Symmetries of weight 6 multiple polylogarithms and Goncharov's programme

Steven Charlton
Max Planck Institute for Mathematics, Bonn

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Outline

Definitions

Definition

Multiple polylogarithm (MPL) is

$$\mathrm{Li}_{k_1,k_2,\ldots,k_d}(x_1,x_2,\ldots,x_d) := \sum_{0<m_1<m_2<\cdots<m_d} \frac{x_1^{m_1}\cdots x_d^{m_d}}{m_1^{k_1}\cdots m_d^{k_d}}\,, \quad |x_i|<1\,.$$

- $\mathrm{Li}_1(x) = -\log(1-x)$
- Weight is $n = k_1 + \cdots + k_d$,
- Depth is d.

Applications in: number theory, differential geometry, hyperbolic geometry, high-energy physics, . . .

Goal: Understand the nature, structure and properties of the depth filtration.

Proposition ($\mathrm{Li}_{1,1}$ is depth 1, popularised by Goncahrov, Zagier)

For $|xy|<1, |y|<1$, *power series identity holds*

$$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\left(\frac{y(x-1)}{1-y}\right) - \mathrm{Li}_2\left(\frac{-y}{1-y}\right) - \mathrm{Li}_2(xy)\,.$$

Depth reductions

Proposition ($\mathrm{Li}_{1,1,1}$ is depth 1, Goncharov, Zhao, ...)

$$\begin{aligned}\mathrm{Li}_{1,1,1}(x,y,z) = & -\mathrm{Li}_3\left(\tfrac{1-xyz}{1-x}\right) - \mathrm{Li}_3\left(\tfrac{1-xyz}{xy(1-z)}\right) + \mathrm{Li}_3\left(\tfrac{(y-1)(1-xyz)}{(1-x)y(1-z)}\right) + \mathrm{Li}_3(xy) \\ & - \mathrm{Li}_3\left(\tfrac{y(1-x)}{y-1}\right) + \mathrm{Li}_3(1-x) - \mathrm{Li}_3\left(\tfrac{y(1-z)}{y-1}\right) + \mathrm{Li}_3\left(\tfrac{-y}{1-y}\right) + \text{products}\end{aligned}$$

Question: Can weight 4 be reduced to depth 1?
Apparently not (Wojtkowiak: no polynomial expression)... but:

Proposition ($\mathrm{Li}_{3,1}$ satisfies relations, modulo depth 1, Zagier, Gangl)

$$\begin{aligned}\mathrm{Li}_{3,1}\left(\tfrac{1-x}{y}, y\right) + \mathrm{Li}_{3,1}\left(\tfrac{x}{y}, y\right) = & -\tfrac{1}{2}\mathrm{Li}_4\left(\tfrac{(1-x)y}{x(1-y)}\right) - \tfrac{1}{2}\mathrm{Li}_4\left(\tfrac{xy}{(1-x)(1-y)}\right) + \tfrac{1}{2}\mathrm{Li}_4\left(\tfrac{(1-y)y}{(1-x)x}\right) \\ & - \mathrm{Li}_4\left(\tfrac{1-y}{1-x}\right) - \mathrm{Li}_4\left(\tfrac{1-y}{x}\right) + \mathrm{Li}_4\left(-\tfrac{y}{1-y}\right) \\ & - \mathrm{Li}_4(1-x) - \mathrm{Li}_4(x) + 2\,\mathrm{Li}_4(y) + \text{products}\end{aligned}$$

Objective: How to predict, understand, find and explain such reductions and obstructions?

Outline

Lie coalgebra of multiple polylogarithms

Recall: Iterated integral along path between a and b,

$$I(a; x_1, \ldots, x_n; b) \coloneqq \int_{a<t_1<\cdots<t_n<b} \frac{\mathrm{d}t_1}{t_1 - x_1} \wedge \frac{\mathrm{d}t_2}{t_2 - x_2} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n - x_n} \,.$$

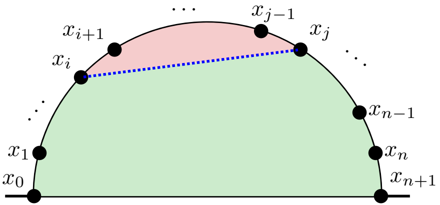
Term-by-term integration shows

$$\mathrm{Li}_{k_1,k_2,\ldots,k_d}(x_1, \ldots, x_d) = (-1)^d I(0; \underbrace{1, 0, \ldots, 0}_{k_1}, \underbrace{x_1, 0, \ldots, 0}_{k_2}, \underbrace{x_1 x_2, 0, \ldots, 0}_{k_3}, \ldots, \underbrace{x_1 \cdots x_{d-1}, 0, \ldots, 0}_{k_d}; x_1 x_2 \cdots x_d) \,.$$

- Goncharov upgraded these to "framed mixed Tate motives" in a Hopf algebra with Δ

In Lie coalgebra $\mathcal{L}$ of irreducibles (mod products) cobracket $\delta \coloneqq \Delta - \Delta^{\mathrm{op}}$ (mod products) is:

$$\delta\, \mathrm{I}^{\mathcal{L}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{i<j} \mathrm{I}^{\mathcal{L}}(x_i; x_{i+1}, \ldots, x_{j-1}; x_j) \wedge \mathrm{I}^{\mathcal{L}}(x_0; x_1, \ldots, x_i, x_j, \ldots, x_n; x_{n+1})$$



Depth is motivic // Cobracket sees depth

Note: $\delta(\text{depth } d) = \text{depth } d \wedge \text{weight } 1 + \sum_{\substack{i+j=d \\ i,j\geq 1}} \text{depth } i \wedge \text{depth } j$

Why? Since: $\mathrm{I}^{\mathcal{L}}(a; \underbrace{0,\ldots,0}_{n\geq 2}; b) = \frac{1}{n!} \mathrm{I}^{\mathcal{L}}(a;0;b)^n = 0 \pmod{\text{products}}$,

Example:

$$\Delta \, \mathrm{Li}^{\mathcal{L}}_{3,1}(x,y) = \log^{\mathcal{L}}(x) \wedge \mathrm{Li}^{\mathcal{L}}_{2,1}(x,y) - \mathrm{Li}^{\mathcal{L}}_1(y) \wedge \mathrm{Li}^{\mathcal{L}}_3(x) + \mathrm{Li}^{\mathcal{L}}_1(y) \wedge \mathrm{Li}^{\mathcal{L}}_3(xy) + \mathrm{Li}^{\mathcal{L}}_1(xy) \wedge \mathrm{Li}^{\mathcal{L}}_3(x) - \mathrm{Li}^{\mathcal{L}}_1(xy) \wedge \mathrm{Li}^{\mathcal{L}}_3(y) - \mathrm{Li}^{\mathcal{L}}_2(y) \wedge \mathrm{Li}^{\mathcal{L}}_2(xy)$$

$$\Delta \, \mathrm{Li}^{\mathcal{L}}_4(x) = \log^{\mathcal{L}}(x) \wedge \mathrm{Li}_3(x) \,.$$

Observation (truncated cobracket $\overline{\delta}$)

Write $\overline{\delta}$ to mean ignore "weight 1", then

$$\overline{\delta} \, \mathrm{Li}^{\mathcal{L}}_{3,1}(x,y) = -\mathrm{Li}^{\mathcal{L}}_2(y) \wedge \mathrm{Li}^{\mathcal{L}}_2(xy) \,, \qquad \overline{\delta} \, \mathrm{Li}^{\mathcal{L}}_4(x) = 0 \,.$$

Conclusion: $\mathrm{Li}^{\mathcal{L}}_{3,1}(x,y) \neq \sum \mathrm{Li}^{\mathcal{L}}_4$'s

Iterating $\overline{\delta}$ // Goncharov's Depth Conjecture

$$\overline{\delta}\,\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) = -\,\mathrm{Li}^{\mathcal{L}}_{2}(y) \wedge \mathrm{Li}^{\mathcal{L}}_{3,1}(xy,z) + \mathrm{Li}^{\mathcal{L}}_{3,1}\Big(yz, \tfrac{1}{y}\Big) \wedge \mathrm{Li}^{\mathcal{L}}_{2}(xyz) - \mathrm{Li}^{\mathcal{L}}_{3}(y) \wedge \mathrm{Li}^{\mathcal{L}}_{2,1}(xy,z)$$
$$-\,\mathrm{Li}^{\mathcal{L}}_{2,1}\Big(yz, \tfrac{1}{y}\Big) \wedge \mathrm{Li}^{\mathcal{L}}_{3}(xyz) + \mathrm{Li}^{\mathcal{L}}_{4}\text{'s} \wedge \text{weight } 2 + \mathrm{Li}^{\mathcal{L}}_{n}\text{'s} \wedge \mathrm{Li}^{\mathcal{L}}_{m}\text{'s}\,.$$

Define iteration: $\overline{\delta}^{[2]} := (\Delta \otimes \mathrm{id}) \circ \Delta \pmod{\text{products, weight 1}}$, to pick out genuine "depth 1 $\wedge$ depth 2" part of $\overline{\delta}$

Then: $$\overline{\delta}^{[2]}\,\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) = -\,\mathrm{Li}^{\mathcal{L}}_{2}(y) \otimes \mathrm{Li}^{\mathcal{L}}_{2}(z) \otimes \mathrm{Li}^{\mathcal{L}}_{2}(xyz) \pmod{\text{shuffles}}\,.$$

But: $$\overline{\delta}^{[2]}\,\mathrm{Li}^{\mathcal{L}}_{a,b}(x,y) = 0\,.$$

Conclusion: $\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) \neq \sum$ depth 2

Goncharov Depth Conjecture (simplified)

A linear combination of MPL's has depth $< k$ exactly when $\overline{\delta}^{[k-1]}$ vanishes.

Expectations and conjectures

Recall and note

$$\underbrace{\overline{\delta}\,\mathrm{Li}^{\mathcal{L}}_{3,1}\big(\tfrac{x}{y},y\big)}_{=: f_4(x,y)} = \mathrm{Li}^{\mathcal{L}}_2(x) \wedge \mathrm{Li}^{\mathcal{L}}_2(y) \qquad \overline{\delta}^{[2]}\,\underbrace{\mathrm{Li}^{\mathcal{L}}_{4,1,1}\big(\tfrac{1}{xyz},x,y\big)}_{=: f_6(x,y,z)} = -\,\mathrm{Li}^{\mathcal{L}}_2(x) \otimes \mathrm{Li}^{\mathcal{L}}_2(y) \otimes \underbrace{\mathrm{Li}^{\mathcal{L}}_2(z^{-1})}_{= -\,\mathrm{Li}^{\mathcal{L}}_2(z)}$$

As f_4, f_6 behave like $\mathrm{Li}^{\mathcal{L}}_2$ in each argument, expect reductions using $\mathrm{Li}^{\mathcal{L}}_2$ identities:

$$\mathrm{Li}^{\mathcal{L}}_2(1) \equiv 0 \pmod{\text{products}}\,, \qquad \mathrm{Li}^{\mathcal{L}}_2(x) + \mathrm{Li}^{\mathcal{L}}_2(\overbrace{1-x}^{\text{or } 1/x}) \equiv 0 \pmod{\text{products}}$$

$$\mathrm{Li}^{\mathcal{L}}_2(V(y,z)) \equiv 0 \pmod{\text{products}}\,, \quad V(y,z) \coloneqq -[y] + [z] - \big[\tfrac{z}{y}\big] + \big[\tfrac{1-z}{1-y}\big] - \big[\tfrac{y(1-z)}{(1-y)z}\big]$$

- Nielsen-type: $f_{2k}(1, x_2, \ldots, x_k) \overset{?}{\equiv} 0 \pmod{\text{dp} < k} \qquad k = 2, 3$
- Zagier-type: $f_{2k}(x_1, x_2, \ldots, x_k) + f_{2k}(\underbrace{1-x_1}_{\text{or } 1/x_1}, x_2, \ldots, x_k) \overset{?}{\equiv} 0 \pmod{\text{dp} < k}$
- Gangl-type: $f_{2k}(V(y,z), x_2, \ldots, x_k) \overset{?}{\equiv} 0 \pmod{\text{dp} < k}$

Weight 4 known // Recall: $f_4(x,y) = \mathrm{Li}^{\mathcal{L}}_{3,1}(\frac{x}{y}, y)$

Proposition (Nielsen-type, Kölbig, Lewin, Wojtkowiak)

$$f_4(1,x) = -\mathrm{Li}^{\mathcal{L}}_4(1-x) + 2\,\mathrm{Li}^{\mathcal{L}}_4(x) + \mathrm{Li}^{\mathcal{L}}_4\Big(\frac{x}{x-1}\Big)$$

$\leftsquigarrow$ Related to $S_{2,2}(x)$ Nielsen polylogarithm

Proposition (Zagier-type, Zagier ∼2000, Gangl)

$$\mathrm{Li}^{\mathcal{L}}_{3,1}(\tfrac{1-x}{y}, y) + \mathrm{Li}^{\mathcal{L}}_{3,1}(\tfrac{x}{y}, y) = \sum \mathrm{Li}^{\mathcal{L}}_4\text{'s from introduction} + \varepsilon.$$

Theorem (Gangl-type, Gangl ∼2011, later Goncharov-Rudenko, Matveiakin-Rudenko)

$$f_4(\underbrace{V(y,z)}_{-[y]+[z]-[\frac{z}{y}]+[\frac{1-z}{1-y}]-[\frac{y(1-z)}{(1-y)z}]}, x) = \sum_{i=1}^{122+\varepsilon} \mathrm{Li}^{\mathcal{L}}_4(h_i(x,y,z))\,, \quad h_i(x,y,z) \in \mathbb{Q}(x,y,z)$$

Gangl: Judicious computer experimentation to find candidate $h_i(x,y,z)$ involving ratios of products of four cross-ratios, and algebraically search for identity

Weight 6 // Recall: $f_6(x,y) = \mathrm{Li}^{\mathcal{L}}_{4,1,1}(\frac{1}{xyz}, x, y)$

These results are now also known in weight 6

Theorem (Gangl-type, Matveiakin-Rudenko 2022)

$$f_6(\underbrace{V(y,z)}_{-[y]+[z]-[\frac{z}{y}]+[\frac{1-z}{1-y}]-[\frac{y(1-z)}{(1-y)z}]}, x_2, x_3) \equiv 0 \quad (\text{mod } \mathsf{dp} \leq 2 \text{ \& Zagier-type})$$

Theorem (Nielsen-type & Zagier-type, C 2024, arXiv:2405.13853)

$$f_6(1,y,z) \equiv 0 \quad (\text{mod } \mathsf{dp} \leq 2)$$

$$f_6(x,y,z) + f_6(1-x,y,z) \equiv 0 \quad (\text{mod } \mathsf{dp} \leq 2)$$
$$f_6(x,y,z) + f_6(x^{-1},y,z) \equiv 0 \quad (\text{mod } \mathsf{dp} \leq 2)$$

Consequence: Goncharov's depth conjecture holds for weight 6 depth 3 $(k=3)$.

Remark: Computer algebra to investigate specialisations/degenerations and how to combine identities for the proof, also to track hundreds of terms for explicit results

Outline

Quadrangular polylogarithms

Rudenko defined $\mathrm{QLi}_{n+k}(x_0, \ldots, x_{2n+1})$: a family of polylogarithmic functions on $\mathfrak{M}_{0,2n+2}$, with good singularities.

Description via sum over certain quadrangles inside a $2n+2$-gon

- Introduce $abcd = [x_a, x_b, x_c, x_d] := \frac{(x_a - x_b)(x_c - x_d)}{(x_b - x_c)(x_d - x_a)}$ (cyclic) cross-ratio
- Label vertices of a $(2n+2)$-gon P with $x_0, \ldots, x_{2n+1}$
- A quadrilateral $x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}$ gives rise to a cross-ratio
- Any quadrangular Q determines a dual tree t_Q
- Arborification map attaches depth n MPL with arguments $[x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}]$ to t_Q

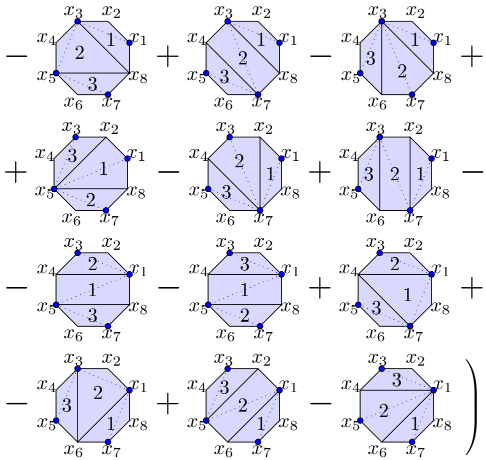
Key identity

$\mathbf{Q}_{2k}$:

$$\sum_{i=1}^{2k+3} (-1)^i \, \mathrm{QLi}_{2k}(x_1, \ldots, \widehat{x_i}, \ldots, x_{2k+2}) \equiv 0 \pmod{\mathrm{dp} < k}$$

Quadrangular polylogarithm: weight 6, 8 points // $[a,b,c,d] = \frac{(a-b)(c-d)}{(b-c)(d-a)}$

$\mathrm{QLi}_6(x_1,\ldots,x_8) = f_6 \Bigg($



$\Bigg) + \text{depth} \leq 2$

Term 1: $-f_6(1238, 3458, 5678) = -f_6([x_1,x_2,x_3,x_8],[x_3,x_4,x_5,x_8],[x_5,x_6,x_7,x_8])$

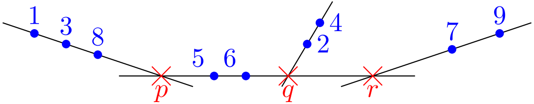
$\mathbf{Q}_6:$ $\sum_{i=1}^{9} (-1)^i\, \mathrm{QLi}_6(x_1,\ldots,\widehat{x_i},\ldots,x_9) \equiv 0 \pmod{\mathsf{dp} \leq 2}$

Degenerating identities // Stable curves

Recall

The Deligne-Mumford compactification of $\mathfrak{M}_{0,n}$ is described by (genus 0) stable curves

- Components are isomorphic to $\mathbb{P}^1$
- Only singular points are simple double points
- Number of marked & singular points per components ≥ 3



Idea: The points x_1, x_3, x_8 all degenerate to p

But: There is a always projective transformation moving (x_1, x_3, x_8, p) to $(\infty, 0, 1, z)$

So: Points x_1, x_3, x_8 split off as a separate $\mathbb{P}^1$

Calculate: Set $x_i = \lambda_1 y_i + p$, $i = 1, 3, 8$, (similar for $i = 2, 4$ and $i = 7, 9$), with $\lambda_j \to 0$

Cross-ratios well-defined: $[x_1,x_3,x_8,x_i] = [x_1,x_3,x_8,p]$ and $[x_1,x_3,x_2,x_4] = [x_1,x_3,p,p] = 0$

Outline

General identities // Degenerate Nielsen-type

- Via general theory of multiple polylogarithms, or from $\mathbf{Q}_6$ directly

(inverse) $$f_6(x, y, z) + f_6(x^{-1}, y^{-1}, z^{-1}) \equiv 0 \pmod{\text{dp } 2}$$

(reverse) $$f_6(x, y, z) - f_6(z, y, x) \equiv 0 \pmod{\text{dp } 2}$$

$(2 ⧢ 1)$ $$f_6(x_1, x_2, y) + f_6(x_1, y, x_2) + f_6(y, x_1, x_2) \equiv 0 \pmod{\text{dp } 2}$$

- Use computer algebra to investigate all specialisations for useful identities

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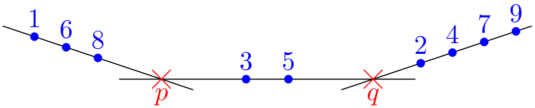
Specialise $\mathbf{Q}_6$ to

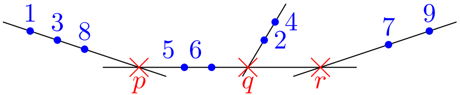


Lemma (Degenerate Nielsen-type)

With $A = pq35$, $$f_6(1, 1, A) \equiv 0 \pmod{\text{dp } 2}$$

Nielsen-type symmetries

Specialise $\mathbf{Q}_6$ to

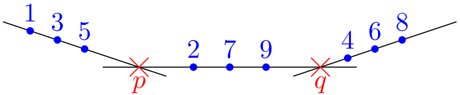


Lemma (Nielsen-type Symmetry 1)

With $A = pq5r, B = 56pr,$

$$f_6(1, A, B) \equiv f_6\left(1, \underbrace{\frac{A(1-B)}{1-AB}}_{pq6r}, \underbrace{\frac{AB-1}{AB}}_{p56q}\right) \pmod{\text{dp } 2}$$

Specialise $\mathbf{Q}_6$ to



Lemma (Nielsen-type Symmetry 2)

With $A = 2pq7, B = 9p27,$

$$f_6(1, A, B) \equiv -f_6\left(1, A, \underbrace{\frac{1-AB}{A(1-B)}}_{q927}\right) \pmod{\text{dp } 2}.$$

Nielsen-type symmetries

Play these symmetries against each other

- Write $g(x_1, \ldots, x_5) \coloneqq f_6(1, [x_3, x_1, x_4, x_2], [x_5, x_1, x_3, x_2])$ and notice

(inverse) $$g(x_1, x_2, x_3, x_4, x_5) \equiv -g(x_2, x_1, x_3, x_4, x_5) \pmod{\text{depth } 2},$$

(sym 1) $$g(x_5, p, r, q, x_6) \equiv g(p, x_6, q, r, x_5) \pmod{\text{depth } 2},$$

(sym 2) $$g(p, x_7, x_2, q, x_9) \equiv -g(q, x_2, x_7, p, x_9) \pmod{\text{depth } 2}.$$

Nielsen-type reduction

Play these symmetries against each other

- Write $g(x_1, \ldots, x_5) \coloneqq f_6(1, [x_3, x_1, x_4, x_2], [x_5, x_1, x_3, x_2])$ and notice

$$
\begin{aligned}
& & & g(x_1, x_2, x_3, x_4, x_5) \\
&\text{(inverse)} & \equiv & -g(x_2, x_1, x_3, x_4, x_5) \\
&\text{(sym 2)} & \equiv & \;\; g(x_4, x_3, x_1, x_2, x_5) \\
&\text{(sym 1, thrice)} & \equiv & \;\; g(x_4, x_3, x_2, x_1, x_5) \\
&\text{(sym 2)} & \equiv & -g(x_1, x_2, x_3, x_4, x_5) \pmod{\text{dp } 2} \,.
\end{aligned}
$$

Theorem (Nielsen-type reduction)

$$f_6(1, X, Y) \equiv 0 \pmod{\text{dp } 2}$$

Then: Repeat computer algebra to investigate all degenerations to find Zagier-type

Summary

- Depth reductions of MPL's
  - Reduction of $\mathrm{Li}_{1,1}, \mathrm{Li}_{1,1,1}$ to Li_2, Li_3, respectively
  - Obstruction for $\mathrm{Li}_{3,1}$ to depth 1
- Goncharov's Depth Conjecture
  - Cobracket on (motivic) MPL's, truncation, iteration
  - Cobracket conjecturally detects depth
  - Conjectured Nielsen-, Zagier-, & Gangl-type reductions of $\mathrm{Li}_{4,1,1}$
- Tools for the proof of weight 6 reductions
  - 8 point quadrangular polylogarithm identity
  - Specialisations and degenerations, stable curves
- Sketch proof of Nielsen-type reduction
  - Nielsen-type symmetries
  - Play symmetries against each other
  - Establish Nielsen-type reduction

Zagier-type reduction, overview

Lemma (Symmetry 1, $A = p67q, B = pq56, C = p45q$, from $29 \cup_p 4567 \cup_q 138$)

$$f_6(A, B, C) \equiv -f_6(1-A\,,\ \tfrac{B}{B-1}\,,\ 1-C) \pmod{\text{dp } 2}$$

Lemma (Four term relation, $A = p67q, B = pq56, C = p45q$, from $29 \cup_p 4678 \cup_q 135$)

$$f_6(A, \tfrac{C}{B}, \tfrac{1-C}{1-B}) + f_6(A, \tfrac{C}{B}, \tfrac{1-B}{1-C}) - f_6(A, \tfrac{1}{B}, 1-C) - f_6(A, \tfrac{1}{B}, C) \equiv 0 \pmod{\text{dp } 2}$$

Lemma (Symmetry 2, $A = 67qp, B = 4q6p, C = q34p$, from $29 \cup_p 3467 \cup_q 158$)

$$2f_6(A, B, C) + 2f_6(A, \tfrac{B}{B-1}, \tfrac{1}{1-C}) \equiv 0 \pmod{\text{depth } 2}$$

- Consider $W(a,b,c) := f_6(a, \frac{1}{b}, c) + f_6(a, \frac{1}{b}, 1-c)$,
- See $W(a,b,c) \equiv -W(a, \frac{c}{b}, \frac{1-c}{1-b}) \pmod{\text{dp } 2}$,
- Order is 5 implies $W(a,b,c) \equiv 0 \pmod{\text{dp } 2}$.

Nielsen-type Symmetry 1

Note: $$\mathrm{Li}^{\mathcal{L}}_{3\,;\,1,1,1}(x,y,z) \coloneqq -\,\mathrm{I}^{\mathcal{L}}(0;0,0,0,1,x,xy;xyz)$$
$$= f_6(x,y,z) + \mathrm{Li}^{\mathcal{L}}_{4,2}(\tfrac{1}{xyz}, yz) + \mathrm{Li}^{\mathcal{L}}_{5,1}(\tfrac{1}{xy}, x) - 5\,\mathrm{Li}^{\mathcal{L}}_{6}(xy) - \mathrm{Li}^{\mathcal{L}}_{6}(x) + 5\,\mathrm{Li}^{\mathcal{L}}_{6}(yz)$$

$$\mathrm{Li}^{\mathcal{L}}_{3\,;\,1,1,1}(1,A,B) - \mathrm{Li}^{\mathcal{L}}_{3\,;\,1,1,1}\left(1, \tfrac{A(1-B)}{1-AB}, -\tfrac{1-AB}{AB}\right) =$$
$$\mathrm{Li}^{\mathcal{L}}_{4,2}\left(-\left[\tfrac{B-1}{B},\tfrac{B}{B-1}\right]-\left[\tfrac{B-1}{(1-A)B},\tfrac{B}{B-1}\right]+\left[\tfrac{B}{B-1},\tfrac{AB-1}{AB}\right]-\left[\tfrac{(1-A)B}{B-1},\tfrac{1}{AB}\right]+\left[\tfrac{(1-A)B}{B-1},\tfrac{B-1}{(1-A)B}\right]\right)$$
$$+\,\mathrm{Li}^{\mathcal{L}}_{5,1}\left(-2\left[\tfrac{A(1-B)}{A-1},\tfrac{A-1}{A(1-B)}\right]-2\left[\tfrac{B-1}{(1-A)B},\tfrac{B}{B-1}\right]-\left[\tfrac{AB}{A-1},\tfrac{1}{B}\right]-\left[\tfrac{AB}{A-1},\tfrac{AB-1}{AB}\right]\right.$$
$$+2\left[\tfrac{B}{B-1},\tfrac{AB-1}{AB}\right]-2\left[\tfrac{(1-A)B}{B-1},\tfrac{1}{AB}\right]+4\left[\tfrac{(1-A)B}{B-1},\tfrac{B-1}{(1-A)B}\right]-\left[AB,\tfrac{1}{B}\right]$$
$$\left.-2\left[AB,\tfrac{1}{AB}\right]+2\left[\tfrac{A-1}{A},\tfrac{A}{A-1}\right]+\left[A,\tfrac{1}{A}\right]-2\left[\tfrac{B-1}{B},\tfrac{B}{B-1}\right]\right)$$
$$+\,\mathrm{Li}^{\mathcal{L}}_{6}\left(-5\left[\tfrac{B-1}{(1-A)B}\right]+4\left[\tfrac{A-1}{A(1-B)}\right]-\left[\tfrac{AB}{AB-1}\right]+10\left[\tfrac{A-1}{AB}\right]\right.$$
$$\left.+10\left[\tfrac{1}{AB}\right]-\left[\tfrac{1}{1-A}\right]-\left[\tfrac{1}{A}\right]+6\left[\tfrac{A-1}{A}\right]+6\left[\tfrac{B-1}{B}\right]\right).$$

Nielsen-type Symmetry 2

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,A,B) + \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(1,A,\tfrac{1-AB}{A(1-B)}\right) =$$

$$\mathrm{Li}^{\mathcal{L}}_{4,2}\Big(+\big[1,\tfrac{A}{A-1}\big]+\big[\tfrac{A-1}{A},\tfrac{A}{A-1}\big]-\big[AB,\tfrac{1}{B}\big]+\big[AB,\tfrac{1}{AB}\big]+\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{1-B}\big]$$
$$-\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{A(1-B)}\big]-\big[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\big]-\big[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\big]\Big)$$
$$+\mathrm{Li}^{\mathcal{L}}_{5,1}\Big(+\big[\tfrac{A(1-B)}{(1-A)(1-AB)},\tfrac{1-AB}{A(1-B)}\big]-\big[\tfrac{A(1-B)}{(1-A)(1-AB)},\tfrac{AB-1}{AB}\big]+\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{1-B}\big]-\big[(1-A)B,\tfrac{1-AB}{(1-A)B}\big]$$
$$+\big[\tfrac{A-1}{A(1-B)},\tfrac{B-1}{(1-A)B}\big]-2\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{A(1-B)}\big]-2\big[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\big]-2\big[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\big]$$
$$-\big[\tfrac{1}{1-B},\tfrac{1-B}{1-AB}\big]+\big[(1-A)B,\tfrac{1}{B}\big]-2\big[AB,\tfrac{1}{B}\big]+\big[AB,\tfrac{1}{AB}\big]+2\big[\tfrac{A-1}{A},\tfrac{A}{A-1}\big]-2\big[1-A,\tfrac{1}{1-A}\big]-2\big[A,\tfrac{1}{A}\big]\Big)$$
$$+\mathrm{Li}^{\mathcal{L}}_{6}\Big(+5\big[\tfrac{A(1-B)}{(1-A)(1-AB)}\big]+2\big[\tfrac{(1-A)B}{1-AB}\big]+2\big[\tfrac{A-1}{A(1-B)}\big]+5\big[\tfrac{1}{(1-A)B}\big]-2\big[\tfrac{B-1}{(1-A)B}\big]+\big[\tfrac{A(1-B)}{1-AB}\big]$$
$$-\big[\tfrac{1-B}{1-AB}\big]+8\big[\tfrac{1}{AB}\big]-\big[\tfrac{1}{1-AB}\big]+5\big[\tfrac{1}{1-A}\big]+9\big[\tfrac{1}{A}\big]-4\big[\tfrac{A-1}{A}\big]-2\big[\tfrac{1}{1-B}\big]-3\big[\tfrac{1}{B}\big]\Big).$$

Nielsen-type reduction

$$\mathrm{Li}^{\mathcal{L}}_{3\,;\,1,1,1}(1,A,B) =$$

$$\tfrac{1}{2}\,\mathrm{Li}^{\mathcal{L}}_{4,2}\Big(-\left[\tfrac{A(1-B)}{1-AB},\tfrac{1-AB}{A(1-B)}\right]-\left[\tfrac{(1-A)B}{1-AB},\tfrac{1-AB}{(1-A)B}\right]+\left[\tfrac{1-AB}{A(1-B)},\tfrac{1-B}{1-AB}\right]+\left[\tfrac{1-AB}{(1-A)B},\tfrac{1-A}{1-AB}\right]+\left[\tfrac{A-1}{A(1-B)},\tfrac{B-1}{(1-A)B}\right]$$

$$-\left[\tfrac{A(1-B)}{A-1},\tfrac{A-1}{A(1-B)}\right]-\left[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\right]-\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{A(1-B)}\right]-2\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\right]+\left[\tfrac{1}{1-AB},\tfrac{1-AB}{1-A}\right]$$

$$+\left[\tfrac{AB}{AB-1},\tfrac{AB-1}{AB}\right]+\left[\tfrac{1-AB}{1-A},\tfrac{1}{1-AB}\right]-\left[\tfrac{A}{A-1},\tfrac{A-1}{A(1-B)}\right]-\left[\tfrac{A}{A-1},\tfrac{AB-1}{AB}\right]-\left[\tfrac{1}{1-B},\tfrac{1-B}{1-AB}\right]-\left[AB,\tfrac{1}{B}\right]$$

$$-\left[1-B,\tfrac{A-1}{A(1-B)}\right]+\left[AB,\tfrac{1}{AB}\right]+2\left[\tfrac{A-1}{A},\tfrac{A}{A-1}\right]-\left[1-A,\tfrac{1}{1-A}\right]+\left[1-B,\tfrac{1}{1-B}\right]-\left[1,\tfrac{1}{1-A}\right]+\left[1,\tfrac{A}{A-1}\right]\Big)$$

$$+\tfrac{1}{2}\,\mathrm{Li}^{\mathcal{L}}_{5,1}\Big(-\left[\tfrac{A(1-B)}{(A-1)(1-AB)},\tfrac{1-AB}{1-B}\right]-\left[\tfrac{(1-A)B}{(B-1)(1-AB)},\tfrac{1-AB}{1-A}\right]+\left[\tfrac{(A-1)(1-AB)}{A(1-B)},\tfrac{1}{1-AB}\right]+\left[\tfrac{(B-1)(1-AB)}{(1-A)B},\tfrac{1}{1-AB}\right]$$

$$-3\left[\tfrac{A(1-B)}{1-AB},\tfrac{1-AB}{A(1-B)}\right]-4\left[\tfrac{(1-A)B}{1-AB},\tfrac{1-AB}{(1-A)B}\right]-\left[\tfrac{1-AB}{(1-A)(1-B)},\tfrac{1-A}{1-AB}\right]-\left[\tfrac{1-AB}{(1-A)(1-B)},\tfrac{1-B}{1-AB}\right]+2\left[\tfrac{A-1}{A(1-B)},\tfrac{B-1}{(1-A)B}\right]$$

$$-4\left[\tfrac{A(1-B)}{A-1},\tfrac{A-1}{A(1-B)}\right]-2\left[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\right]-2\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{A(1-B)}\right]-4\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\right]+\left[\tfrac{1-AB}{A(1-B)},\tfrac{1-B}{1-AB}\right]$$

$$+\left[\tfrac{1-AB}{(1-A)B},\tfrac{1-A}{1-AB}\right]+4\left[\tfrac{AB}{AB-1},\tfrac{AB-1}{AB}\right]+2\left[\tfrac{1}{1-AB},\tfrac{1-AB}{1-A}\right]-\left[\tfrac{1-B}{1-AB},\tfrac{1-AB}{1-B}\right]+2\left[\tfrac{1-AB}{1-A},\tfrac{1}{1-AB}\right]-\left[\tfrac{AB}{A-1},\tfrac{AB-1}{AB}\right]$$

$$+\left[\tfrac{(1-A)B}{B-1},\tfrac{1}{AB}\right]-2\left[\tfrac{A}{A-1},\tfrac{A-1}{A(1-B)}\right]-2\left[\tfrac{A}{A-1},\tfrac{AB-1}{AB}\right]-3\left[\tfrac{1}{1-B},\tfrac{1-B}{1-AB}\right]-\left[\tfrac{B}{B-1},\tfrac{AB-1}{AB}\right]-2\left[1-B,\tfrac{A-1}{A(1-B)}\right]+\left[B,\tfrac{1}{B}\right]$$

$$-\left[\tfrac{AB}{A-1},\tfrac{1}{B}\right]+\left[(1-A)B,\tfrac{1}{B}\right]-\left[(1-A)B,\tfrac{1-AB}{(1-A)B}\right]+\left[AB,\tfrac{1}{AB}\right]+4\left[1-B,\tfrac{1}{1-B}\right]-3\left[AB,\tfrac{1}{B}\right]+6\left[\tfrac{A-1}{A},\tfrac{A}{A-1}\right]-2\left[1-A,\tfrac{1}{1-A}\right]\Big)$$

$$+\tfrac{1}{2}\,\mathrm{Li}^{\mathcal{L}}_{6}\Big(-10\left[\tfrac{(1-A)B}{(B-1)(1-AB)}\right]+10\left[\tfrac{(1-A)(1-B)}{1-AB}\right]-10\left[\tfrac{A(1-B)}{(A-1)(1-AB)}\right]+3\left[\tfrac{A(1-B)}{1-AB}\right]+3\left[\tfrac{(1-A)B}{1-AB}\right]-2\left[\tfrac{1-B}{1-AB}\right]+\left[\tfrac{A-1}{A(1-B)}\right]$$

$$+5\left[\tfrac{1}{(1-A)B}\right]-2\left[\tfrac{B-1}{(1-A)B}\right]+10\left[\tfrac{A-1}{AB}\right]+7\left[\tfrac{AB}{AB-1}\right]+14\left[\tfrac{1}{AB}\right]-2\left[\tfrac{1}{1-A}\right]+6\left[\tfrac{1}{A}\right]-8\left[\tfrac{1}{1-B}\right]+4\left[\tfrac{B-1}{B}\right]-5\left[\tfrac{1}{B}\right]\Big).$$