathcal{O}(\Omega); u \mapsto \mathrm{Li}_u$.

Shuffle relation

For any $u, v \in \mathbb{C}\langle x_0, x_1\rangle$, $\mathrm{Li}_{u \sqcup\!\sqcup v} = \mathrm{Li}_u \mathrm{Li}_v$.

So, $\mathrm{Li}_\bullet : (\mathbb{C}\langle x_0, x_1\rangle, \sqcup\!\sqcup) \to \mathcal{O}(\Omega)$ is a ring homomorphism.

Motivation

We want relations in case that $k_1, \ldots, k_d$ are all non-positive!

In the first place, MPL *with non-positive indices (=NPMPL)* is defined as follows: for $(k_1, \ldots, k_d) \in \mathbb{N}^{\infty}$,

$$\mathrm{Li}^{-}_{(k_1,\ldots,k_d)} := \mathrm{Li}_{(-k_1,\ldots,-k_d)} = \sum_{n_1 > \cdots > n_d > 0} n_1^{k_1} \cdots n_d^{k_d} z^{n_1}.$$

If $\theta := z\partial_z, \lambda := z/(1-z) = \mathrm{Li}^{-}_{0}$, then

$$\mathrm{Li}^{-}_{(k_1,\ldots,k_d)} = \theta^{k_1}(\lambda \cdot \theta^{k_2}(\cdots \lambda \cdot \theta^{k_d}(\lambda)) \cdots).$$

Therefore, all NPMPLs are 1-variable rational function on Ω, precisely they forms $zf(z)/(1-z)^m$ for some $m \in \mathbb{N}_{>0}$ and some $f(z) \in \mathbb{C}[z]$.

Motivation

Consider the similar correspondence: for $(k_1, \ldots, k_d) \in \mathbb{N}^\infty$,

$$X^* x_1 \ni x_0^{k_1} x_1 \cdots x_0^{k_d} x_1 \mapsto \theta^{k_1}(\lambda \cdot \theta^{k_2}(\cdots \lambda \cdot \theta^{k_d}(\lambda)) \cdots) \in \mathbb{C}(z) \subset \mathcal{O}(\Omega),$$

and a $\mathbb{C}$-linear extension

$$\mathrm{Li}_\bullet^- : \mathbb{C}\langle x_0, x_1 \rangle x_1 \to \mathbb{C}(z); w \mapsto \mathrm{Li}_w^-$$

for any word $w = x_0^{k_1} x_1 \cdots x_0^{k_d} x_1 \in X^* x_1$.

$$\begin{array}{ccc} \mathbb{C}\langle x_0, x_1 \rangle x_1 & \xrightarrow{\mathrm{Li}_\bullet^-} & \mathbb{C}(z) \\ {\scriptstyle x_0 \cdot -}\downarrow & & \downarrow{\scriptstyle \theta} \\ C\langle x_0, x_1 \rangle x_1 & \xrightarrow{\mathrm{Li}_\bullet^-} & \mathbb{C}(z) \end{array} \qquad \begin{array}{ccc} \mathbb{C}\langle x_0, x_1 \rangle x_1 & \xrightarrow{\mathrm{Li}_\bullet^-} & \mathbb{C}(z) \\ {\scriptstyle x_1 \cdot -}\downarrow & & \downarrow{\scriptstyle \lambda \cdot -} \\ C\langle x_0, x_1 \rangle x_1 & \xrightarrow{\mathrm{Li}_\bullet^-} & \mathbb{C}(z) \end{array}$$

Motivation

Examples of NPMPL

- $\mathrm{Li}^{-}_{x_0x_1x_0x_1} = \mathrm{Li}^{-}_{(1,1)} = \theta(\lambda\theta(\lambda)) = (\theta(\lambda))^2 + \lambda\theta^2(\lambda)$
  $= z^2/(1-z)^4 + z/(1-z)\cdot(z^2+z)/(1-z)^3$
  $= (z^3+2z^2)/(1-z)^4.$
- $\mathrm{Li}^{-}_{x_1x_0x_1x_0^2x_1} = \mathrm{Li}^{-}_{(0,1,2)} = \lambda(\theta(\lambda\theta^2(\lambda))) = \lambda\theta(\lambda)\theta^2(\lambda) + \lambda^2\theta^3(\lambda)$
  $= z/(1-z)\cdot z/(1-z)^2\cdot(z^2+z)/(1-z)^3$
  $\qquad + z^2/(1-z)^2\cdot(z^3+4z^2+z)/(1-z)^4$
  $= (z^4+z^3)/(1-z)^6 + (z^5+4z^4+z^3)/(1-z)^6$
  $= (z^5+5z^4+2z^3)/(1-z)^6.$

Motivation

Then, we have the natural question as follows.

Question

What relations exist in non-positive case?

$\rightsquigarrow$ Various relations and methods to provide ones have been founded!
Many of them are studied from combinatorial viewpoints.

Goal in this talk

We show relations of a method to obtain $\mathbb{Q}$-linear functional equations by introducing Magnus polynomials in non-commutative ring $\mathbb{C}\langle x_0, x_1\rangle$ (The definition of Magnus Polynomials will be given in a later slide.)

Background

In positive case, really various relations (including a shuffle relation) are obtained by Euler, Zagier, Kaneko, ...

In non-positive case, various relations on MPL and MZV have been also studied by many mathematicians.

For example...

[Guo, Zhang, '08], [Furusho, Komori, Matsumoto, Tsumura, '15] and so on.

Background

A non-positive version of shuffle relation has been studied at various angles.

In [Ebrahim-Fard, Manchon, Singer, '17] and [Duchamp, H. N. Minh, Ngo, '17], they define "non-positive version" of shuffle products from different viewpoints respectively.

Indeed...
[EMS] — the products are defined along a theory of Rota-Baxter algebras.
[DMN] — the products of two elements are defined as splitting ones of each images to an associative algebra.

These papers above provide different methods to obtain relations of MPL or MZV at non-positive indices from different angles respectively.

Preliminaries for main results

R : integral domain with char. 0,
$[,] : R\langle x_0, x_1\rangle \times R\langle x_0, x_1\rangle \to R\langle x_0, x_1\rangle; (u, v) \mapsto [u, v] := uv - vu$,
If $n \in \mathbb{N}$, then a polynomial $x_1^{(n)} \in R\langle x_0, x_1\rangle$ is defined as follows:

- $x_1^{(0)} := x_1$,
- $x_1^{(n+1)} := [x_0, x_1^{(n)}]$.

Definition: Magnus polynomial (Magnus, 1937)

Let $\mathbf{k} = (k_1, \ldots, k_d; k_\infty) \in \mathbb{N}^\infty \times \mathbb{N}$ $(d \geq 0)$.
Magnus polynomial $M^{(\mathbf{k})}$ for an index $\mathbf{k}$ is defined as follows

$$M^{(\mathbf{k})} := x_1^{(k_1)} \cdots x_1^{(k_d)} x_0^{k_\infty}.$$

Note that $M^{(\mathbf{k})} = x_0^{k_\infty}$ if $d = 0$.

Main results

Let $R = \mathbb{C}$, and recall $\mathbb{C}$-linear map $\mathrm{Li}^-_\bullet : \mathbb{C}\langle x_0, x_1 \rangle x_1 \to \mathbb{C}(z)$.

Theorem (K, master thesis, February '25)

For $\mathbf{k} = (k_1, \ldots, k_d; k_{d+1}) \in \mathbb{N}^\infty \times \mathbb{N}$,

$$\mathrm{Li}^-_{M^{(\mathbf{k})}x_1} = \mathrm{Li}^-_{k_1} \cdots \mathrm{Li}^-_{k_{d+1}}.$$

The hint of this proof is combinatorial amounts appearing in [DMN] and [Nakamura, '23]:

$$\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots\binom{s_1+\cdots+s_{r-1}-k_1-\cdots-k_{r-2}}{k_{r-1}}.$$

Main results

Example 1

For $M^{(1;1)} = [x_0, x_1]x_0 = x_0x_1x_0 - x_1x_0^2$,

$$\begin{aligned} \mathrm{Li}^-_{M^{(1;1)}x_1} &= \mathrm{Li}^-_{(1,1)} - \mathrm{Li}^-_{(0,2)} \\ &= \theta(\lambda\theta(\lambda)) - \lambda\theta^2(\lambda) && \text{(by definition)} \\ &= (\theta(\lambda))^2 + \lambda\theta^2(\lambda) - \lambda\theta^2(\lambda) && \text{(by Leibniz rule of } \theta\text{)} \\ &= (\mathrm{Li}^-_1)^2 \end{aligned}$$

Main results

Corollary: kernel of Li^- and permutation

For $\sigma \in \mathfrak{S}_{d+1}$, $(M^{(\mathbf{k})} - M^{(\sigma(\mathbf{k}))})x_1 \in \mathrm{Ker}\,\mathrm{Li}^-_\bullet$.
Here, $\sigma(\mathbf{k}) = (k_{\sigma(1)}, \ldots, k_{\sigma(d)}; k_{\sigma(d+1)})$.

This proposition gives a new method to obtain $\mathbb{Q}$-linear functional equations between NPMPLs!

Let us calculate an easy but non-trivial example in next page.

Example

Example 2

$(M^{(0,1;2)} - M^{(1,2;0)})x_1 \in \mathrm{Ker}\,\mathrm{Li}^-_\bullet$.

- $M^{(0,1;2)}x_1 = x_1[x_0, x_1]x_0^2x_1$
  $= x_1x_0x_1x_0^2x_1 - x_1^2x_0^3x_1$
- $M^{(1,2;0)}x_1 = [x_0, x_1][x_0, [x_0, x_1]]x_1$
  $= x_0x_1x_0^2x_1^2 - 2x_0x_1x_0x_1x_0x_1 + x_0x_1^2x_0^2x_1 - x_1x_0^3x_1^2 +$
  $2x_1x_0^2x_1x_0x_1 - x_1x_0x_1x_0^2x_1$

So, $(M^{(0,1;2)} - M^{(1,2;0)})x_1 = 2x_1x_0x_1x_0^2x_1 - x_1^2x_0^3x_1 - x_0x_1x_0^2x_1^2 +$
$2x_0x_1x_0x_1x_0x_1 - x_0x_1^2x_0^2x_1 + x_1x_0^3x_1^2 - 2x_1x_0^2x_1x_0x_1$,
and this means that the following equation holds:

$$2\,\mathrm{Li}^-_{(0,1,2)} + 2\,\mathrm{Li}^-_{(1,1,1)} + \mathrm{Li}^-_{(0,3,0)} = \mathrm{Li}^-_{(0,0,3)} + \mathrm{Li}^-_{(1,2,0)} + \mathrm{Li}^-_{(1,0,2)} + 2\,\mathrm{Li}^-_{(0,2,1)}$$

Reference

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