Multiple divided Bernoulli polynomials and numbers

Olivier Bouillot,
Gustave Eiffel University, France

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Definition:

The numbers $\mathcal{Z}e^{s_1,\cdots,s_r}$ defined by

$$\mathcal{Z}e^{s_1,\cdots,s_r} = \sum_{0<n_r<\cdots<n_1} \frac{1}{{n_1}^{s_1}\cdots{n_r}^{s_r}} ,$$

where $s_1,\cdots,s_r \in \mathbb{C}$ such that $\Re(s_1+\cdots+s_k) > k$, $k \in [\![1\,;\,r]\!]$, are called multiple zeta values.

Fact: There exists at least three different ways to renormalize multiple zeta values at negative integers.

$$\mathcal{Z}e^{0,-2}_{MP}(0) = \frac{7}{720} \quad , \quad \mathcal{Z}e^{0,-2}_{GZ}(0) = \frac{1}{120} \quad , \quad \mathcal{Z}e^{0,-2}_{FKMT}(0) = \frac{1}{18} .$$

Question: Is there a group acting on the set of all possible multiple zeta values renormalisations?

Main goal: Define multiple Bernoulli numbers in relation with this.

Outline

Ecalle's concrete definition:

A **mould** is a function with a varying number of variables.

Mathematical definition:

A *mould* is a function defined over a free monoid $\Omega^\star$ of (finite) sequences (or words) constructed over the alphabet Ω (or sometimes over a subset of $\Omega^\star$) with values in a commutative algebra $\mathbb{C}$.

Typical example : *The Multizetas Values !*

Notations:

	Functional notations	Mould notations
Evaluation	$f(x)$	$M^{\underline{s}}$
Name	f	$M^\bullet \in \mathcal{M}_{\mathbb{C}}^\bullet(\Omega)$

Main idea of mould calculus - The so-called Mould/comould's contractions.

Moulds might be contracted with dual objects, called **comoulds** (which are also functions with a variable number of variables) :

Definition:

The *mould-comould contraction* of a mould $M^\bullet$ and a comould $B_\bullet$ is:

$$\sum_{\bullet} M^\bullet B_\bullet := \sum_{\underline{\omega} \in \Omega^\star} M^{\underline{\omega}} B_{\underline{\omega}}$$

(*if the sum is well-defined...*)

For analytical reasons, a mould-comould contraction might be understood to be an **algebra automorphism** or a **derivation**.

Important remark:

Mould's operations and symmetries come from such an interpretation.

First abstraction - Formal mould/comould contraction

- To each letter $\omega \in \Omega$, we define a symbol a_ω, which will be, when necessary, specialized to B_ω.

  $\rightsquigarrow$ The symbols a_ω do not commute.

  $\rightsquigarrow$ The symbols a_ω are extended to words:

$$a_{\omega_1 \cdots \omega_r} = a_{\omega_1} \cdots a_{\omega_r} \ .$$

- To each mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$, we define a series $s(M^\bullet) \in \mathbf{C}\langle\langle \mathrm{A} \rangle\rangle$, where $\mathrm{A} = \{a_\omega \ ; \ \omega \in \Omega\}$ by:

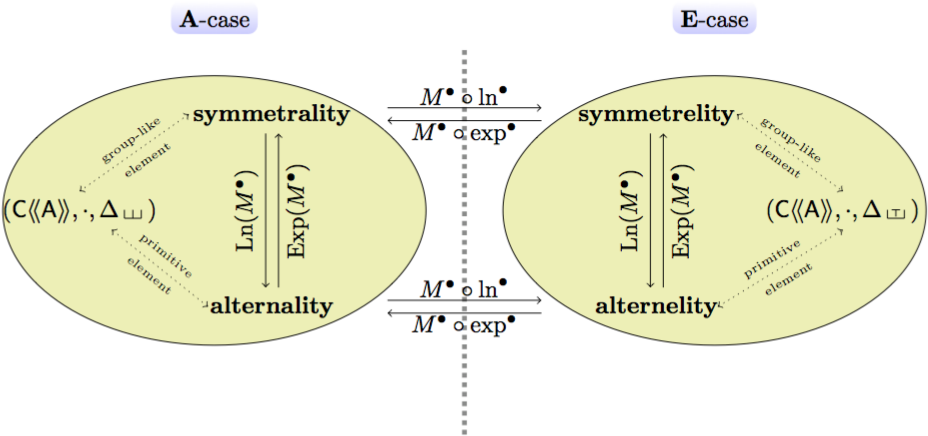
$$s(M^\bullet) = \sum_{\underline{\omega} \in \Omega^\star} M^{\underline{\omega}} \, a_{\underline{\omega}} := \sum_\bullet M^\bullet \ a_\bullet \ .$$

  s is called the **formal mould/comould contraction**.

  $\rightsquigarrow$ If φ is a specialization map (not necessarily morphism) defined by $\varphi(a_{\underline{\omega}}) = B_{\underline{\omega}}$, then:

$$\varphi(s(M^\bullet)) = \sum_\bullet M^\bullet \ B_\bullet \ .$$

Mould operations anf primary symmetries



This defines $+$ and $\times$ of moulds as you can imagine.
The composition $\circ$ is a more complicated operation. . ., which mimics a change of alphabet.

Examples of mould multiplication.

- **Example of mould product computation:** $P^\bullet = M^\bullet \times N^\bullet$

$$\begin{aligned} P^{\emptyset} &= M^{\emptyset} N^{\emptyset} \\ P^{\omega_1} &= M^{\omega_1} N^{\emptyset} + M^{\emptyset} N^{\omega_1} \\ P^{\omega_1,\omega_2} &= M^{\omega_1,\omega_2} N^{\emptyset} + M^{\omega_1} N^{\omega_2} + M^{\emptyset} N^{\omega_1,\omega_2} \end{aligned}$$

- For all $n \in \mathbb{N}^*$, let us consider the mould $\mathcal{I}_n^\bullet$ defined over the alphabet $\Omega = \mathbb{N}^*$ by:

$$\mathcal{I}_n^{\underline{s}} = \begin{cases} \dfrac{1}{n^{s_1}} & , \text{ if } l(\underline{s}) = 1 \ . \\ 0 & , \text{ otherwise.} \end{cases}$$

A new expression of the mould of multiple zeta values $\mathcal{Z}e^\bullet$ is the following factorisation:

Proposition: (B., 18)

$$\mathcal{Z}e^\bullet = \cdots \times (1^\bullet + \mathcal{I}_3^\bullet) \times (1^\bullet + \mathcal{I}_2^\bullet) \times (1^\bullet + \mathcal{I}_1^\bullet) = \prod_{n>0}^{\searrow} (1^\bullet + \mathcal{I}_n^\bullet) \ .$$

where the last product is a convergent one if we restrict ourself to sequences $\underline{s} \in \Omega_{CV}^\star = \{(s_1, \cdots, s_r) \in \mathbb{N}_1^\star \ ; \ s_1 \geq 2\}$.

Formal moulds and secondary symmetries

Another point of view on moulds: A mould is a collection of functions $(f_0, f_1, f_2, \cdots)$, where $f_n : \Omega^n \longmapsto \mathbf{C}$.

Definition:

A formal mould is a collection of formal series $(S_0, S_1, S_2, \cdots)$, where S_n is a formal power series in n indeterminates (and consequently, S_0 is constant)

Notation: $\mathcal{FM}^\bullet_{\mathbf{C}} = \{\text{formal mould with values in the algebra } \mathbf{C}\}$.

What is the difference between a mould and a formal mould?

Mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$ where $\Omega = (X_1, X_2, \cdots)$.	Formal mould $M^\bullet \in \mathcal{FM}^\bullet_{\mathbf{C}}$
No link between M^{X_1,X_2} and M^{X_2,X_1} !!!	M^{X_1,X_2} and M^{X_2,X_1} are related by the **substitution** of the indeterminates.

Nevertheless, $\mathcal{FM}^\bullet_{\mathbf{C}} \subset \mathcal{M}^\bullet_{\mathbf{C}}(X_1, X_2, \cdots)$.

Definition:

If a formal mould satisfies some symmetry, we say it is a *secondary symmetries*.

Definition:

With a mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\mathbb{N})$, we associate two formal moulds $Mog^\bullet$ and $Meg^\bullet$ defined by:

$$\begin{cases} Mog^{X_1,\cdots,X_r} = \displaystyle\sum_{s_1,\cdots,s_r\in\mathbb{N}^*} M^{s_1,\cdots,s_r} X_1^{s_1-1}\cdots X_r^{s_r-1} \,. \\ Meg^{X_1,\cdots,X_r} = \displaystyle\sum_{s_1,\cdots,s_r\in\mathbb{N}} M^{s_1,\cdots,s_r} \frac{X_1^{s_1}}{s_1!}\cdots\frac{X_r^{s_r}}{s_r!} \,. \end{cases}$$

This produces two operators on moulds:

$$og : \begin{array}{ccc} \mathcal{M}^\bullet_{\mathbb{C}}(\Omega) & \longrightarrow & \mathcal{FM}^\bullet_{\mathbb{C}}(\mathrm{X}) \\ M^\bullet & \longmapsto & Mog^\bullet \end{array} \quad , \quad eg : \begin{array}{ccc} \mathcal{M}^\bullet_{\mathbb{C}}(\Omega) & \longrightarrow & \mathcal{FM}^\bullet_{\mathbb{C}}(\mathrm{X}) \\ M^\bullet & \longmapsto & Meg^\bullet \,. \end{array}$$

Proposition: (B., 15)

og and eg are algebra morphisms:
$og\big(M^\bullet \times N^\bullet\big) = og(M^\bullet) \times og(N^\bullet)$ and $eg\big(M^\bullet \times N^\bullet\big) = eg(M^\bullet) \times eg(N^\bullet)$.

Main objective: Adapt the Hopf algebraic setting to the case of formal moulds.

Let us consider:

- $\mathbf{X} = \{X_1, X_2, \cdots\}$ an infinite set of indeterminates.
- $\widehat{\mathbf{X}}$ an extended alphabet: **A**-case: $\widehat{\mathrm{X}} = \mathbf{X}$.

  E-case: $\widehat{\mathbf{X}} = \mathbf{X} \cup \bigcup_{r \geq 2} \left\{ \sum_{i=1}^{r} X_i , X_1, \cdots, X_r \in \mathrm{X} \right\}$
- $\boldsymbol{\mathcal{A}} = \{A_x \ ; \ x \in \widehat{\mathbf{X}}\}$, with symbols that do not commute.
- $\boldsymbol{\Delta}_{\uplus} : \mathbb{C}[\![\widehat{\mathbf{X}}]\!]\langle\!\langle \boldsymbol{\mathcal{A}} \rangle\!\rangle \longrightarrow \mathbb{C}[\![\widehat{\mathbf{X}}]\!]\langle\!\langle \boldsymbol{\mathcal{A}} \rangle\!\rangle \otimes \mathbb{C}[\![\widehat{\mathbf{X}}]\!]\langle\!\langle \boldsymbol{\mathcal{A}} \rangle\!\rangle$ defined by:

$$\boldsymbol{\Delta}_{\uplus}(A_x) = A_x \otimes 1 + \sum_{\substack{u,v \in \widehat{\mathrm{X}} \\ u+v=x}} A_u \otimes A_v + 1 \otimes A_x$$

and extended to words of $\mathcal{A}^\star$ such that $\boldsymbol{\Delta}$ is a morphism for the concatenation product and then by $\mathbb{C}[\![\widehat{\mathrm{X}}]\!]$-linearity to $\mathbb{C}[\![\widehat{\mathrm{X}}]\!]\langle\!\langle \boldsymbol{\mathcal{A}} \rangle\!\rangle$.

Lemma: (B., 15)

Let us define

$$\begin{array}{rccccrccc} \eta: & \mathbb{C}[\![\widehat{\mathsf{X}}]\!] & \longrightarrow & \mathbb{C}[\![\widehat{\mathsf{X}}]\!]\langle\!\langle \boldsymbol{\mathcal{A}} \rangle\!\rangle & \text{and} & \varepsilon: & \mathbb{C}[\![\widehat{\mathsf{X}}]\!]\langle\!\langle \boldsymbol{\mathcal{A}} \rangle\!\rangle & \longrightarrow & \mathbb{C}[\![\widehat{\mathsf{X}}]\!] \\ & S & \longmapsto & S \cdot 1 & & & S & \longmapsto & \langle S|1\rangle \ . \end{array}$$

So, $(\mathbb{C}[\![\widehat{\mathsf{X}}]\!]\langle\!\langle \boldsymbol{\mathcal{A}} \rangle\!\rangle, \cdot, \eta, \boldsymbol{\Delta}_{\uplus}, \varepsilon)$ is a bialgebra.

Secondary formal mould/comould contraction

To a formal mould $FM^{\bullet} \in \mathcal{FM}^{\bullet}_{\mathsf{C}}$, we associate a series $S(FM^{\bullet}) \in \mathbf{C}[\![\mathbf{X}]\!]\langle\!\langle \mathsf{A} \rangle\!\rangle$ by:

$$S(FM^{\bullet}) = \sum_{\underline{\omega} \in \mathsf{A}^{\star}} FM^{\underline{\omega}} A_{\underline{\omega}} := \sum_{\bullet} FM^{\bullet} A_{\bullet} \ .$$

Generics theorem for secondary symmetries

Theorem: (Ecalle, $\sim 90's$, see [SNAG] *in French!*)

Let $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\mathbb{N})$ be a mould.

1. $M^\bullet$ is symmetr<u>**a**</u>l if, and only if, $Mog^\bullet$ is symmetr<u>**a**</u>l .
2. $M^\bullet$ is altern<u>**a**</u>l if, and only if, $Mog^\bullet$ is altern<u>**a**</u>l .
3. $M^\bullet$ is symmetr<u>**e**</u>l if, and only if, $Meg^\bullet$ is symmetr<u>**i**</u>l .
4. $M^\bullet$ is altern<u>**e**</u>l if, and only if, $Meg^\bullet$ is altern<u>**i**</u>l .

Theorem: (B., 2015)

Let $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\mathbb{N})$ be a mould.

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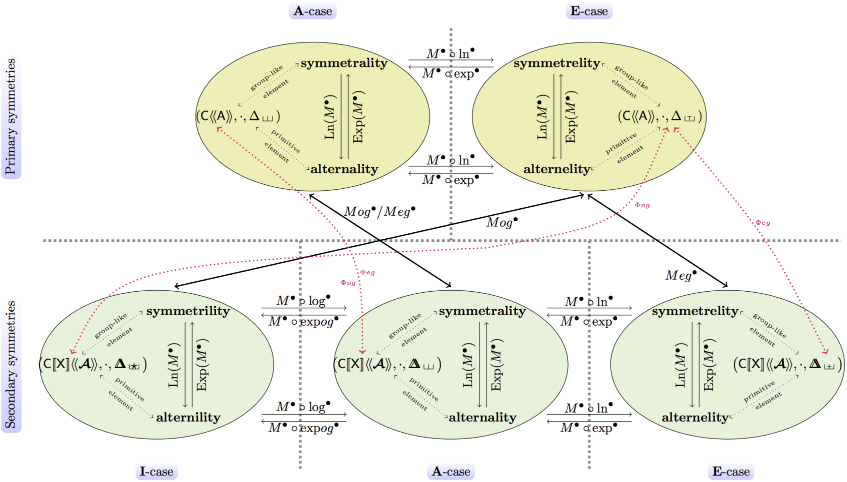
2. $M^\bullet$ is altern<u>**a**</u>l if, and only if, $Meg^\bullet$ is altern<u>**a**</u>l .
3. $M^\bullet$ is symmetr<u>**e**</u>l if, and only if, $Meg^\bullet$ is symmetr<u>**e**</u>l .
4. $M^\bullet$ is altern<u>**e**</u>l if, and only if, $Meg^\bullet$ is altern<u>**e**</u>l .

Summary of mould calculus



Choose of a paradigm

From now on,

- all computations will be done USING NONCOMMUTATIVE SERIES,
- keeping in mind THE MOULD CALCULUS FRAMEWORK.

And...

... let's go to Bernoulli polynomials!

Outline

Outline

Two Equivalent Definitions of Bernoulli Polynomials / Numbers

Bernoulli numbers:	**Bernoulli polynomials:**
By a generating function:	By a generating function:
$\dfrac{t}{e^t - 1} = \sum_{n\geq 0} b_n \dfrac{t^n}{n!}$.	$\dfrac{te^{xt}}{e^t - 1} = \sum_{n\geq 0} B_n(x) \dfrac{t^n}{n!}$.
By a recursive formula:	By a recursive formula:
$\begin{cases} b_0 = 1 \ , \\ \forall n \in \mathbb{N} \ , \ \sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 \ . \end{cases}$	$\begin{cases} B_0(x) = 1 \ , \\ \forall n \in \mathbb{N} \ , \ B'_{n+1}(x) = (n+1)B_n(x) \ , \\ \forall n \in \mathbb{N}^* \ , \ \int_0^1 B_n(x) \, dx = 0 \ . \end{cases}$
First examples:	First examples:
$b_n = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \cdots$	$\begin{aligned} B_0(x) &= 1 \ , \\ B_1(x) &= x - \tfrac{1}{2} \ , \\ B_2(x) &= x^2 - x + \tfrac{1}{6} \ , \\ &\vdots \end{aligned}$

Elementary properties satisfied by the Bernoulli polynomials and numbers

P1 $b_{2n+1} = 0 \;\; if \;\; n > 0.$

P2 $B_n(0) = B_n(1) \;\; if \;\; n > 1.$

P3 $$\sum_{k=0}^{m} \binom{m+1}{k} b_k = 0, \;\; m > 0.$$

P4 $$\begin{cases} B_n'(z) = nB_{n-1}(z) \text{ if } n > 0. \\ B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k} \text{ for all } n. \end{cases}$$

P5 $B_n(x+1) - B_n(x) = nx^{n-1}$, for all n.

P6 $(-1)^n B_n(1-x) = B_n(x)$, for all n.

P7 $$\sum_{k=0}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}(0)}{n+1} .$$

P8 $$\int_a^x B_n(t)\, dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1} .$$

P9 $$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \text{ for all } m > 0 \text{ and } n \geq 0.$$

Elementary properties satisfied by the Bernoulli polynomials and numbers

P1 $b_{2n+1} = 0$ *if* $n > 0$.

P2 $B_n(0) = B_n(1)$ *if* $n > 1$.

} Have to be extended, *but is not restritive enough.*

P3 $\sum_{k=0}^{m} \binom{m+1}{k} b_k = 0, \; m > 0.$ Has to be extended, *but too particular.*

P4 $\begin{cases} B_n'(z) = nB_{n-1}(z) \text{ if } n > 0. \\ B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k} \text{ for all } n. \end{cases}$

Important property, *but turns out to have a generalization with a corrective term...*

P5 $B_n(x+1) - B_n(x) = nx^{n-1}$, for all n. Has to be extended, *but how???*

P6 $(-1)^n B_n(1-x) = B_n(x)$, for all n. Has to be extended, *but how???*

P7 $\sum_{k=0}^{N-1} k^p = \frac{B_{n+1}(N) - B_{n+1}(0)}{n+1}$. *Does not depend of the Bernoulli numbers...*

P8 $\int_a^x B_n(t)\, dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1}$. *Has a generalization using the derivative of a multiple Bernoulli polynomial instead of the Bernoulli polynmials.*

P9 $B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right)$ for all $m > 0$ and $n \geq 0$. *???*

On the Hurwitz Zeta Function

Definition:

The Hurwitz Zeta Function is defined, for $\Re e\ s > 1$, and $z \in \mathbb{C} - \mathbb{N}_{<0}$, by:

$$\zeta(s,z) = \sum_{n>0} \frac{1}{(n+z)^s} \ .$$

Property:

$s \longmapsto \zeta(s,z)$ can be analytically extended to a meromorphic function on $\mathbb{C}$, with a simple pole located at 1.

Property:

H1 $$\begin{cases} \dfrac{\partial \zeta}{\partial z}(s,z) = -s\zeta(s+1,z). \\ \zeta(s,x+y) = \displaystyle\sum_{n\geq 0} \binom{-s}{n} \zeta(s+n,x)y^n. \end{cases}$$

H2 $\zeta(s,z-1) - \zeta(s,z) = z^{-s}$.

H3 $\zeta(-n,z) = -\dfrac{B_{n+1}(z)}{n+1}$ for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$.

Definition of Hurwitz Multiple Zeta Functions

$\mathcal{H}e^{s_1,\cdots,s_r}(z) = \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}$, if $z \in \mathbb{C} - \mathbb{N}_{<0}$ and $(s_1,\cdots,s_r) \in (\mathbb{N}^*)^r$, such that $s_1 \geq 2$.

Lemma 1: (B., J. Ecalle, 2012)

For all sequences $(s_1,\cdots,s_r) \in (\mathbb{N}^*)^r$, $s_1 \geq 2$, we have:

$$\mathcal{H}e^{s_1,\cdots,s_r}(z-1) - \mathcal{H}e^{s_1,\cdots,s_r}(z) = \mathcal{H}e^{s_1,\cdots,s_{r-1}}(z) \cdot z^{-s_r} \ .$$

Lemma 2:

The Hurwitz Multiple Zeta Functions multiply by the stuffle product (of $\mathbb{N}^*$).

Outline

Main Goal

Heuristic:

$$Be^{s_1,\cdots,s_r}(z) = \text{Multiple (Divided) Bernoulli Polynomials} = \mathcal{H}e^{-s_1,\cdots,-s_r}(z)\ .$$

$$be^{s_1,\cdots,s_r} = \text{Multiple (Divided) Bernoulli Numbers} = \mathcal{H}e^{-s_1,\cdots,-s_r}(0)\ .$$

We want to define $Be^{s_1,\cdots,s_r}(z)$ such that:

- their properties are similar to Hurwitz Multiple Zeta Functions' properties.
- their properties generalize these of Bernoulli polynomials.

Main Goal:

Find some polynomials $Be^{s_1,\cdots,s_r}$ such that:

$$\begin{cases} Be^{n}(z) = \dfrac{B_{n+1}(z)}{n+1}\ , \text{ where } n \geq 0\ , \\ Be^{n_1,\cdots,n_r}(z+1) - Be^{n_1,\cdots,n_r}(z) = Be^{n_1,\cdots,n_{r-1}}(z)z^{n_r}\ , \text{ for } n_1,\cdots,n_r \geq 0\ , \\ \text{the } Be^{n_1,\cdots,n_r} \text{ multiply by the stuffle product.} \end{cases}$$

An algebraic construction

Notation 1:

Let $\mathbf{X} = \{X_1, \cdots, X_n, \cdots\}$ be a (commutative) alphabet of indeterminates ; $\widehat{\mathbf{X}}$ its corresponding extended alphabet.

We denotes:

$$\mathcal{B}eeg^{Y_1,\cdots,Y_r}(z) = \sum_{n_1,\cdots,n_r\geq 0} Be^{n_1,\cdots,n_r}(z)\frac{Y_1^{n_1}}{n_1!}\cdots\frac{Y_r^{n_r}}{n_r!} \ ,$$

for all $r \in \mathbb{N}^*$, $Y_1, \cdots, Y_r \in \widehat{\mathbf{X}}$.

Remark: $\mathcal{B}eeg^{Y_1,\cdots,Y_r}(z+1) - \mathcal{B}eeg^{Y_1,\cdots,Y_r}(z) = \mathcal{B}eeg^{Y_1,\cdots,Y_{r-1}}(z)e^{zY_r}$.

Notation 2:

Let $\mathrm{A} = \{a_1, \cdots, a_n, \cdots\}$ be a non-commutative alphabet.
We denotes:

$$\mathfrak{B}(z) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \mathcal{B}eeg^{X_{k_1},\cdots,X_{k_r}}(z)a_{k_1}\cdots a_{k_r} \in \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\!\langle \mathrm{A}\rangle\!\rangle \ .$$

Remark: $\mathfrak{B}(z+1) = \mathfrak{B}(z)\cdot\left(1 + \sum_{k>0} e^{zX_k}\, a_k\right)$

Reformulation of the main goal

From secondary symmetries of mould calculus:

$$\begin{aligned} Be^{n_1,\cdots,n_r} &\text{ multiply the stuffle on non-negative integers} \\ &\iff \mathcal{B}e^{Y_1,\cdots,Y_r} \text{ multiply the stuffle on X} \\ &\iff \mathfrak{B} \text{ is group-like in } \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle. \end{aligned}$$

Reformulation of the main goal

Find some polynomials $B^{n_1,\cdots,n_r}$ such that:

$$\begin{cases} \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k}-1} - \dfrac{1}{X_k} \ , \\ \mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \mathfrak{E}(z) \ , \text{ where } \mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k} a_k \ , \\ \mathfrak{B} \text{ is a “group-like” element of } \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle \ . \end{cases}$$

Outline

A singular solution

Remainder: $\mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k}\, a_k.$

From a false solution to a singular solution...

$$\mathcal{S}(z) = \overset{\searrow}{\prod_{n>0}} \mathfrak{E}(z-n) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \frac{e^{z(X_{k_1}+\cdots+X_{k_r})}}{\prod_{i=1}^{r}(e^{X_{k_1}+\cdots+X_{k_i}}-1)} a_{k_1}\cdots a_{k_r} \text{ is a}$$

false solution to system
$$\begin{cases} \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k}-1} - \dfrac{1}{X_k}\ , \\ \mathfrak{B}(z+1) = \mathfrak{B}(z)\cdot\mathfrak{E}(z)\ , \\ \mathfrak{B} \text{ is a ``group-like'' element of } \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle\ . \end{cases}$$

Explanations: 1.
$$\begin{aligned} \mathfrak{B}(z) &= \cdots = \mathfrak{B}(z-n)\cdot\mathfrak{E}(z-n)\cdots\mathfrak{E}(z-1) \\ &= \cdots = \left(\lim_{n\longrightarrow+\infty} \mathfrak{B}(z-n)\right)\cdot \overset{\longleftarrow}{\prod_{n>0}} \mathfrak{E}(z-n)\ . \end{aligned}$$

2. $\mathcal{S}(z) \in \mathbb{C}[z](\!(\widehat{\mathbf{X}})\!)\langle\!\langle \mathrm{A} \rangle\!\rangle$, $\mathcal{S}(z) \notin \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle$.

Question: How to find a correction of $\mathcal{S}$, to send it into $\mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle$.

Another solution

Fact: If $\Delta(f)(z) = f(z-1) - f(z)$, $\ker \Delta \cap z\mathbb{C}[z] = \{0\}$.

Consequence: There exist a unique family of polynomials such that:

$$\begin{cases} Be_0^{n_1,\cdots,n_r}(z+1) - Be_0^{n_1,\cdots,n_r}(z) = Be_0^{n_1,\cdots,n_{r-1}}(z)z^{n_r} \ . \\ Be_0^{n_1,\cdots,n_r}(0) = 0 \ . \end{cases}$$

This produces a series $\mathfrak{B}_0 \in \mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle$ defined by:

$$\mathfrak{B}_0(z) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \mathcal{B}eeg_0^{X_{k_1},\cdots,X_{k_r}}(z)\ a_{k_1}\cdots a_{k_r} \ .$$

Lemma: (B., 2013)

1. The noncommutative series $\mathfrak{B}_0$ is a "group-like" element of $\mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle$.
2. The coefficients of $\mathfrak{B}_0(z)$ satisfy a recurence relation, where $Y_1, \cdots, Y_r \in \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle$

$$\begin{cases} \mathcal{B}eeg_0^{Y_1}(z) = \dfrac{e^{zY_1}-1}{e^{Y_1}-1} \\ \mathcal{B}eeg_0^{Y_1,\cdots,Y_r}(z) = \dfrac{\mathcal{B}eeg_0^{Y_1+Y_2,Y_3,\cdots,Y_r}(z) - \mathcal{B}eeg_0^{Y_2,Y_3,\cdots,Y_r}(z)}{e^{Y_1}-1} \end{cases}$$

3. The series $\mathfrak{B}_0$ can be expressed in terms of $\mathcal{S}$: $\mathfrak{B}_0(z) = \big(\mathcal{S}(0)\big)^{-1} \cdot \mathcal{S}(z)$.

Characterization of the set of solutions

Reminder: A family of multiple Bernoulli polynomials produces a series $\mathfrak{B}$ such that:

$$\begin{cases} \mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \mathfrak{E}(z) \text{ , where } \mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k} \, a_k \ , \\ \mathfrak{B} \text{ is a ``group-like'' element of } \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\langle \mathrm{A} \rangle\rangle \ , \\ \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k} - 1} - \dfrac{1}{X_k} \ . \end{cases}$$

Proposition: (B. 2013)

Any familly of polynomials which are solution of the previous system comes from a noncommutative series $\mathfrak{B} \in \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\langle \mathrm{A} \rangle\rangle$ such that there exists $\mathfrak{b} \in \mathbb{C}[\![\widehat{\mathbf{X}}]\!]\langle\langle \mathrm{A} \rangle\rangle$ satisfying:

1. $\langle \mathfrak{b} | A_k \rangle = \dfrac{1}{e^{X_k} - 1} - \dfrac{1}{X_k}$
2. $\mathfrak{b}$ is "group-like"
3. $\mathfrak{B}(z) = \mathfrak{b} \cdot \mathfrak{B}_0 = \mathfrak{b} \cdot \big(\mathcal{S}(0)\big)^{-1} \cdot \mathcal{S}(z)$.

Theorem: (B., 2013)

The subgroup of "group-like" series of $\mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\langle \mathrm{A} \rangle\rangle$, with vanishing coefficients in length 1, acts on the set of all possible multiple Bernoulli polynomials, *i.e.* on the set of all possible *algebraic* renormalization.

Outline

New Goal:

From $\mathfrak{B}(z) = \mathfrak{b} \cdot \mathfrak{B}_0$, determine a suitable series $\mathfrak{b}$ such that the reflexion formula

$$(-1)^n B_n(1-z) = B_n(z) \ , n \in \mathbb{N}$$

has a nice generalization.

For a generic series $s \in \mathbb{C}[z][\![\widehat{\mathbf{X}}]\!]\langle\langle \mathrm{A} \rangle\rangle$,

$$s(z) = \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{X_{k_1}, \cdots, X_{k_r}}(z)\, a_{k_1} \cdots a_{k_r} \ ,$$

we consider:

$$\begin{aligned} \overline{s}(z) &= \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{X_{k_r}, \cdots, X_{k_1}}(z)\, a_{k_1} \cdots a_{k_r} \\ \widetilde{s}(z) &= \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{-X_{k_1}, \cdots, -X_{k_r}}(z)\, a_{k_1} \cdots a_{k_r} \end{aligned}$$

The reflection equation for $\mathfrak{B}_0(z)$

Proposition: (B. 2014)

Let $sg = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} (-1)^r a_{k_1} \cdots a_{k_r} = \left(1 + \sum_{n>0} a_n\right)^{-1}$. Then,

$$\widetilde{\mathcal{S}}(0) = \left(\overline{\mathcal{S}}(0)\right)^{-1} \cdot sg \quad \text{and} \quad \widetilde{\mathcal{S}}(1-z) = \left(\overline{\mathcal{S}}(z)\right)^{-1} .$$

Corollary 1: (B. 2014)

For all $z \in \mathbb{C}$, we have: $sg \cdot \widetilde{\mathfrak{B}}_0(1-z) = \left(\overline{\mathfrak{B}}_0(z)\right)^{-1}$.

Example:

$$\begin{aligned} \mathcal{B}_0^{-X,-Y,-Z}(1-z) &= -\mathcal{B}_0^{X,Y,Z}(z) - \mathcal{B}_0^{X+Y,Z}(z) - \mathcal{B}_0^{X,Y+Z}(z) \\ &\quad -\mathcal{B}_0^{X+Y+Z}(z) + \mathcal{B}_0^{Y,Z}(z) + \mathcal{B}_0^{Y+Z}(z) . \end{aligned}$$

The generalization of the reflection formula

Corollary 2: (B. 2014)

$$\widetilde{\mathfrak{B}}(1-z)\cdot\overline{\mathfrak{B}}(z)=\widetilde{\mathfrak{b}}\cdot sg^{-1}\cdot\overline{\mathfrak{b}}\ . \tag{1}$$

Remark: $\widetilde{\mathcal{S}}(0)\cdot sg^{-1}\cdot\overline{\mathcal{S}}(0)=1.$

Heuristic:

A reasonable candidate for a multi-Bernoulli polynomial comes from the coefficients of a series $\mathfrak{B}(z)=\mathfrak{b}\cdot\mathfrak{B}_0(z)$ where $\mathfrak{b}$ satisfies:

1. $\langle\mathfrak{b}|a_k\rangle=\dfrac{1}{e^{X_k}-1}-\dfrac{1}{X_k}$
2. $\mathfrak{b}$ is "group-like"
3. $\widetilde{\mathfrak{b}}\cdot sg^{-1}\cdot\overline{\mathfrak{b}}=1$.

Outline

Resolution of an equation

Goal: Characterise the solutions of $\begin{cases} \widetilde{\mathfrak{u}} \cdot sg^{-1} \cdot \overline{\mathfrak{u}} = 1 \ . \\ \mathfrak{u} \text{ is "group-like"} \ . \end{cases}$

Proposition: (B., 2014)

Let us denote $\sqrt{sg} = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \frac{(-1)^r}{2^{2r}} \binom{2r}{r} a_{k_1} \cdots a_{k_r}$ ..

Any "group-like" solution $\mathfrak{u}$ of $\widetilde{\mathfrak{u}} \cdot sg^{-1} \cdot \overline{\mathfrak{u}} = 1$ comes from a "primitive" series $\mathfrak{v}$ satisfying

$$\overline{\mathfrak{v}} + \widetilde{\mathfrak{v}} = 0 \ ,$$

and is given by:

$$\mathfrak{u} = exp(\mathfrak{v}) \cdot \sqrt{sg} \ .$$

If moreover $\langle \mathfrak{u} | a_k \rangle = \frac{1}{e^{X_k} - 1} - \frac{1}{X_k}$, then necessarily, we have:

$$\langle \mathfrak{v} | a_k \rangle = \frac{1}{e^{X_k} - 1} - \frac{1}{X_k} + \frac{1}{2} := f(X_k) \ .$$

New goal: Find a nice series $\mathfrak{v}$ satisfying:

1. $\mathfrak{v}$ is "primitive". 2. $\overline{\mathfrak{v}} + \widetilde{\mathfrak{v}} = 0$. 3. $\langle \mathfrak{v} | a_k \rangle = \dfrac{1}{e^{X_k} - 1} - \dfrac{1}{X_k} + \dfrac{1}{2} = f(X_k)$.

Remark: $\langle \mathfrak{v} | a_k \rangle$ is an odd formal series in $X_k \in \mathrm{X}$.

Generalization: $\widetilde{\mathfrak{v}} = -\mathfrak{v}$, so $\overline{\mathfrak{v}} = \mathfrak{v}$.

$\Longrightarrow \langle \mathfrak{v} | a_{k_1} a_{k_2} \rangle = -\dfrac{1}{2} f(X_{k_1} + X_{k_2})$, but does not determine $\langle \mathfrak{v} | a_{k_1} a_{k_2} a_{k_3} \rangle$.

A restrictive condition:

A natural condition is to have:

$$\text{there exists } \alpha_r \in \mathbb{C} \text{ such that } \langle \mathfrak{v} | a_{k_1} \cdots a_{k_r} \rangle = \alpha_r f(X_{k_1} + \cdots + X_{k_r}) \ .$$

Now, there is a unique "primitive" series $\mathfrak{v}$ satisfying this condition and the new goal:

$$\langle \mathfrak{v} | a_{k_1} \cdots a_{k_r} \rangle = \frac{(-1)^{r-1}}{r} f(X_{k_1} + \cdots + X_{k_r}) \ .$$

Definition : (B., 2014)

The series $\mathfrak{B}(z)$ and $\mathfrak{b}$ defined by

$$\begin{cases} \mathfrak{B}(z) & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \cdot (\mathcal{S}(0))^{-1} \cdot \mathcal{S}(z) \\ \mathfrak{b} & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \end{cases}$$

are noncommutative series of $\mathbb{C}[z][\![X]\!]\langle\langle A\rangle\rangle$ whose coefficients are respectively the exponential generating functions of *multiple Bernoulli polynomials* and *multiple Bernoulli numbers*.

Example:

The exponential generating function of bi-Bernoulli polynomials and numbers are respectively:

$$\sum_{n_1,n_2\geq 0} B^{n_1,n_2}(z)\frac{X^{n_1}}{n_1!}\frac{Y^{n_2}}{n_2!} = -\frac{1}{2}f(X+Y)+\frac{1}{2}f(X)f(Y)-\frac{1}{2}f(X)+\frac{3}{8} + f(X)\ \frac{e^{zY}-1}{e^Y-1}-\frac{1}{2}\ \frac{e^{zY}-1}{e^Y-1} + \frac{e^{z(X+Y)}-1}{(e^X-1)(e^{X+Y}-1)}-\frac{e^{zY}-1}{(e^X-1)(e^Y-1)}\ .$$

Examples of explicit expression for multiple Bernoulli numbers:

Consequently, we obtain explicit expressions like, for $n_1, n_2, n_3 > 0$:

$$b^{n_1,n_2} = \frac{1}{2}\left(\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1} - \frac{b_{n_1+n_2+1}}{n_1+n_2+1}\right).$$

$$\begin{aligned} b^{n_1,n_2,n_3} = &+\frac{1}{6}\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1}\frac{b_{n_3+1}}{n_3+1} \\ &-\frac{1}{4}\left(\frac{b_{n_1+n_2+1}}{n_1+n_2+1}\frac{b_{n_3+1}}{n_3+1} + \frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+n_3+1}}{n_2+n_3+1}\right) \\ &+\frac{1}{3}\frac{b_{n_1+n_2+n_3+1}}{n_1+n_2+n_3+1}. \end{aligned}$$

Remark: If $n_1 = 0$, $n_2 = 0$ or $n_3 = 0$, the expressions are not so simple...

Table of Multiple Bernoulli Numbers in length 2

$b^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

Conclusion

1. We have respectively defined the Multiple (divided) Bernoulli Polynomials and Multiple (divided) Bernoulli Numbers by:

$$\begin{cases} \mathfrak{B}(z) & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \cdot (\mathcal{S}(0))^{-1} \cdot \mathcal{S}(z) \\ \mathfrak{b} & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \end{cases}$$

where $\mathfrak{v}$ is defined by: $\begin{cases} \langle \mathfrak{v} | a_k \rangle & = & \dfrac{1}{e^{X_k} - 1} - \frac{1}{X_k} + \frac{1}{2} := f(X_k) \\ \langle \mathfrak{v} | a_{k_1} \cdots a_{k_r} \rangle & = & \dfrac{(-1)^{r-1}}{r} f(X_{k_1} + \cdots + X_{k_r}) \end{cases}$

They both multiply the stuffle.

2. The Multiple Bernoulli Polynomials satisfy a nice generalization of:
   - the nullity of b_{2n+1} if $n > 0$.
   - the symmetry $B_n(1) = B_n(0)$ if $n > 1$.
   - the difference equation $\Delta(B_n)(x) = nx^{n-1}$.
   - the reflection formula $(-1)^n B_n(1-x) = B_n(x)$.

References

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- O. Bouillot: *From primary to secondary mould symmetries* (2018) (with $\geq$ 80 complete examples)

THANK YOU FOR YOUR ATTENTION !

On mould composition

Let us suppose that the alphabet $(\Omega, +)$ has an additive semi-group structure. Let us denote $\omega_1 + \cdots + \omega_r$ by $||\underline{\omega}||$ for all sequences $\underline{\omega} \in \Omega^\star$.

Definition:

Let $M^\bullet$ and $N^\bullet$ be two moulds of $\mathcal{M}^\bullet_{\mathbb{C}}(\Omega)$ such that $N^\emptyset = 0$.
Then, the mould composition $C^\bullet = M^\bullet \circ N^\bullet$ is defined for all sequences $\underline{\omega} \in \Omega^\star$ by:

$$(M^\bullet \circ N^\bullet)^{\underline{\omega}} = \begin{cases} M^\emptyset & \text{, if } \underline{\omega} = \emptyset \\ \displaystyle\sum_{k>0} \sum_{\substack{\underline{\omega}^1, \cdots, \underline{\omega}^k \in \Omega^\star - \{\emptyset\} \\ \underline{\omega}^1 \cdots \underline{\omega}^k = \underline{\omega}}} M^{||\underline{\omega}^1||, \cdots, ||\underline{\omega}^k||} N^{\underline{\omega}^1} \cdots N^{\underline{\omega}^k} & \text{, otherwise} \end{cases}$$

Let us consider two constant-type moulds $M^\bullet$ and $N^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\Omega)$, *i.e.* such that $N^\emptyset = 0$ and defined by $M^{\underline{\omega}} = m_r$ and $N^{\underline{\omega}} = n_r$ for all sequences $\underline{\omega} \in \Omega^\star$ of length r.

If well-defined, the composition $C^\bullet = M^\bullet \circ N^\bullet$ is a constant-type mould. Then, denoting $\mathcal{M} = \sum_{r \geq 0} m_r X^r \in \mathbb{C}[\![X]\!]$, $\mathcal{N} = \sum_{r > 0} n_r X^r \in X\mathbb{C}[\![X]\!]$ and $\mathcal{C} := \mathcal{M} \circ \mathcal{N} = \sum c_r X^r \in \mathbb{C}[\![X]\!]$, then $c_r = C^{\underline{\omega}}$, if $l(\underline{\omega}) = r$.

Examples of mould composition

Let us suppose that the alphabet $(\Omega, +)$ has an additive semi-group structure.

- Let $M^\bullet$, $N^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\Omega)$ such that $N^\emptyset = 0$ and ω_1, ω_2 and $\omega_3 \in \Omega$.
  We then have:

$$\left(M^\bullet \circ N^\bullet\right)^\emptyset = M^\emptyset \, . \qquad \left(M^\bullet \circ N^\bullet\right)^{\omega_1} = M^{\omega_1} N^{\omega_1} \, .$$

$$\left(M^\bullet \circ N^\bullet\right)^{\omega_1,\omega_2} = M^{\omega_1+\omega_2} N^{\omega_1,\omega_2} + M^{\omega_1,\omega_2} N^{\omega_1} N^{\omega_2} \, .$$

$$\begin{aligned}\left(M^\bullet \circ N^\bullet\right)^{\omega_1,\omega_2,\omega_3} = {} & M^{\omega_1+\omega_2+\omega_3} N^{\omega_1,\omega_2,\omega_3} + M^{\omega_1+\omega_2,\omega_3} N^{\omega_1,\omega_2} N^{\omega_3} + \\ & M^{\omega_1,\omega_2+\omega_3} N^{\omega_1} N^{\omega_2,\omega_3} + M^{\omega_1,\omega_2,\omega_3} N^{\omega_1} N^{\omega_2} N^{\omega_3} \, .\end{aligned}$$

- Let $\mathcal{Z}ea^\bullet = \mathcal{Z}e^\bullet \circ (exp^\bullet - 1^\bullet)$, where $exp^\bullet$ and $1^\bullet$ are the constant-type mould coming from the exponential map and the constant 1 map.
  Then:

$$\mathcal{Z}ea^\emptyset = 1 \, . \qquad \mathcal{Z}ea^p = \mathcal{Z}e^p \, .$$

$$\mathcal{Z}ea^{p,q} = \mathcal{Z}e^{p,q} + \frac{1}{2}\mathcal{Z}e^{p+q} \, .$$

$$\mathcal{Z}ea^{p,q,r} = \mathcal{Z}e^{p,q,r} + \frac{1}{2}\left(\mathcal{Z}e^{p+q,r} + \mathcal{Z}e^{p,q+r}\right) + \mathcal{Z}e^{p+q+r} \, .$$

Remark: $\mathcal{Z}ea^{p,q} + \mathcal{Z}ea^{q,p} = \mathcal{Z}ea^p \mathcal{Z}ea^q$ and $\mathcal{Z}ea^{p,q,r} + \mathcal{Z}ea^{p,r,q} + \mathcal{Z}ea^{r,p,q} = \mathcal{Z}ea^{p,q}\mathcal{Z}ea^r$

Algebraic structure with composition

Let us suppose that the alphabet $(\Omega, +)$ has an additive semi-group structure.

Proposition: Algebraic structure (Ecalle, 81 / complete detailed proof in B. 18)

$(\mathcal{M}^\bullet_\mathbb{C}(\Omega), +, ., \times, \circ)$ is an *algebra with composition i.e.* that

1. $(\mathcal{M}^\bullet_\mathbb{C}(\Omega), +, ., \times)$ is a C-algebra;
2. the internal operation $\circ : \mathcal{M}^\bullet_\mathbb{C}(\Omega) \times \mathcal{M}^\bullet_\mathbb{C}(\Omega) \longrightarrow \mathcal{M}^\bullet_\mathbb{C}(\Omega)$ is:
   - associative;
   - unitary;
   - distributive relatively to the addition;
   - left-distributive relatively to the multiplication.

Composition stability properties

Proposition: (Ecalle, 81 / / complete detailed proof in B., 18)

Let us assume that $(\Omega, +)$ is a commutative semi-group, so that the mould composition is well-defined.

We have the following stability properties:

1. symmetral $\circ$ alternel $\in$ symmetrel;
2. symmetrel $\circ$(symmetral $-1^\bullet$) $\in$ symmetral;
3. ...

Corollary

$\mathcal{Z}ea^\bullet$ is a symmetral mould, *i.e.* multiply the **shuffle** product! *(Really, not the stuffle!)*.