Representations of Knot Groups

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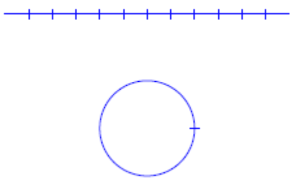

The simplest example: the action of $\mathbb{Z}$ on $\mathbb{R}$ by translations. Each integer n corresponds to the translation $x \to x + n$. The quotient space is the circle S^1. The circle inherits a metric from the standard metric on the line.

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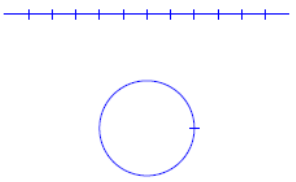

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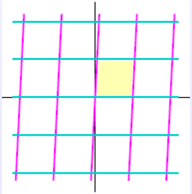
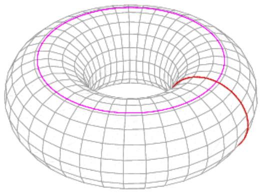
Geometric structure on a 2-manifold

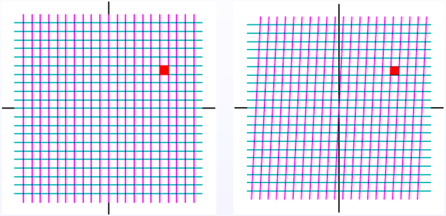
A torus is $S^1 \times S^1$. Euclidean plane is $\mathbb{R} \times \mathbb{R}$. We obtain the torus by the action of the product $\mathbb{Z} \times \mathbb{Z}$ on the plane by translations. This corresponds to a tiling of the plane by parallelograms, whose sides are identified to obtain the torus.





Geometric structures on 2-manifolds

Such a parallelogram is a fundamental region for the group action. Here are two different actions of $\mathbb{Z} \times \mathbb{Z}$ on the Euclidean plane by translations. Each one provides a Euclidean geometric structure on the torus.



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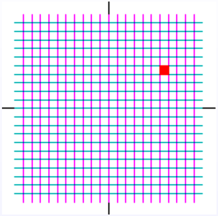
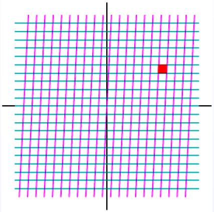
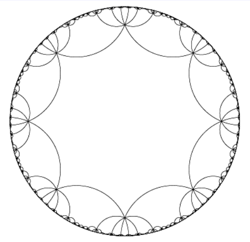
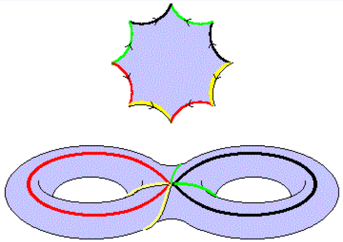




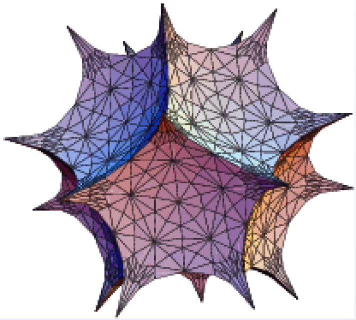
A fundamental region does not have to be flat, *i.e.* Euclidean. Tessellate a **hyperbolic plane** with octagons. A genus-2 surface (doughnut with two holes) is obtained by gluing edges of a hyperbolic octagon in pairs. The surface is a quotient of $\mathbb{H}^2$.

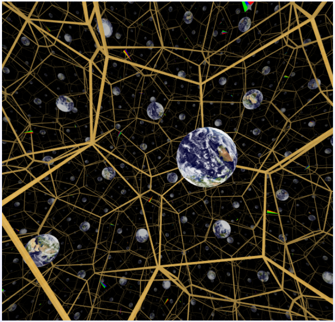




Hyperbolic structures on 3-manifolds

Similarly, factor out $\mathbb{H}^3$ by a suitable group of isometries Γ. This leads to a tiling of $\mathbb{H}^3$ by hyperbolic polyhedra. The quotient manifold is obtained from a single tile by identifying pairs of faces. The manifold inherits a hyperbolic metric from $\mathbb{H}^3$.



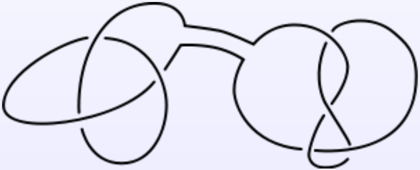


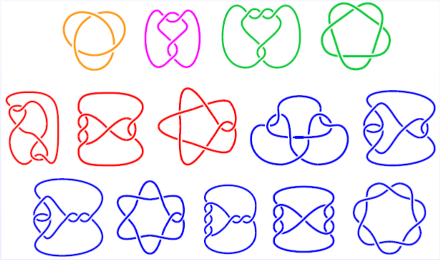
We are interested in manifolds whose volume is finite (e.g. hyperbolic knot complements in 3-sphere). Mostow-Prasad Rigidity Th.: for such a manifold the hyperbolic **metric is unique** as long as it is complete.

A *hyperbolic knot/link* is such that its complement in a 3-sphere is a hyperbolic manifold.

How many knots are hyperbolic?

A knot/link is *prime* if it cannot be written as a connected sum of 2 knots/links. Every knot or a (non-split) link can be uniquely decomposed as a knot sum of prime knots/links (Schubert, Hashizume).





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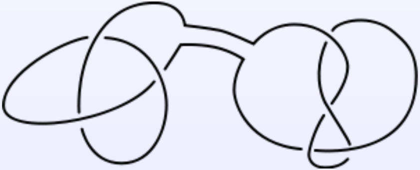

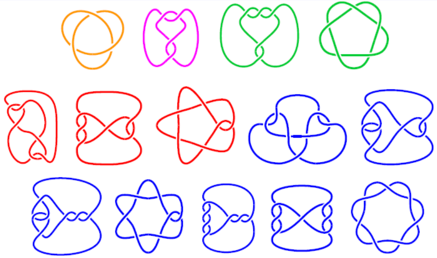

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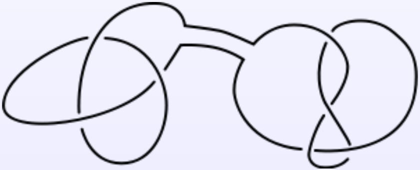

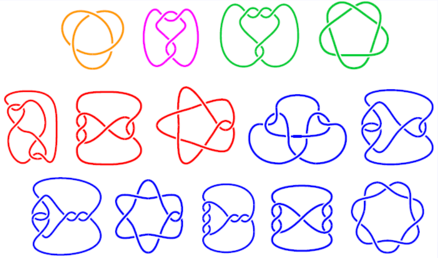

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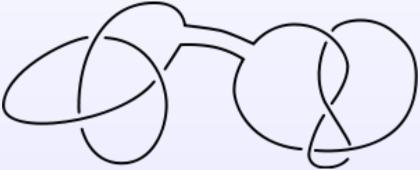

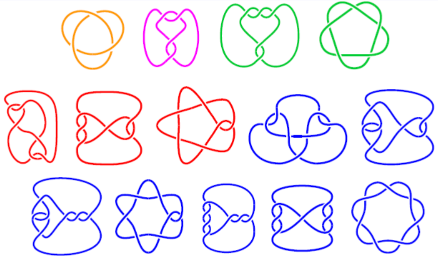

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Of the 1,701,935 prime knots up to 16 crossings, 32 are non-hyperbolic.
Of the 8,053,378 prime knots with 17 crossings, 30 are non-hyperbolic.

Geometric structures and representations

Let M be a hyperbolic 3-manifold. The set of all representations of $\pi_1(M)$ into (P)SL(2, $\mathbb{C}$) is the (P)SL(2, $\mathbb{C}$)-representation variety $R(\Gamma)$ of the 3-manifold. Conjugate representations correspond to the same geometric structure. So a character variety $X(\Gamma) = \{\chi_\rho : \ \rho \in R(\Gamma)\}$ is useful, where the character function $\chi_\rho : \Gamma \to \mathbb{C}$ is $\chi_\rho(\gamma) = \text{trace}(\rho(\gamma))$.

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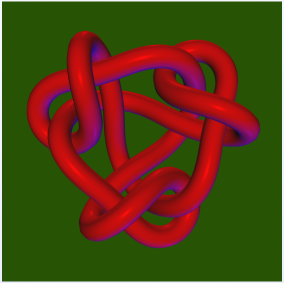
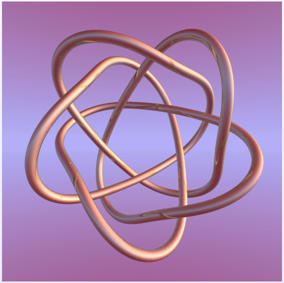
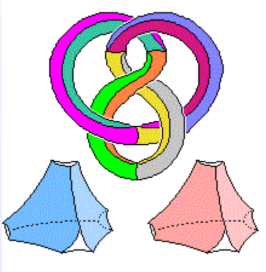


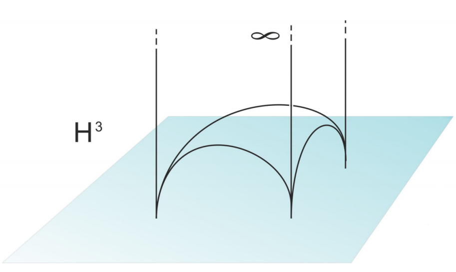


The unique complete hyperbolic structure of M corresponds to the discrete faithful representation. A component of $X(\Gamma)$ that contains such a representation is a *canonical component*. If M is a hyperbolic knot in S^3, this component is a complex curve.

Computing representations: prior work

SnapPea: the first method for computing the hyperbolic structure of 3–manifolds (Thurston, Weeks). It decomposes the manifold into tetrahedra with vertices in the boundary of $\mathbb{H}^3$. Delete these vertices, leaving cross-sectional triangles instead. The cross-sectional triangles form the torus boundary of the link. This was generalized to compute geometric representations (Garoufalidis-Goerner-Zickert).





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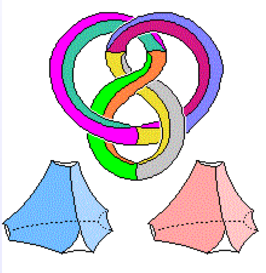
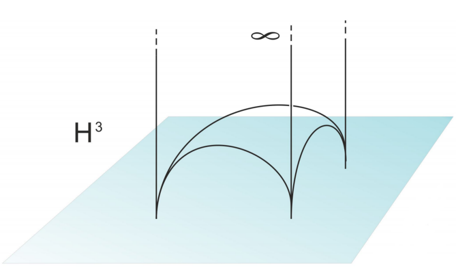




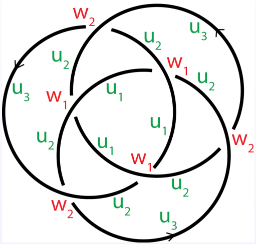
Two other methods for computing *parabolic* representations: Thislethwaite-T.(2012) and Kim-Kim-Yoon(2018). Computations of varieties for some families of knots that admit nice representations: Macasieb-Peterson-Luijk, Chen, Tran.

An alternative method giving equations for the variety

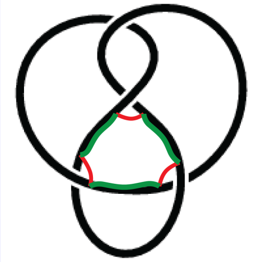
The study of varieties of knots has a long history (e.g. Riley's work). There are many open questions about the connections between the topology of the character variety and the topology/geometry of the respective manifold. E.g. Culler-Shalen showed that ideal points of the character variety of a 3-manifold M "detect" essential surfaces embedded in M.

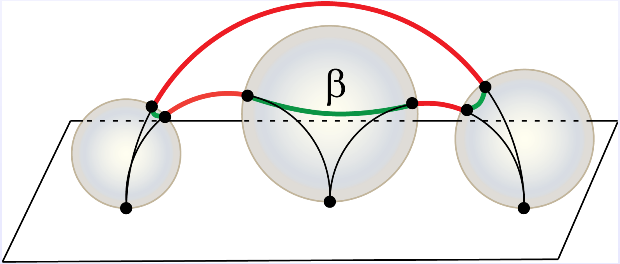
Once there is a simpler method producing equations for the variety based solely on a knot diagram, the above questions might be easier to tackle. We will describe such a method (Petersen-T.). It gives equations for the canonical component of the character variety of a knot. It generalizes earlier work on parabolic representations (Thistlethwaite-T.).

Equations for representations (Thistlethwaite-T., T.-Petersen)



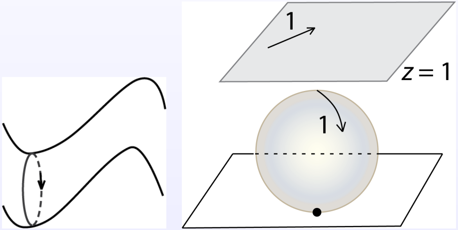
Take a link diagram that satisfies a few mild restrictions. Consider its region bounded by red arcs from an overpass to an underpass, and green arcs on the boundary torus. Assign a complex label to every arc. The labels contain geometric information. We use hyperbolic isometries rotating a preimage of a region in $\mathbb{H}^3$ to write the equations for the labels.





Parabolic representations: a meridian and crossing arcs

A simple closed curve traveling once around the boundary torus of a link is a *meridian*. Its preimage in $\mathbb{H}^3$ lies on a horosphere. Parameterize Euclidean translations on each horosphere by complex numbers so that the meridional translation corresponds to 1.



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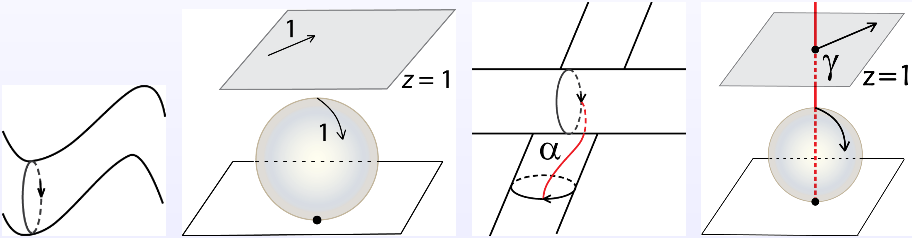
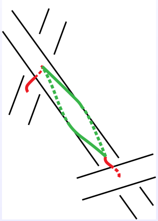
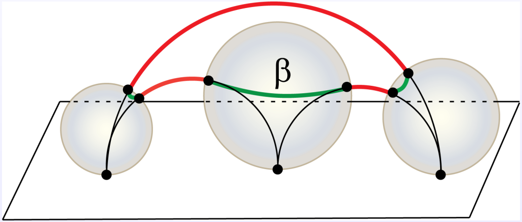


A crossing arc from an overpass to an underpass if a knot diagram is homotopic to a unique geodesic in knot complement. This geodesic is a red arc α on the picture. It has γ as the preimage in $\mathbb{H}^3$. The modulus of the crossing label determines the hyperbolic cusp-to-cusp distance along the arc γ, and the argument of the label is the angle between the meridianal translations on horospheres.

Parabolic representations: edges

Consider a preimage of a green arc. Its preimage in $\mathbb{H}^3$ is the arc β on the corresponding horosphere. The arc β travels between the points where the preimages of the neighboring crossings arcs pierce the horosphere. The edge label is the Euclidean translation along β. Its orientation is inherited from the orientation of the link.





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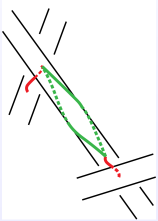
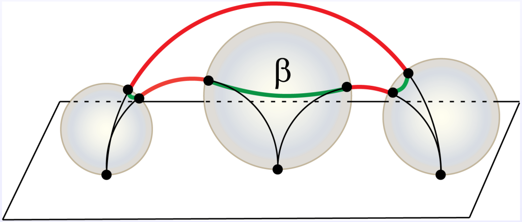
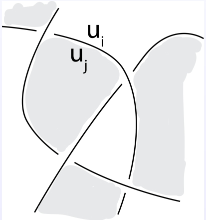
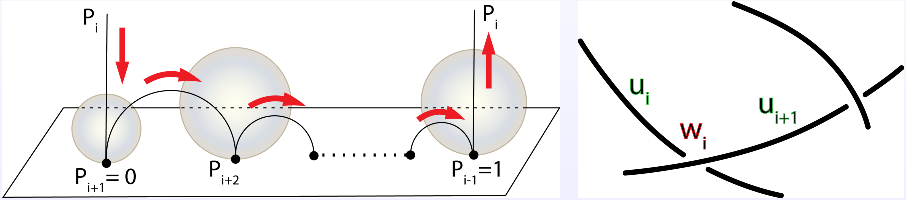






Color the regions of the link diagram in black and white as a checkerboard. Each edge gives rise to two arcs: on the boundary of the black region, and on the boundary of the white one. Traveling along one arc and returning along another corresponds to a meridian, which is 1. Hence, $u_i - u_j$ is 1, -1 or 0. This provides the first set of equations.

A region of a link diagram corresponds to the cyclic sequence of horospheres in $\mathbb{H}^3$. There is an isometry of $\mathbb{H}^3$ which maps three consecutive centers P_{i-1}, P_i, P_{i+1} to P_i, P_{i+1}, P_{i+2}. The isometry is $\rho_i : z \to \frac{-\xi_i}{z-1}$, where $\xi_i = \frac{|P_{i-1}P_i||P_{i+1}P_{i+2}|}{|P_{i-1}P_{i+1}||P_iP_{i+2}|}$ is the cross-ratio of distances between 4 points, called a shape parameter. One can write the parameter ξ_i in terms of the edge and crossing labels as $\xi_i = \frac{\pm w_i}{u_i u_{i+1}}$, where the sign depends on the link orientation.



Since the polygon closes up, the composite $\rho_k \circ \ldots \circ \rho_1 = 1$. If we represent the Möbius transformations by 2×2 matrices, we see that the product

$$\begin{pmatrix} 0 & -\xi_k \\ 1 & -1 \end{pmatrix} \ldots \begin{pmatrix} 0 & -\xi_1 \\ 1 & -1 \end{pmatrix}$$

is a scalar multiple of the identity matrix.

From the matrix entries we read off three independent polynomial relations for every region in the complex labels. One can write a general formula for the relations that depends only on the number of sides in a region. E.g., for a 3–sided region, each of the three shape parameters $\xi_i = \pm \frac{w_i}{u_i u_{i+1}} = 1.$

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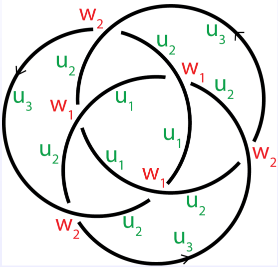

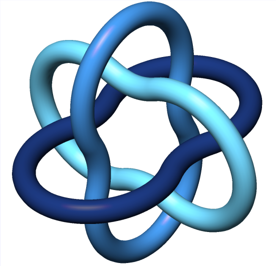

Parabolic representations: example. For the Borromean Rings,

$$\frac{w_1}{u_1^2} = \frac{w_1}{u_2^2} = \frac{w_1}{-(u_1+1)(u_2+1)} = \frac{w_2}{(u_1+1)^2} = \frac{w_1}{(u_1+1)^2} = \frac{-w_2}{u_1 u_3} = 1 \text{ . Hence,}$$

$$u_1 = u_2 = u_3 = \frac{1}{2}(-1+i) , \quad w_1 = -\frac{i}{2} = -w_2 .$$

The solutions give parabolic representations including the discrete faithful one.

Connection to the Wirtinger generators of the link group

The character variety is often obtained by using Wirtinger generators and relations for a knot group. Wirtinger generators are the loops going from a basepoint around every overpass of the link diagram.

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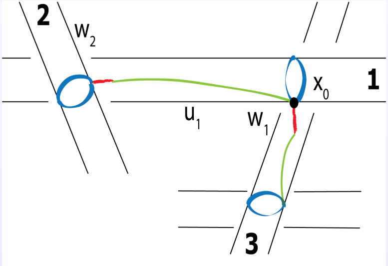
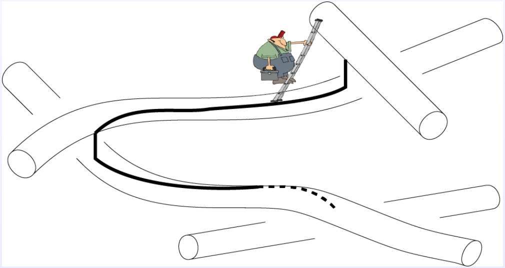
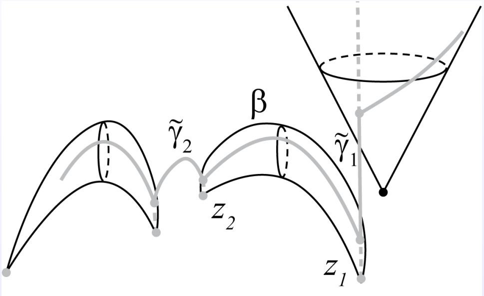


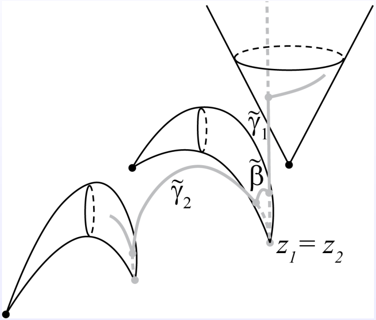


Imagine a playground consisting of knotted tubes, and ladders joining underpasses with overpasses. By traveling from the basepoint to one of the crossings along a loop (and composing the respective matrices on the way), we obtain $(P)SL(2, C)$-matrices for the Wirtinger generators.

Character variety: generalizing beyond parabolic representations

Lemma. Consider two consecutive crossing arcs in a region of a link diagram. Their preimages in $\mathbb{H}^3$ share an ideal point. I.e. the situation on the right occurs. The situation on the left does not occur.





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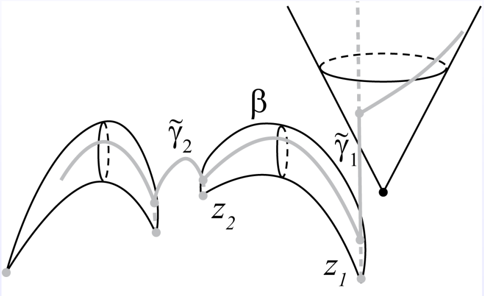
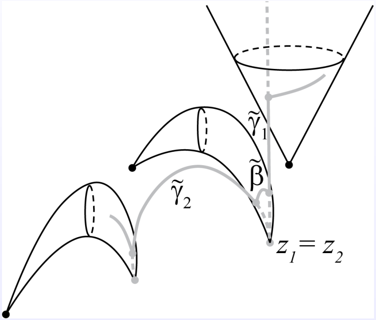
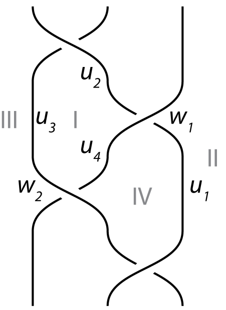


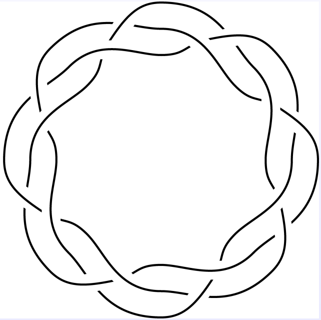


Alternatively, we obtain an analog of a shape parameter written in terms of the entries of edge, crossing and meridian matrices. For a k-sided region of a link diagram, one can then multiply k matrices $\begin{pmatrix} 0 & -\xi_k \\ 1 & -1 \end{pmatrix} \dots \begin{pmatrix} 0 & -\xi_1 \\ 1 & -1 \end{pmatrix}$, and set this equal to an identity matrix, as in parabolic case. This also yields simple equations for the variety of the canonical curve.

Example: character variety of an infinite family of braids $(\sigma_1(\sigma_2)^{-1})^n$, $n > 2$

Region I: $W_1 U_2 W_2 U_3^{-1} W_2 U_4 = \pm k_1 I.$
Region II: $(W_1 U_1^{-1})^n = \pm k_2 I.$
Region III: $(W_2 M^{-1} U_3)^n = \pm k_3 I.$
Region IV: $W_1 M^{-1} U_1 W_1 (M^{-1} U_4)^{-1} W_2 (M^{-1} U_2)^{-1} = \pm k_4 I.$
Here I is the identity matrix and $k_j, j = 1, 2, 3, 4$, is a scalar multiple.





Note that we are using the symmetry of the link diagram. As a result, we obtain formulas for equations in terms of matrix entries for an arbitrary n. These are equations for the character variety of the canonical component.

Number-theoretic invariants of 3-manifolds

Two hyperbolic 3-manifolds are commensurable if there exists a common finite-sheeted cover. One of the ideas motivating a study of hyperbolic 3-manifolds from the number-theoretical point of view is to classify manifolds up to commensurability. For this, the invariant trace field is often used, which is a number field over $\mathbb{Q}$ generated by the squares of the traces of the discrete faithful representation. It is a topological and commensurability invariant of the manifold (A. Reid).

Theorem (Neumann-T.). The complex labels associated to a link diagram or a handlebody decomposition generate the invariant trace field of the hyperbolic 3-manifold.

Corollary: there are several geodesic arcs (often, just one arc) in the manifold, and the angle and distance along these arcs/arc generate the field. The approach described above allows to compute the invariant trace field from a link diagram, and in many cases exactly (i.e. obtain a polynomial). Previously, a program Snap has been sued for this, which involved an intelligent guess of the respective algebraic number.