Extension by continuity of the domain of Poly- and Hyper- logarithms.

Stars of the Plane.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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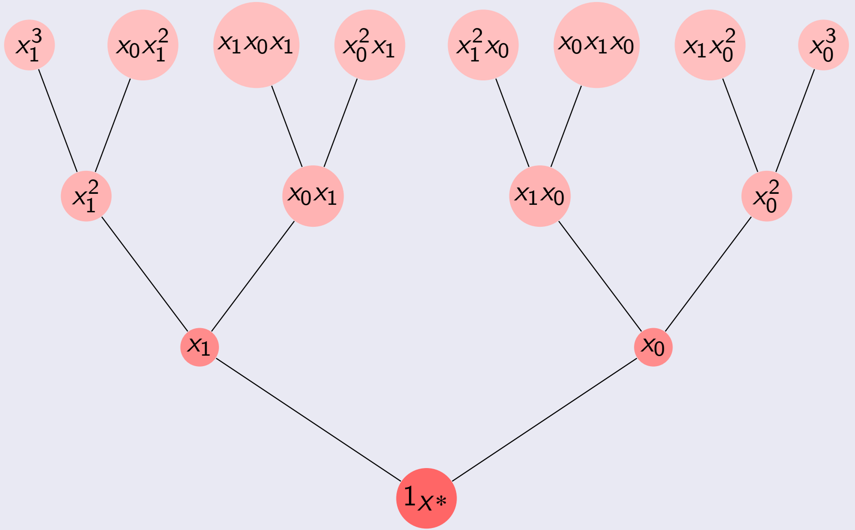
Introduction

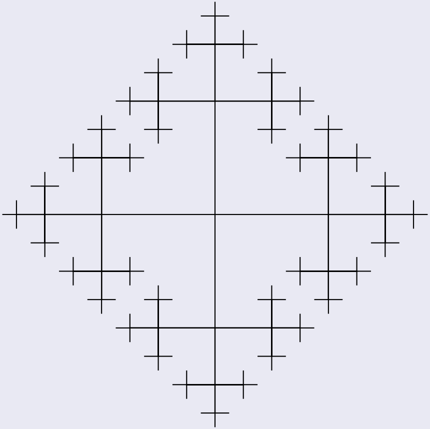
1. The story of automata theory (in the large, i.e. Eilenberg-Schützenberger machines) is all about states, ations (command letters), alphabets, transitions and multiplicities (outputs).
2. In this review, we will see several sets of states
   1. (Free) monoid on the alphabet $X = \{x_0, x_1\}$
   2. (If times permits), the free group (on X)

The free monoid $\{x_0, x_1\}^*$.

skip slide PNCDE



Free Group, here $\Gamma(a, b)$.



Factorizations

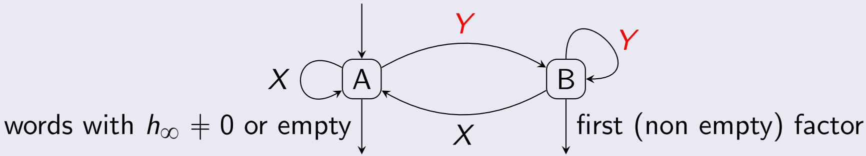
Two years ago (CAP10), one of us (H. Nakamura) began his talk by some combinatorics on words (stringology) i.e. any string (word) on the alphabet $\Sigma = \{X, Y\}$ could be written

$$w = X^{h_1} Y X^{h_2} Y \cdots Y X^{h_d} Y \,|\, X^{h_\infty} \ . \tag{1}$$

Doing this, save the last factor X^{h_∞}, we obtain a factorization into blocs of the form $X^h Y$. We will later write this set $X^* Y = Y + XY + X^2 Y + \ldots$, the (free) monoid they generate $(X^* Y)^* = 1 + (X^* Y)^+$ [a]. The set of all words, therefore, is

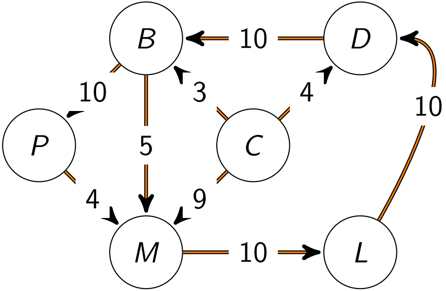
$$(X + Y)^* = (X^* Y)^* X^* = (X^* Y)^+ X^* + X^*, \tag{2}$$

an instance of Lazard's elimination theorem (discussed in CAP 9).
Factorization (1) can be computed by the following (boolean or $\mathbb{N}$-) automaton



[a] Where $S^+ = S + S^2 + \cdots$ and $S^* = 1 + S^+$

A simple transition system: flow charts or flow diagrams



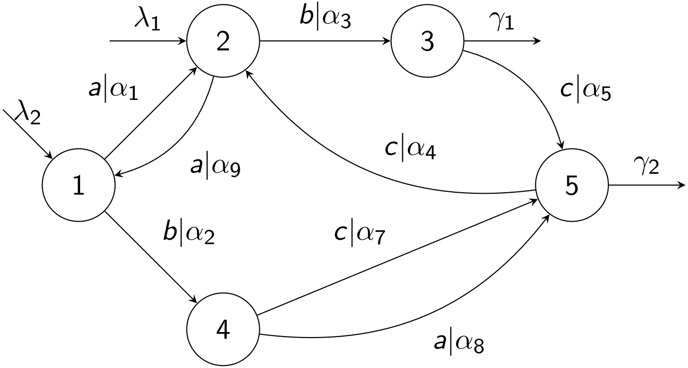
Directed graph weighted by numbers which can be lengths, time (durations), costs, fuel consumption, probabilities. This graph is equivalent to a square matrix. Coefficients are taken in different semirings (i.e. rings without the “minus” operation, as tropical or [min,+]) according to the type of computations to be done. Tropical semirings were so called by MPS school because they were founded by the Hungarian-born Brazilian mathematician and computer scientist Imre Simon. Evaluation is done by multiplications in series and addition in parallel.

Weighted (or multiplicity) automata: the forefathers

Samuel Eilenberg, *Automata, Languages, and Machines (Vol. A & B)* Acad. Press, New York, (1974)

Marcel-Paul Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*

Multiplicity Automaton (Eilenberg, Schützenberger)



Example: Evaluate 2.*bccabc*.

Multiplicity automaton (linear representation) & behaviour

Linear representation

Due to the left-to-right word reading, it is

$$\lambda = \begin{pmatrix}\lambda_2 & \lambda_1 & 0 & 0 & 0\end{pmatrix}, \qquad \gamma = \begin{pmatrix}0 & 0 & \gamma_1 & 0 & \gamma_2\end{pmatrix}^T$$

$$\mu(a) = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 \\ \alpha_9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

$$\mu(c) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 0 & 0 & \alpha_7 \\ 0 & \alpha_4 & 0 & 0 & 0 \end{pmatrix}$$

Multiplicities.

1. Multiplicities are taken within a semiring R. Each time you change R, you change your universe.
2. If $R = \mathbb{B}$, you get the theory of languages, if $R = \mathbb{N}$, you are able to count the paths for example.
3. If R is commutative, you have the theory of rational series and if R is a field, you get a way to compute within Sweedler's duals.
4. If the multiplicities are probabilities, you get stochastic automata.
5. **But** R does not need to be commutative
    1. If $R = \mathbf{k}\langle\Gamma\rangle$ for some alphabet Γ, you get transducers
    2. R can be a semiring of operators, this opens the door to application of rational identities to the plane of transition matrices.

Linear representation & Behaviour

Remark

For a right-to-left word reading, data have to be transposed.

Non commutative series

Series are functions $X^* \to R$ where R is a semiring (i.e. a ring without the "minus" operation as example the tropical semiring). We have different ways to consider a series, namely:

Math: Functions, elements of a dual (total, restricted, Sweedler's &c.)

Computer Sci.: Behaviour of a system (automaton, transducer, grammar &c.)

Physics: Non comm. diff. equations, evaluation of paths, normal orderings &c.

Behaviour of a "word machine", the series $\mathcal{B}(\mathcal{M})$.

$$\langle \mathcal{B}(\mathcal{M}) | w \rangle = \lambda\,\mu(w)\,\gamma = \sum_{\substack{i,j \\ \text{states}}} \lambda(i) \underbrace{\left(\sum weight(p) \right)}_{\text{weight of all paths } (i) \to (j) \text{ with label } w} \gamma(j) \quad (4)$$

Operations and definitions on series (R semiring).

Addition, Scaling: As for functions because $R\langle\langle X\rangle\rangle = R^{X^*}$ (viewed as R-R modules)
Concatenation: $f.g(w) = \sum_{w=uv} f(u)g(v)$
Polynomials: Series s.t. $supp(f) = \{w\}_{f(w)\neq 0}$ is finite.
The set of polynomials will be denoted $R\langle X\rangle$.
Pairing: $\langle S|P\rangle = \sum_{w\in X^*} S(w)P(w)$ (S series, P polynomial)
Summation: $\sum_{i\in I} S_i$ summable iff f or all $w \in X^*$, $i \mapsto \langle S_i|w\rangle$ is finitely supported. In particular, we have

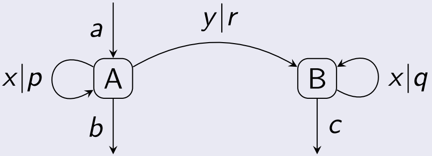
$$\sum_{i\in I} S_i := \sum_{w\in X^*} \Big(\sum_{i\in I}\langle S_i|w\rangle\Big)\, w$$

Remark: This notion is exactly the one of limit of the net of partial sums $(\sum_{i\in F} S_i)_{F\subset_{finite} I}$ with respect to the sup-lattice of finite subsets of I, topology being the product of discrete topologies on R (see [13] "summable").

Operations and definitions on series (R semiring)/2

Star: For all series S s.t. $\langle S|1_{X*}\rangle = 0$, the family $(S^n)_{n\geqslant 0}$ is summable and we set $S^* := \sum_{n\geqslant 0} S^n = 1 + S + S^2 + \cdots$ (if R is a ring, we have $S^* = (1-S)^{-1}$) and the **plus-notation** $S^+ := \sum_{n\geqslant 0} S^n = S + S^2 + \cdots$ (again, if R is a ring we have $S^+ = S.(1-S)^{-1} = (1-S)^{-1}.S$).
Shifts: $\langle u^{-1}S|w\rangle = \langle S|uw\rangle$ and $\langle Su^{-1}|w\rangle = \langle S|wu\rangle$.

Let $\mathcal{M}$ be the automaton (p, q, r, a, b, c can be operators).



$$I = \begin{pmatrix} a & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} p.x & r.y \\ 0 & q.x \end{pmatrix} \; F = \begin{pmatrix} b \\ c \end{pmatrix}$$

$$T^* = \begin{pmatrix} (p.x)^* & (p.x)^*.r.y.(q.x)^* \\ 0 & (q.x)^* \end{pmatrix}$$

$$\mathcal{B}(\mathcal{M}) = I.T^*.F = a.(p.x)^*.b + a.(p.x)^*.r.y.(q.x)^*.b$$

Rational series (Sweedler's duals & Schützenberger's shifts)

skip slide

Theorem A (k field, X finite), see [11].

Let $S \in \mathbf{k}\langle\langle X \rangle\rangle$ TFAE

i) The family $(Su^{-1})_{u \in X^*}$ is of finite rank.

ii) The family $(u^{-1}S)_{u \in X^*}$ is of finite rank.

iii) The family $(u^{-1}Sv^{-1})_{u,v \in X^*}$ is of finite rank.

iv) It exists $n \in \mathbb{N}$, $\lambda \in \mathbf{k}^{1\times n}$, $\mu : X^* \to \mathbf{k}^{n\times n}$ (a multiplicative morphism) and $\gamma \in \mathbf{k}^{n\times 1}$ such that, for all $w \in X^*$

$$(S, w) = \lambda\mu(w)\gamma \qquad (5)$$

v) The series S is in the closure of $\mathbf{k}\langle X \rangle$ for $(+, conc, ^*)$ within $\mathbf{k}\langle\langle X \rangle\rangle$.

Definition

A series which fulfills one of the conditions of Theorem A will be called *rational*. The set of these series will be denoted by $k^{rat}\langle\langle X \rangle\rangle$. In the theory of Hopf algebras it is Sweedler's dual of $\mathbf{k}\langle X \rangle$.

Remarks

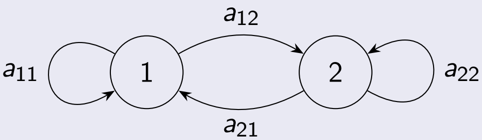
1. (i $\leftrightarrow$ iii) needs $\mathbf{k}$ to be a field.
2. (iv) needs X to be finite, (iv $\leftrightarrow$ v) is known as the theorem of Kleene-Schützenberger (M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.)
3. For the sake of Combinatorial Physics (where the alphabets can be infinite), **(iv)** has been extended to infinite alphabets and replaced by **iv')** The series S is in the rational closure of $\mathbf{k}^X$ (linear series) within $\mathbf{k}\langle\langle X\rangle\rangle$.
4. When $\mathbf{k}$ is a ring, the rational closure of a subset $P \subset \mathbf{k}\langle\langle X\rangle\rangle$ is exactly the inverse-closed subalgebra of $\mathbf{k}\langle\langle X\rangle\rangle$ generated by P.
5. In the vein of (v) expressions like ab^* or identities like $(ab^*)^*a^* = (a+b)^*$ (Lazard's elimination) will be called rational.

Sweedler's duals & Kleene-Schüzenberger's Theorem./2

6. For the needs of CS, an analogue of Theorem A has been proved for **k** a commutative semiring (see [16, 12, 14]) where "is of finite rank" is replaced *mutatis mutandis* by "is contained in a shift-invariant submodule of finite type".
7. Contrariwise to the case when **k** is a field, the property of being a submodule of finite type is not hereditary (as soon as we only have a ring). It can then happen that the module generated by the shifts of a rational series be not of finite type. The case $\mathbf{k} = \mathbb{N},\ S = a^*a^* = \sum_{n\geqslant 0}(n+1)a^n$ is typical: when one computes the shifts *on the series* $S = a^*a^* = \sum_{n\geqslant 0}(n+1)a^n$ (considered as a function), we get a shift-invariant module of infinite type whereas, following Eilenberg [11], when we perform them on its rational expression a^*a^*, we get a FS automaton.
8. This theorem is linked to the following subjects: Representative functions on X^* (see Eiichi Abe [1], Chari & Pressley [4]), Sweedler's duals [9] &c).

Words and paths

Powers of a (generic) transfer matrix



$$T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$T^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{22}^2 + a_{21}a_{12} \end{pmatrix}$$

$$T^n = \begin{pmatrix} \sum n\text{-paths } 1 \to 1 & \sum n\text{-paths } 1 \to 2 \\ \sum n\text{-paths } 2 \to 1 & \sum n\text{-paths } 2 \to 2 \end{pmatrix}$$

Star notation and Mc Naughton-Yamada formulae.

We set $T^+ := \sum_{n\geqslant 1} T^n$, $T^* := 1 + T^+ = 1 + T + T^2 + \cdots = \sum_{n\geqslant 0} T^n$. This matrix T^* is the (unique) solution $R \in \mathbf{k}\langle\langle a_{ij} \rangle\rangle$ of the self-reproducing equations

$$R = I + TR = I + RT$$

Mac Naughton-Yamada (with multiplicities) formulae.

Expressions

$$\text{With } T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ we have } T^* = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ with} \tag{6}$$

$$A_{11} = (a_{11} + a_{12}a_{22}^*a_{21})^* \qquad A_{12} = A_{11}a_{12}a_{22}^* \ (\text{or } = a_{11}^*a_{12}A_{22})$$
$$A_{21} = A_{22}a_{21}a_{11}^* \ (\text{or } = a_{22}^*a_{21}A_{11}) \qquad A_{22} = (a_{22} + a_{21}a_{11}^*a_{12})^* \tag{7}$$

Applications of "word machines".

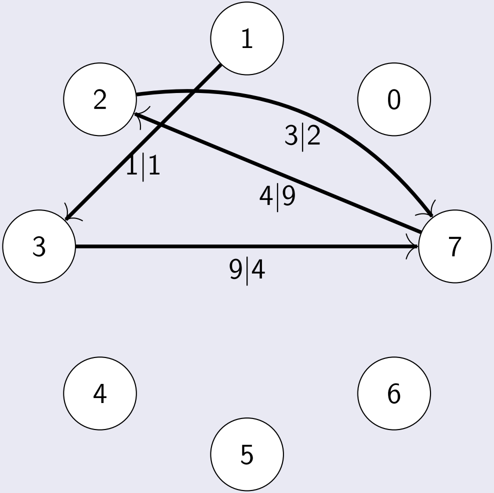
These expressions have many incarnations/applications. Among them

- Sweedler's duals (and explicit/combinatorial computations within them)
- NCDE and, in particular, Hyper- (and Poly-) logarithms (today)
- Noncommutative geometry

Remarks

1. If the multiplicities of slide 12 are taken in some $\Sigma \times \mathbf{k}\langle\Gamma\rangle$ (resp. $\Sigma \times \Gamma$), we have a finite-state (resp. letter-to-letter) transducer.
2. Σ (resp. Γ) is called (and understood as) input (resp. output) alphabet.
3. If, in all loops, multiplicities belong to $\mathbf{k}_+\langle\langle\Gamma\rangle\rangle$ (i.e. series with no constant term), it is always possible to compute the star of the transfer matrix.
4. In a more general way, if multiplicities are taken in an augmented ring $(\mathcal{A}, \epsilon)$ which is complete (i.e. Hausdorff and complete with the topology defined by $\{(\mathcal{A}_+)^n\}_{n\geqslant 0}$) and $a_{ij} \in \mathcal{A}_+$ the generic matrix T possesses a star (computable by formulas Eq. 7). This is the case of many rings of formal series ($\mathbf{k}[[X]]$, $\mathbf{k}[[M]]$).
5. One obtains rational identities by factoring the sets of paths differently (see dual expressions of A_{12}, A_{21} in formulas formulas Eq. 7).

Application 1: Transducer



1	1	9	4	3	8			
	3	9			1	4	9	2
		7	4					
			2	3				
				7				

With this simple transducer, we see that "states" can mean "cases". Here $\Sigma = \Gamma = \{0, \cdots 9\}$.

Application 2: Difference and differential equations

1. We have seen the shifts which give rise to a calculus on rational expressions, that we recall here
   1. x^{-1} is (left and right) linear
   2. $x^{-1}(E.F) = x^{-1}(E).F + \langle E|1_{X*}\rangle x^{-1}(F)$
   3. $x^{-1}(E^*) = x^{-1}(E).E^*$

   but not only, as transpose of right and left multiplication, they operate on series and can be used to set difference equations.
2. In the same way, we can consider differential equations of the type

$$\mathbf{d}(S) = MS \ ; \ \langle S|1_{X*}\rangle = 1_{\mathcal{A}} \tag{8}$$

   where $\mathbf{d}(S) = \sum_{w\in X*}(\langle S|w\rangle)'.w$ (term by term differentiation) and M, the multiplier, is a series without constant term. The case when $M = \sum_{x\in X} u_x\, x$ (homogeneous of degree one) is of particular interest and is used to better understand iterated integrals.

Construction of a solution: Picard iterations.

1. In the case when $(\mathcal{A}, d)$ admits a section (then $(\mathcal{A}, d, \textstyle\int)$), one can construct a particular solution of

$$\begin{cases} \mathbf{d}(S) &= M.S \text{ with } M \in \mathcal{A}_+\langle\langle X\rangle\rangle \\ \langle S|1_{X^*}\rangle &= 1_{\mathcal{A}} \end{cases} \tag{9}$$

using Picard iterations.

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int M.S_n \tag{10}$$

Then, it is not difficult to see that S_n admits a limit S^{Pic} which satisfies (9).
The complete set of solutions of (9) is $S^{Pic}.\mathbb{C}\langle\langle X\rangle\rangle$.

Example of iterated integrals. ⏩ return ⏩ Lie group

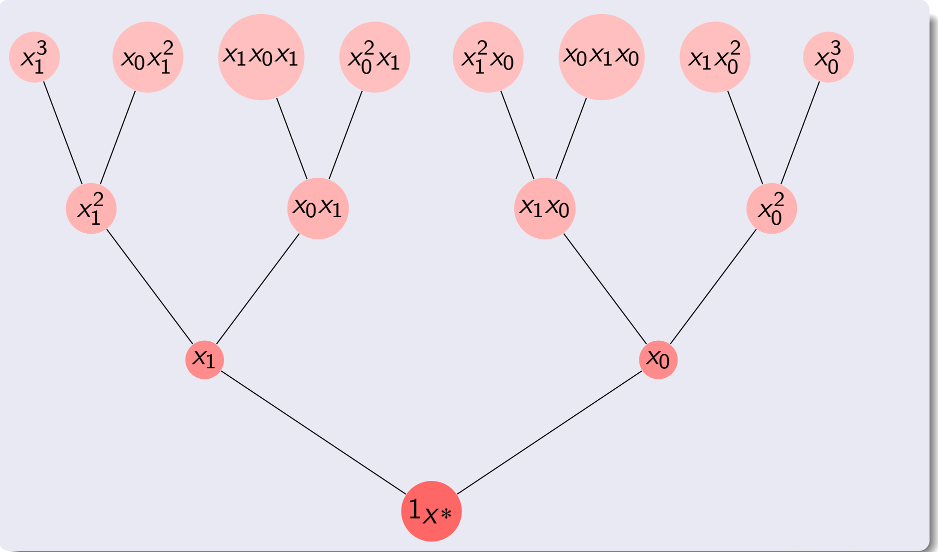
2. For example, let us consider a perturbated version of the polylogarithmic system (here $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$, $h \in \mathcal{H}(\Omega)$ and $S \in \mathcal{H}(\Omega)\langle\langle x_0, x_1 \rangle\rangle$)

$$\begin{cases} \mathbf{d}(S) = \left(\frac{x_0}{z} + \frac{x_1}{1-z} + h(z).[x_0, x_1]\right).S & (NCDE\text{-}Per1) \\ S(z_0) = 1_{X^*} & (Init.\ Cond.) \end{cases} \quad (11)$$

$S_{z_0}^{Pic}(z)$ satisfies and can be computed by the following recursion

$$\langle S|w\rangle[z] = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_{z_0}^{z} \langle S|u\rangle[s] \frac{ds}{s} & \text{if} \quad w = x_0 u \\ \int_{z_0}^{z} \frac{ds}{1-s} = \log(\frac{1-z_0}{1-z}) & \text{if} \quad w = x_1 \\ \langle S|x_0 x_1 u\rangle[z] + \int_{z_0}^{z} \langle S|u\rangle[s].h(s)\, ds & \text{if} \quad w = x_1 x_0 u \\ \int_{z_0}^{z} \langle S|x_1 u\rangle[s]\, \frac{ds}{1-s} & \text{if} \quad w = x_1 x_1 u \end{cases}$$

Computation by levels and from left to right.



(Very) quick review of Polylogarithms.

3. Here we consider $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$
4. Classical polylogarithms are defined, for $k \geqslant 1, |z| < 1$, by

$$-\log(1-z) = \mathrm{Li}_1 = \sum_{n \geqslant 1} \frac{z^n}{n^1};\ \mathrm{Li}_2 = \sum_{n \geqslant 1} \frac{z^n}{n^2};\ \ldots;\ \mathrm{Li}_k(z) := \sum_{n \geqslant 1} \frac{z^n}{n^k}$$

5. Multiple polylogarithms extend classical ones twofold, they are indexed by words (i.e. lists) and satisfy the following system

$$\begin{cases} \mathbf{d}(S) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right).S & (NCDE) \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 \log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} & (Asympt.\ Init.\ Cond.) \end{cases} \tag{12}$$

from the general theory (differential Galois group of NCDE + Lazard elimination), this system has a unique solution over Ω which is precisely Li (called G_1 in [6]).

Explicit construction of Li.

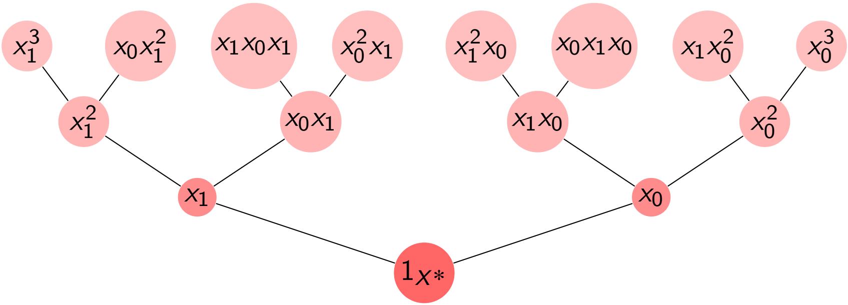
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\langle \mathrm{Li} \,|w\rangle[z] = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \langle \mathrm{Li}\,|u\rangle[s]\frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \langle \mathrm{Li}\,|u\rangle[s]\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \langle \mathrm{Li}\,|u\rangle[s]\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \end{cases}$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (improper for the fourth line) are well defined and (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ is Li (G_1 in [1]).



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $t_0 = 0$

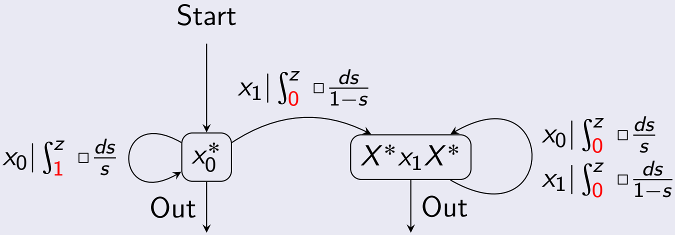
$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geqslant 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geqslant 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geqslant 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geqslant 1}\frac{z^{n_1}}{n_1^2n_2} \quad ; \quad \langle S|x_0^n\rangle = \frac{\log^n(z)}{n!}$$

Computation of integrators by transducer

The two cases of the transducer are given by the languages x_0^* and $X^*x_1X^*$ and the generating series Li by the behaviour of the transducer



$$T = \begin{pmatrix} x_0|\int_1^z \square \frac{ds}{s} & 0 \\ x_1|\int_0^z \square \frac{ds}{1-s} & x_0|\int_0^z \square \frac{ds}{s} + x_1|\int_0^z \square \frac{ds}{1-s} \end{pmatrix}$$

$$\text{Alphabet} \;:\; \Sigma = \{x_0, x_1\} \times \mathrm{End}(W) \simeq \mathrm{End}(W).\{x_0, x_1\} \text{ with } W \subset \mathcal{H}(\Omega) \quad (13)$$

The space W.

1. We define $\mathcal{H}_0$ as the space of $f \in \mathcal{H}(\Omega)$ admitting an analytic continuation around zero. This space embeds naturally in $\mathcal{H}(\Omega)$. Then we define W as the algebra generated by $\mathcal{H}_0(\Omega)$ and $\log(z)$.
2. Due to the fact that $f \in W \setminus \{0\} \Longrightarrow f \sim_0 \alpha_k . z^k$ for some k and $\alpha_k \neq 0$, it is an easy exercise to see that W is a free $\mathcal{H}_0$-module with basis $\{\log^n(z)\}_{n \geqslant 0}$. We also remark that W is closed by all the integrators. More precisely, with splitting $\mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathbb{C}.1_\Omega$ w.r.t. the evaluation at zero (i.e. $\mathcal{H}_0^+ = \ker(\delta_0)$) we see that

$$W = W_+ \bigoplus \underbrace{\left(\oplus_{n\geqslant 0} \mathbb{C}.\log^n(z)\right)}_{W_r (=\text{rightmost branch})} = W_+ \bigoplus W_r \ . \tag{14}$$

   1. the integrator $\int_1^z, \square \frac{ds}{s}$ acts within W_r
   2. W_+ is made of sums $z^p \log^q(z)$ with $p \geqslant 1$ so that the other integrators (with lower bound 0) act in W_+
   3. $\int_0^z \square \frac{ds}{1-s}$ sends W_r to W_+.

Computation of the behaviour/1

Linear representation

Due to the fact that the action is on the left (i.e. right-left reading of the word), we have (with the alphabet $\mathrm{End}(W).\{x_0, x_1\}$)

$$\lambda = \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1_\Omega \\ 0 \end{pmatrix}$$

$$T = \begin{pmatrix} \int_{1}^{z} \square \frac{ds}{s}.x_0 & 0 \\ \int_{0}^{z} \square \frac{ds}{1-s}.x_1 & \int_{0}^{z} \square \frac{ds}{s}.x_0 + \int_{0}^{z} \square \frac{ds}{1-s}.x_1 \end{pmatrix}$$

Computation of the star/1

Applying formulas of Eq. (7), we get

$$T^* = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & 0 \\ a_{22}^* a_{21} a_{11}^* & a_{22}^* \end{pmatrix}$$

Computation of the star/2

This star can be factored, considering that

$$T = \begin{pmatrix} \int_{1}^{z} \square \frac{ds}{s}.x_0 & 0 \\ \int_{0}^{z} \square \frac{ds}{1-s}.x_1 & \int_{0}^{z} \square \frac{ds}{s}.x_0 + \int_{0}^{z} \square \frac{ds}{1-s}.x_1 \end{pmatrix} =$$

$$\begin{pmatrix} \int_{1}^{z} \square \frac{ds}{s} & 0 \\ 0 & \int_{0}^{z} \square \frac{ds}{s} \end{pmatrix} .x_0 + \begin{pmatrix} 0 & 0 \\ \int_{0}^{z} \square \frac{ds}{1-s} & \int_{0}^{z} \square \frac{ds}{1-s} \end{pmatrix} .x_1 =$$

$$T_0.x_0 + T_1.x_1$$

and using formula (2), we get

$$T^* = \Big((T_0.x_0)^* T_1.x_1\Big)^* (T_0.x_0)^* = \Big((T_0.x_0)^* T_1.x_1\Big)^+ (T_0.x_0)^* + (T_0.x_0)^* \tag{15}$$

About the asymptotic condition

3. We then have

$$\mathrm{Li} = \begin{pmatrix} 1 & 1 \end{pmatrix} T^* \begin{pmatrix} 1_\Omega \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \Big((T_0.x_0)^* T_1.x_1 \Big)^+ (T_0.x_0)^* \begin{pmatrix} 1_\Omega \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \end{pmatrix} (T_0.x_0)^* \begin{pmatrix} 1_\Omega \\ 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix} \Big((T_0.x_0)^* T_1.x_1 \Big)^+ (T_0.x_0)^* \begin{pmatrix} 1_\Omega \\ 0 \end{pmatrix}}_{\mathrm{Li}^+ \text{ only words s.t. } |w|_{x_1} > 0} + e^{x_0 \log(z)} \tag{16}$$

In this way $\mathrm{Li} = \mathrm{Li}^+ + e^{x_0 \log(z)}$ and we get

$$\lim_{z -> 0} e^{-x_0 \log(z)} \mathrm{Li} = \lim_{z -> 0} \mathrm{Li}\ e^{-x_0 \log(z)} = 1 \tag{17}$$

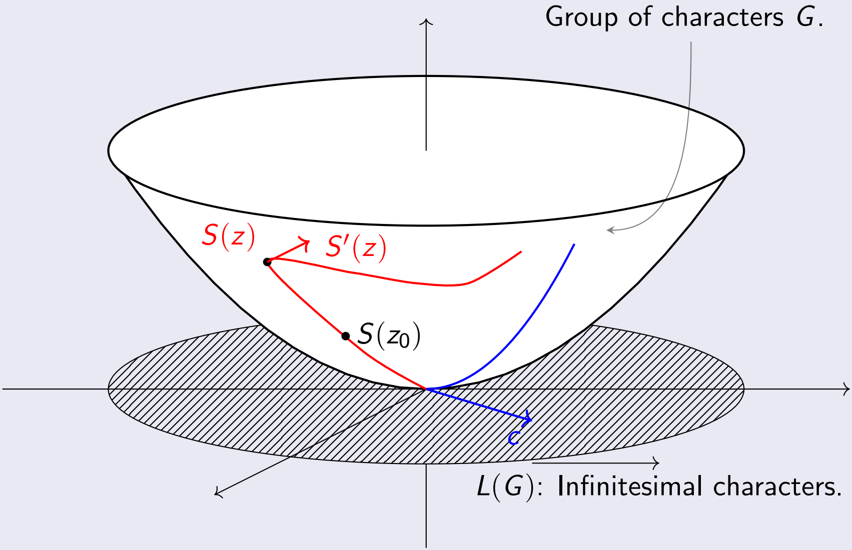
this allows to prove unicity by means of the differential Galois group of (12).

About the asymptotic condition/2

4. $\mathrm{Li} = G_1$ is a shuffle character (due to the fact that the multiplier and the asymptotic condition are grouplike i.e. characters).
5. For $a \notin]-\infty, 0]$, the integrator $\int_1^z \square \frac{ds}{s}$ can be replaced by $\int_a^z \square \frac{ds}{s}$, one then finds a series G_a which fullfils system (12) where the asymptotic initial condition is modified to
$\lim_{\substack{z\to 0\\ z\in\Omega}} S(z)e^{-x_0(\log(z)-\log(a))} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle}$.
6. Due to the fact that, on the one hand the asymptotic counterterm $e^{-x_0(\log(z)-\log(a))}$ is grouplike (i.e. a shuffle character) and, on the other hand the multiplier is primitive (i.e. a shuffle infinitesimal character), one easily sees that all G_a are shuffle characters.
7. Computing $\langle G_a|x_0^*\rangle = \sum_{n\geqslant 0}\langle G_a|x_0^n\rangle = e^{(\log(z)-\log(a))} = z/a$, one sees that all shuffle characters G_a are different[a].

[a] More generally, the possibility of setting a series in the RHS place of a scalar product has been explored in [8].

The Lie group of characters. ⏩ return



Domain of Li (definition)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S=\sum_{n\geqslant 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geqslant 0} Li_{S_n}(z)$ converges **unconditionally** for the compact convergence in Ω (see [8]). One sets

$$Li_S(z) := \sum_{n\geqslant 0} Li_{S_n}(z) \tag{18}$$

Due to the nuclearity of $\mathcal{H}(\Omega,\mathbb{C})$, one can prove that $Dom(Li;\Omega)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle x_0, x_1\rangle\rangle$.

The ladder (outer frame)

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\ \mathrm{Li}_\bullet\ } & \mathcal{H}(\Omega) \\
\downarrow & & \downarrow \\
Dom(Li;\Omega) & \xrightarrow{\ \mathrm{Li}^{(1)}_\bullet\ } & \mathcal{H}(\Omega)
\end{array}$$

Coefficients in the Ladder

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \overset{\mathrm{Li}_\bullet}{\hookrightarrow\!\!\!\twoheadrightarrow} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \overset{\mathrm{Li}_\bullet^{(1)}}{\twoheadrightarrow} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \overset{\mathrm{Li}_\bullet^{(2)}}{\longrightarrow} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Were, for every additive subgroup $(H,+) \subset (\mathbb{C},+)$, $\mathcal{C}_H$ has been set to the following subring of $\mathbb{C}$

$$\mathcal{C}_H := \mathbb{C}\{z^\alpha(1-z)^{-\beta}\}_{\alpha,\beta\in H}\ . \quad (19)$$

Examples

$$Li_{x_0^*}(z) = z,\ \ Li_{x_1^*}(z) = (1-z)^{-1}$$

$$Li_{(\alpha x_0+\beta x_1)^*}(z) = Li_{(\alpha x_0)^*\, ш\, (\beta x_1)^*}(z) = z^\alpha(1-z)^{-\beta}$$

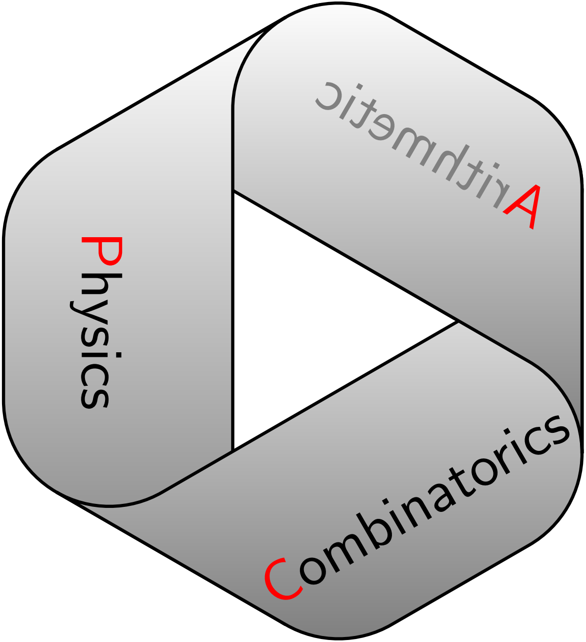
Concluding remarks

1. We have indicated the structure of automaton with multiplicities in a (non necessarily commutative) semiring R, following the original thought of Eilenberg and Schützenberger.
2. The computation of its behaviour, a generating series, entails that of the star of a matrix (in general with noncommutative coefficients).
3. When one specializes R to $R = \Sigma \times \mathbf{k}$ ($\mathbf{k}$ a ring of operators), one gets a powerful notion of Σ-action which is powerful enough to, for example, generate Hyperlogarithms and, through Lazard elimination, explain the asymptotic initial conditions.
4. When one specializes R to $R = \Sigma \times \mathbf{k}$ ($\mathbf{k}$ a commutative semiring), one gets the classical structure of automaton with multiplicities in $\mathbf{k}$, rational series, rational calculus.

Concluding remarks/2

5. If, moreover, $\mathbf{k}$ is a field, one can use the this rational calculus to compute within every Sweedler's dual of a $\mathbf{k}$ Hopf or bi-algebra.
6. The trick is the following. Let $\sigma : X \longrightarrow \mathcal{A}$ be an (indexed) generating family of $\mathcal{A}$, $\mu : \mathbf{k}\langle X\rangle \longrightarrow \mathcal{A}$ the corresponding (onto) morphism and $\mu^* : \mathcal{A}^* \hookrightarrow \mathbf{k}\langle\langle X\rangle\rangle$ its transpose. Then, due to the formula $\mu^*(f_{\mu(u)}) = \mu^*(f)_u$ we have $\mu^*(\mathcal{A}^\circ) = \mathbf{k}^{rat}\langle\langle X\rangle\rangle \cap Im(\mu^*)$ which allows the rational calculus within $\mathcal{A}^\circ$.

THANK YOU FOR YOUR ATTENTION !


Arithmetic
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Combinatorics

Annex: Formalization of the result of Slide 23

The general theorem is the following. It can be generalized in many directions (differential algebra, analysis &c.)

Theorem (A)

Let X be an alphabet, $\Omega \subset \mathbb{C}$ a connected open subset and $\mathcal{H}(\Omega)$, the $\mathbb{C}$-algebra of (complex valued) holomorphic functions on Ω.

$$[\Sigma] \begin{cases} \mathbf{d}(S) = M.S & (NCDE\text{-}Gen) \\ S(z_0) = 1_{X^*} & (Init.\ Cond.) \end{cases} \tag{20}$$

Where the multiplier $M \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ has constant term zero. Then

1. *Due to the fact that $\langle M | 1_{X^*} \rangle = 0$, the system $[\Sigma]$ admits a unique solution $S_{[\Sigma]}$.*
2. *If the multiplier is primitive (i.e. a Lie series, see [3, 17]) then, for all $z \in \Omega$, $S_{[\Sigma]}(z)$ is group-like.*

Proof

Firstly Picard iterations of Slide 22, with lower bound z_0, can be applied to prove existence of a solution of $[\Sigma]$. Let us call $S_{[\Sigma]}$ this solution. Unicity is obtained remarking that since any other solution is of the form $S_{[\Sigma]}.C$ where $C \in \mathbb{C}\langle\langle X \rangle\rangle$, condition $S(z_0) = 1_{X*}$ forces C to be 1_{X*}.
If, moreover, the multiplier is primitive, we have to apply the theory of differential equations with unknown $S \in \widehat{\mathcal{H}(\Omega)[M]}$ (where $\widehat{\mathcal{H}(\Omega)[M]}$ is the total algebra of $X^* \otimes X^* = X^* \times X^*$, direct product of the free monoid with itself), with coefficients in $\mathcal{H}(\Omega)$ (see e.g. [2, 20]). We can extend the derivation $\frac{d}{dz}$ of $\mathcal{H}(\Omega)$ as a derivation on these "double series". One then checks easily that $T = \Delta_{ш}(S)$ ($S = S_{[\Sigma]}$) and $S \otimes S$ satify the same differential equation with the same initial condition $T(z_0) = 1_{X*} \otimes 1_{X*}$ and we are done.

Remark. – If, in $[\Sigma]$, the initial condition "$(Init.\ Cond.)$" is replaced by any limiting condition of type $\lim_{z->z_0} S(z).T(z) = 1$ where T is group-like, then any solution S of the system is group-like. This proves that Polylogarithms, which satisfy system (12), have a group-like generating series.

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