Harmonic Hecke eigenlines and Mazur's problem

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p-adic modular forms in the sense of Serre

Definition

A power series f is a p**-adic modular form** if there is a sequence of classical modular forms f_i such that $v_p(f - f_i) \to \infty$ as $i \to \infty$.

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Let $\zeta(s)$ be the Riemann zeta-function. Then for $k \geq 1$, the weight $2k$ **Eisenstein series** is given by

$$G_{2k}(z) := \frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

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Remark

For $2k \geq 4$, $G_{2k}(z)$ is a weight $2k$ modular form on $SL_2(\mathbb{Z})$.

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First examples of p-adic modular forms come from Eisenstein series.

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Theorem (Serre, 1972)

Let $\zeta^{(p)}(s)$ be the p-adic zeta-function and

$$\sigma_k^{(p)}(n) := \sum_{\substack{d|n \\ \gcd(d,p)=1}} d^k.$$

Then for $k \geq 1$, we have that

$$G_{2k}^{(p)}(z) = \frac{1}{2}\zeta^{(p)}(1-2k) + \sum_{n=1}^{\infty} \sigma_{2k-1}^{(p)}(n) q^n$$

is a p-adic modular form of weight $2k$.

p-adic Eisenstein series

Remark

$G_{k_1}^{(p)}(z) \equiv G_{k_2}^{(p)}(z) \pmod{p^a}$ whenever $k_1 \equiv k_2 \pmod{(p-1)p^{a-1}}$ and $k_1, k_2 \not\equiv 0 \pmod{p-1}$.

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Example

$6 \equiv 10 \pmod 4$ and $6, 10 \not\equiv 0 \pmod 4$ so

$$G_6^{(5)}(z) = \frac{781}{126} + q + 33q^2 + 244q^3 + 1057q^4 + q^5 + \cdots ,$$

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$$G_6^{(5)}(z) = \frac{781}{126} + q + 33q^2 + 244q^3 + 1057q^4 + q^5 + \cdots ,$$

and

$$G_{10}^{(5)}(z) = \frac{488281}{66} + q + 513q^2 + 19684q^3 + 262657q^4 + q^5 + \cdots$$

are congruent modulo 5.

Mazur's question

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The **eigencurve** is a rigid-analytic curve whose points correspond to normalized finite slope p-adic overconvergent modular eigenforms.

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Question (Mazur)

Does an eigencurve-like object exists for harmonic Maass forms?

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Definition

The **eigencurve** is a rigid-analytic curve whose points correspond to normalized finite slope p-adic overconvergent modular eigenforms.

Question (Mazur)

Does an eigencurve-like object exists for harmonic Maass forms?

Remark

The standard constructions of harmonic Maass forms rarely lead to eigenforms:

- Poincaré series,
- Mock theta functions,
- Indefinite theta functions.

Harmonic Maass forms

Definition

For $k \in \mathbb{R}$, the weight k **hyperbolic Laplacian operator** on $\mathbb{H}$ is defined by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = -4y^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} + 2iky \frac{\partial}{\partial \overline{z}}.$$

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A smooth function $f:\mathbb{H}\to\mathbb{C}$ is a weight k **harmonic Maass form with manageable growth** on Γ (denoted $H_k^!(\Gamma)$) if

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- f transforms like a modular form of weight k on Γ,
- $\Delta_k(f) = 0$,
- $f(z) = O(e^{\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$ at all cusps.

Remarks about harmonic Maass forms

Remark

The Fourier expansion of f naturally splits as

$$f(z) = \underbrace{\sum_{n \gg -\infty} c_f^+(n) q^n}_{\substack{\text{holomorphic part} \\ \text{mock modular form}}} + \underbrace{c_f^-(0) y^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n) \Gamma(1-k, -4\pi n y) q^n}_{\text{non-holomorphic part}}.$$

Examples of harmonic Maass forms

Definition

Let $\Gamma_\infty := \pm\left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \mid n \in \mathbb{Z}\right\}$. Given a translation invariant function $\phi(z)$, the weight k level N **Poincaré series** for $\phi(z)$ is

$$\mathbb{P}(\phi; z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \phi|_k \gamma(z).$$

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Example

Ono constructed a Maass-Poincaré series of weight -10 and level 1

$$F^+_{-10,1}(z) = q^{-1} - \frac{65520}{691} - 1842.89472\cdots q - 23274.07545\cdots q^2 + \cdots$$

which is connected to $\Delta(z)$.

Examples of harmonic Maass forms

Definition

Given a quadratic form Q of type $(r-1,1)$, the **theta function associated to** Q with characteristic $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^r$ is the series

$$\Theta_{a,b}(z) = \sum_{n \in a+\mathbb{Z}^r} \rho(n;z) e^{2\pi i B(n,b)} q^{Q(n)}.$$

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Example

For $Q(j,k) = \frac{1}{2}(5j^2 - 2k^2)$, $a = \begin{pmatrix} \frac{1}{10} \\ 0 \end{pmatrix}$, and $b = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}$ we have

$$\Theta^+_{a,b}(z) = 2q^{\frac{1}{40}} \left(\sum_{\substack{n+j \geq 0 \\ n-j \geq 0}} - \sum_{\substack{n+j<0 \\ n-j<0}} \right) (-1)^j q^{\frac{5n^2}{2} + \frac{n}{2} - j^2},$$

which is related to Ramanujan's fifth order mock theta function.

Applications of harmonic Maass forms

- Partitions (Bruinier-Ono, Dyson, Atkin-Swinnerton-Dyer,...).
- Singular moduli (Borcherds, Zagier, Duke-Imamoğlu-Tóth,...).
- Derivatives of L-functions (Gross-Zagier, Bruinier-Ono,...).
- Donaldson invariants (Göttsche-Zagier, Malmendier-Ono,...).
- Kac-Wakimoto characters (Bringmann-Ono, Dabholkar-Murty-Zagier,...).
- Moonshine (Borcherds, Harvey, Duncan-Griffin-Ono,...).

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3. $p^{2(1-\kappa)}\xi_\kappa(f|T_\kappa(p^2)) = \xi_\kappa(f)|T_{2-\kappa}(p^2)$ if $\kappa \in \frac{1}{2} + \mathbb{Z}$.

Pullback of integer weight Eisenstein series

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- For $k > 0$, define

$$G(z,-2k) := \frac{(2k)!\zeta(2k+1)}{(2\pi)^{2k}} + \frac{(-1)^{k+1}y^{1+2k}2^{1+2k}\pi\zeta(-2k-1)}{2k+1}$$
$$+ (-1)^k(2\pi)^{-2k}(2k)!\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}(n)}{n^{2k+1}}q^n$$
$$+ (-1)^k(2\pi)^{-2k}\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}(n)}{n^{2k+1}}\Gamma(1+2k,4\pi ny)q^{-n}.$$

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Let $T_r^{\chi}(v) := \sum_{a|v} \mu(a)\chi(a)a^{r-1}\sigma_{2r-1}(v/a)$.

Set $(-1)^r N = Dv^2$ with D the discriminant of $\mathbb{Q}(\sqrt{D})$ and $\chi_D = \left(\frac{D}{\cdot}\right)$.

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Definition

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$$c_r(N) = \begin{cases} i^{2r+1} L(1+r,\chi_D)\frac{1}{v^{2r+1}} T_{r+1}^{\chi_D}(v) & N > 0 \\ i^{2r-1}\zeta(1+2r) + \frac{2^{2r+4} i\pi^{2r+1} y^{r+\frac{1}{2}}\zeta(-1-2r)}{(2r-3)\Gamma(2r+1)} & N = 0 \\ \frac{\pi^{3/2} L(-r,\chi_D) T_{r+1}^{\chi_D}(v)\Gamma\left(\frac{r+a}{2}\right)}{N^{r+\frac{1}{2}}\Gamma\left(\frac{r+1+a}{2}\right)\Gamma\left(r+\frac{1}{2}\right)}\Gamma\left(r+\frac{1}{2}, -4\pi N y\right) & N < 0, \end{cases}$$

where $a = 0$ if r is odd and $a = 1$ if r is even.

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where $a=0$ if r is odd and $a=1$ if r is even.

Then, for $r \geq 1$, define $\mathcal{H}\left(z,-r+\frac{1}{2}\right) := \sum_{N\in\mathbb{Z}} c_r(N)q^N$.

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1. *For* $k \in \mathbb{N}$*, we have that* $G(z, -2k) \in H^!_{-2k}(SL_2(\mathbb{Z}))$*. Furthermore,* $G(z, -2k)$ *has eigenvalue* $1 + \frac{1}{p^{2k+1}}$ *under the Hecke operator* $T(p)$*.*

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2. *For $r \in \mathbb{N}$, we have that $\mathcal{H}\left(z, -r + \frac{1}{2}\right) \in H^{!}_{-r+\frac{1}{2}}(\Gamma_0(4))$. Furthermore, $\mathcal{H}\left(z, -r + \frac{1}{2}\right)$ has eigenvalue $1 + \frac{1}{p^{2r+1}}$ under the Hecke operator $T(p^2)$.*

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A **weight** k p**-adic harmonic Maass form** is a formal power series

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + c_f^-(0) y^{1-k} + \sum_{0 \neq n \ll \infty} c_f^-(n) \Gamma\left(1-k, -4\pi n y\right) q^n,$$

where $\Gamma(1-k, -4\pi n y)$ is taken as a formal symbol and where the coefficients $c_f^\pm(n)$ are in $\mathbb{C}_p$, such that there exists a series of harmonic Maass forms $f_i(z)$ such that the following properties are satisfied:

1. $\lim_{i\to\infty} n^{1-k_i} c_{f_i}^\pm(n) = n^{1-k} c_f^\pm(n)$ for $n \neq 0$.
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Remark

Here $\lim_{i\to\infty} n^{1-k_i} c_{f_i}^{\pm}(n) = n^{1-k} c_f^{\pm}(n)$ means $v_p(n^{1-k_i} c_{f_i}^{\pm}(n) - n^{1-k} c_f^{\pm}(n))$ tends to ∞.

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Definitions

- Define the usual *p-adic Gamma function* by

$$\Gamma^{(p)}(n) := (-1)^n \prod_{\substack{0<j<n \\ p\nmid j}} j \qquad \text{if } n \in \mathbb{Z},$$

and
$$\Gamma^{(p)}(x) := \lim_{n\to x} \Gamma^{(p)}(n) \qquad \text{if } x \in \mathbb{Z}_p.$$

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- Let $L_p(s,\chi)$ be the *p-adic L-function.*
- Define

$$T_r^{\chi,(p)}(v) := \sum_{\substack{a|v \\ \gcd(a,p)=1}} \mu(a)\chi(a)a^{r-1}\sigma_{2r-1}^{(p)}(v/a).$$

Answer to Mazur's question for integer weights

Theorem 2 (W)

Suppose p is prime. Then the following are true. For each $k \in X := \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$, *we have that*

$$\begin{aligned} G^{(p)}(z,-2k) := {} & \frac{\Gamma^{(p)}(2k+1)\zeta^{(p)}(2k+1)}{(2\pi)^{2k}} \\ & + \frac{(-1)^{k+1}y^{1+2k}2^{1+2k}\pi\zeta^{(p)}(-2k-1)}{2k+1} \\ & + (-1)^k(2\pi)^{-2k}\Gamma^{(p)}(2k+1)\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}^{(p)}(n)}{n^{2k+1}}q^n \\ & + (-1)^k(2\pi)^{-2k}\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}^{(p)}(n)}{n^{2k+1}}\Gamma(1+2k,4\pi ny)q^{-n} \end{aligned}$$

is a weight $-2k$ *p-adic harmonic Maass form.*

Answer to Mazur's question for half-integral weights

Theorem (W)

For each $-r+\frac{1}{2} \in X$, *let*

$$c_r^{(p)}(N) := \begin{cases} i^{2r+1} L_p(1+r,\chi_D) \frac{1}{v^{2r+1}} T_{r+1}^{\chi_D,(p)}(v) & N>0 \\ i^{2r-1}\zeta^{(p)}(1+2r) + \frac{2^{2r+4} i \pi^{2r+1} y^{r+\frac{1}{2}} \zeta^{(p)}(-1-2r)}{(2r-3)\Gamma^{(p)}(2r+1)} & N=0 \\ \frac{\pi^{3/2} L_p(-r,\chi_D) T_{r+1}^{\chi_D,(p)}(v)\Gamma^{(p)}\left(\frac{r+a}{2}\right)}{N^{r+\frac{1}{2}}\Gamma^{(p)}\left(\frac{r+1+a}{2}\right)\Gamma^{(p)}\left(r+\frac{1}{2}\right)} \Gamma\left(r+\frac{1}{2}, -4\pi N y\right) & N<0. \end{cases}$$

Then $\mathcal{H}^{(p)}\left(z, -r+\frac{1}{2}\right) = \sum_{N\in\mathbb{Z}} c_r^{(p)}(N) q^N$ *is a weight* $-r+\frac{1}{2}$ *p-adic harmonic Maass form.*

Two corollaries

Remark

For $k \in \mathbb{Z}$, $G^{(p)}(z, -2k)$ satisfies

$$G^{(p)}(z, -2k) = G(z, -2k) - G(pz, -2k).$$

This implies that $G^{(p)}(z, -2k) \in H^{!}_{-2k}(\Gamma_0(p))$.

Two corollaries

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Remark

The half-integral weight result implies that the Cohen-Eisenstein series are p-adic modular forms in the sense of Serre.

Hecke's trick

Proof sketch

For $k \in \mathbb{Z}$, define

$$\mathcal{G}(z, -2k, s) := \frac{1}{2} \sum_{(0,0) \neq (n,m) \in \mathbb{Z}^2} \frac{(mz+n)^{2k}}{|mz+n|^{2s}},$$

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and

$$\begin{aligned} f(z, -2k, s) &:= \sum_{n=-\infty}^{\infty} (z+n)^{2k} |z+n|^{-2s} \\ &= \sum_{n=-\infty}^{\infty} h_n(y, -2k, s) e^{2\pi i n x} e^{-2\pi n y}, \end{aligned}$$

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For $k \in \mathbb{Z}$, define

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where by the Poisson summation formula

$$h_n(y,-2k,s) = \int_{iy-\infty}^{iy+\infty} z^{2k}|z|^{-2s}e^{-2\pi inz}dz.$$

Construction of the integer weight forms

Proof sketch

- We find

$$\mathcal{G}(z,-2k,s)=\zeta(2s-2k)+\sum_{\substack{n\in\mathbb{Z}\\ m\geq 1}} m^{1+2k-2s}h_{mn}(y,-2k,s)e^{2\pi i n m x}.$$

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- For each $n \in \mathbb{Z}$, $h_n(y,-2k,0) = 0$, so define

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- Compute each coefficient by contour integration to complete the proof.

A result of Zagier

Proposition (Zagier)

There exists a Dirichlet series

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such that if $n = Dv^2$, then

$$E_n(s) = \begin{cases} 0 & \text{if } n \equiv 2,3 \pmod 4 \\ \frac{L(s,\chi_D)T_s^{\chi_D}(v)}{\zeta(2s)v^{2s-1}} & \text{if } n \equiv 0,1 \pmod 4 \\ \frac{\zeta(2s-1)}{\zeta(2s)} & \text{if } n = 0. \end{cases}$$

Construction of the half-integral weight forms

Proof sketch

Let $k = 2r - 1 \in \mathbb{N}$. Define the two Eisenstein series $F\left(z, -\frac{k}{2}, s\right)$ and $E\left(z, -\frac{k}{2}, s\right)$ by

$$F\left(z, -\frac{k}{2}, s\right) = \sum_{\substack{n,m \in \mathbb{Z} \\ n>0 \\ 4|m}} \left(\frac{m}{n}\right) \varepsilon_n^{-k} \frac{(mz+n)^{k/2}}{|mz+n|^{2s}},$$

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and

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$$E\left(z, -\frac{k}{2}, s\right) = \frac{(2z)^{k/2}}{|2z|^{2s}} F\left(\frac{-1}{4z}, -\frac{k}{2}, s\right),$$

where $\left(\frac{m}{n}\right)$ is the *Kronecker symbol* and

$$\varepsilon_n := \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ i & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

Construction of the half-integral weight forms

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Construction of the half-integral weight forms

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Similarly, we have

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Construction of the half-integral weight forms

Proof sketch

Using the Proposition of Zagier, we find

$$a(N) = 2^{\frac{k}{2}+1-2s}\alpha_N\left(y, -\frac{k}{2}, s\right)\frac{1}{2}E^{odd}_{(-1)^r N}\left(-\frac{k}{2}-\frac{1}{2}+2s\right),$$

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Define

$$\mathcal{H}\left(z, -r+\frac{1}{2}\right) := \lim_{s\to 0}\zeta(1+2r-4s)\left[i^{2r-1}F\left(z, -r+\frac{1}{2}, s\right)\right.$$
$$\left. +2^{r-\frac{1}{2}}(1+i^{2r-1})E\left(z, -r+\frac{1}{2}, s\right)\right].$$

Harmonic Maass eigenforms

Proof sketch

Note that

$$\xi_{-2k}(G(z,-2k)) \doteq E_{2k+2}(z),$$

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It is well known that

$$E_{2k+2}(z)|T(p) = (1+p^{2k+1})E_{2k+2}(z),$$

and

$$H_{r+\frac{3}{2}}(z)|T(p^2) = (1+p^{2r+1})H_{r+\frac{3}{2}}(z).$$

Harmonic Maass eigenforms

Proof sketch

We find

$$G(z,-2k)|T(p) - \left(1 + \frac{1}{p^{2k+1}}\right) G(z,-2k),$$

and

$$\mathcal{H}\left(z, -r + \frac{1}{2}\right) \Big| T(p^2) - \left(1 + \frac{1}{p^{2r+1}}\right) \mathcal{H}\left(z, -r + \frac{1}{2}\right)$$

both vanish.

Generalized Bernoulli numbers

Definition

The *generalized Bernoulli numbers* $B(n, \chi)$ are defined by the generating function

$$\sum_{n=0}^{\infty} B(n, \chi) \frac{t^n}{n!} = \sum_{a=1}^{m-1} \frac{\chi(a) t e^{at}}{e^{mt} - 1},$$

Where χ is a Dirichlet character modulo m.

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Where χ is a Dirichlet character modulo m.

Proposition

If k is a positive integer and χ is a Dirichlet character, then

$$L(1-k, \chi) = -\frac{B(k, \chi)}{k}.$$

Kummer's congruences

Proposition

For $n \geq 1$ we have that

$$L_p(1-n,\chi) = -(1-\chi\cdot\omega^{-n}(p)p^{n-1})\frac{B(n,\chi\cdot\omega^{-n})}{n},$$

where ω is the Teichmüller character.

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Remark

If $n \equiv m \pmod{(p-1)p^a}$ and $(p-1) \nmid n, m$ for an odd prime p, then $(1-p^{n-1})\frac{B_n}{n} \equiv (1-p^{m-1})\frac{B_m}{m} \pmod{p^{a+1}}$,
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where a is a nonnegative integer.
If let $\chi \neq 1$ be a primitive Dirichlet character with conductor not divisible by p, then if $n \equiv m \pmod{p^a}$ we have

$$(1-\chi\cdot\omega^{-n}(p)p^{n-1})\frac{B(n,\chi\cdot\omega^{-n})}{n} \equiv (1-\chi\cdot\omega^{-m}(p)p^{m-1})\frac{B(m,\chi\cdot\omega^{-m})}{m}.$$

Congruences for p-adic harmonic Maass forms

Remark

The p-adic zeta function at positive integers does not behave as nicely as at negative integers. However, it is still expected that it satisfies similar congruences modulo some p-adic regulator.

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Example

We have

$$G^{+,(5)}(z,-2) = -\frac{1}{2\pi^2}\left(\zeta^{(5)}(3) + q + \frac{9}{8}q^2 + \frac{28}{27}q^3 + \frac{73}{64}q^4 + \frac{1}{75}q^5 + \cdots\right),$$

and

$$G^{+,(5)}(z,-6) = -\frac{45}{4\pi^6}\left(\zeta^{(5)}(7) + q + \frac{129}{128}q^2 + \frac{2188}{2187}q^3 + \frac{16513}{16384}q^4 + \frac{1}{78125}q^5 + \cdots\right).$$

Summary

Theorem (W)

We have constructed two infinite families of harmonic Maass forms, one integer weight and one half-integral weight. Furthermore, these forms are eigenforms for the Hecke operators $T(p)$ and $T(p^2)$.

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Theorem (W)

We construct two infinite families of p-adic harmonic Maass forms in the sense of Serre. These constructions provide a partial answer to Mazur's question about the existence of an eigencurve for harmonic Maass forms.