The Jensen-Pólya Program for the Riemann Hypothesis and Related Problems

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Joint with Michael Griffin, Larry Rolen, and Don Zagier

Riemann's zeta-function

Definition (Riemann)

For $\mathrm{Re}(s) > 1$, define the **zeta-function** by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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1. *The function $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ (apart from a simple pole at $s = 1$ with residue 1).*
2. *We have the* **functional equation**

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \zeta(1-s).$$

Hilbert's 8th Problem

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Apart from negative evens, the zeros of $\zeta(s)$ *satisfy* $\mathrm{Re}(s) = \frac{1}{2}$.

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"Without doubt, it would be desirable to have a rigorous proof of this proposition; however, I have left this research...because it appears to be unnecessary for the immediate goal of my study...."

Bernhard Riemann (1859)

Important Remarks

Huge Understatement

A proof of RH would clarify our understanding of primes.

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A proof of RH would clarify our understanding of primes.

What is known?

1. *The first "gazillion" zeros satisfy RH* (van de Lune, Odlyzko).
2. $> 41\%$ *of zeros satisfy RH* (Selberg, Levinson, Conrey,...).

Jensen-Pólya Program

J. W. L. Jensen
(1859–1925)

George Pólya
(1887–1985)

Jensen-Pólya Program

Definition

The **Riemann Xi-function** is the entire order 1 function

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

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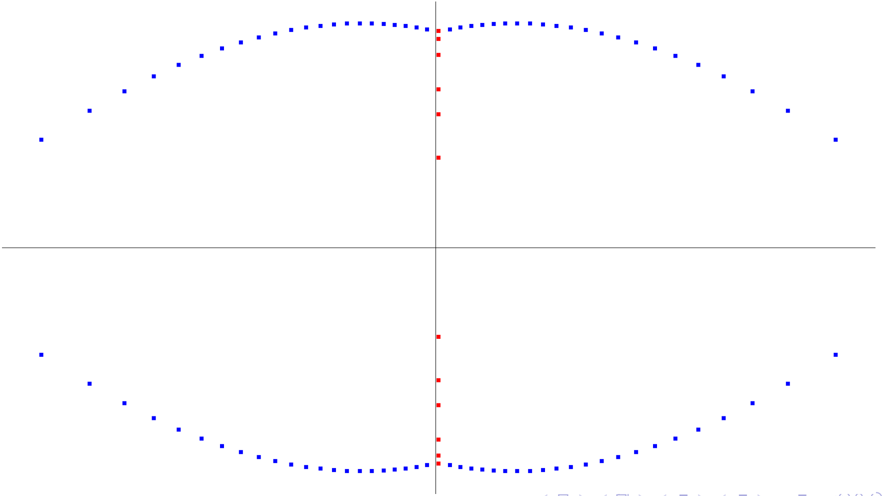
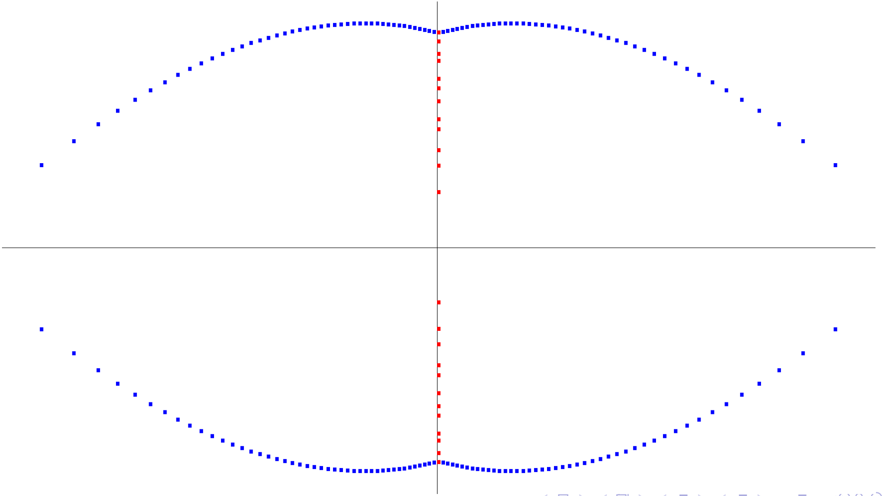
Remark

RH is true $\iff$ *all of the zeros of* $\Xi(z)$ *are purely real.*

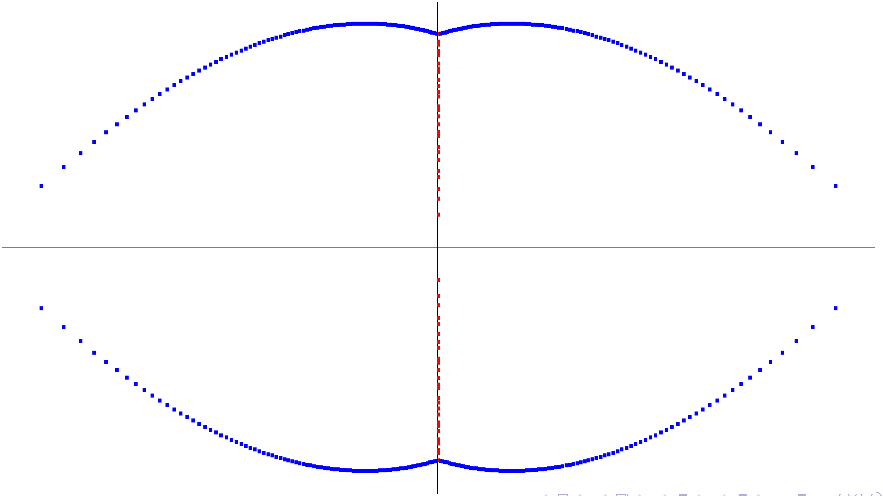
ROOTS OF DEG 100 TAYLOR POLY FOR $\Xi\left(\frac{1}{2}+z\right)$



Roots of Deg 200 Taylor Poly for $\Xi\left(\frac{1}{2}+z\right)$



Roots of Deg 400 Taylor Poly for $\Xi\left(\frac{1}{2}+z\right)$



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- As $d \to +\infty$ the spurious points become more prevalent.

Question

How are the red points distributed as $d \to +\infty$?

Jensen Polynomials

Definition (Jensen)

The **degree** d **and shift** n **Jensen polynomial** for an arithmetic function $a : \mathbb{N} \mapsto \mathbb{R}$ is

$$\begin{aligned} J_a^{d,n}(X) &:= \sum_{j=0}^{d} a(n+j) \binom{d}{j} X^j \\ &= a(n+d)X^d + a(n+d-1)dX^{d-1} + \cdots + a(n). \end{aligned}$$

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$$= a(n+d)X^d + a(n+d-1)dX^{d-1} + \cdots + a(n).$$

Definition

A polynomial $f \in \mathbb{R}[X]$ is **hyperbolic** if all of its roots are real.

Jensen's Criterion

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3. *Nothing for* $d \geq 4$.

New Theorem

"Theorem 1" (Griffin, O, Rolen, Zagier)

For each d at most finitely many $J_{\gamma}^{d,n}(X)$ **are not** *hyperbolic.*

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Theorem (Griffin, O, Thorner)

If $1 \leq d \leq 10^{20}$, then $J_{\gamma}^{d,n}(X)$ is hyperbolic **for all** *n.*

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2. *We* **"locate"** *the real zeros of the* $J_{\gamma}^{d,n}(X)$.
3. *Wagner is generalizing to general L-functions.*

Hermite Polynomials

Definition

The (modified) **Hermite polynomials**

$$\{H_d(X) \ : \ d \geq 0\}$$

are the orthogonal polynomials with respect to $\mu(X) := e^{-\frac{X^2}{4}}$.

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Example (The first few Hermite polynomials)

$$H_0(X) = 1$$
$$H_1(X) = X$$
$$H_2(X) = X^2 - 2$$
$$H_3(X) = X^3 - 6X$$

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1. *Each* $H_d(X)$ *is* **hyperbolic** *with* d **distinct** *roots.*
2. *The zeros of* $H_d(X)$ **interlace**.
3. *If* S_d *denotes the "suitably normalized" zeros of* $H_d(X)$*, then*

$$S_d \longrightarrow \text{ Wigner's Semicircle Law.}$$

RH Criterion and Hermite Polynomials

Theorem 1 (Griffin, O, Rolen, Zagier)

The **renormalized** *Jensen polynomials* $\widehat{J}_{\gamma}^{d,n}(X)$ *satisfy*

$$\lim_{n \to +\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

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For each d at most finitely many $J_{\gamma}^{d,n}(X)$ **are not** *hyperbolic.*

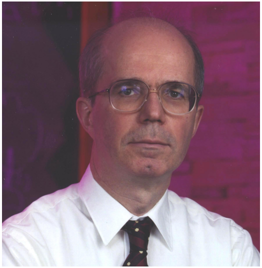
Degree 3 Normalized Jensen polynomials

n	$\widehat{J}_{\gamma}^{3,n}(X)$
100	$\approx 0.9769X^3 + 0.7570X^2 - 5.8690X - 1.2661$
200	$\approx 0.9872X^3 + 0.5625X^2 - 5.9153X - 0.9159$
300	$\approx 0.9911X^3 + 0.4705X^2 - 5.9374X - 0.7580$
400	$\approx 0.9931X^3 + 0.4136X^2 - 5.9501X - 0.6623$
$\vdots$	$\vdots$
10^8	$\approx 0.9999X^3 + 0.0009X^2 - 5.9999X - 0.0014$
$\vdots$	$\vdots$
∞	$H_3(X) = X^3 - 6X$

Random Matrix Model Predictions

Freeman Dyson

Hugh Montgomery



Andrew Odlyzko

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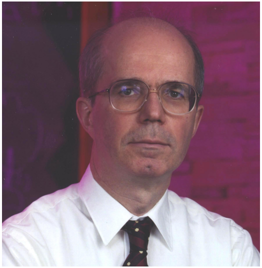

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Gaussian Unitary Ensemble (GUE) (1970s)

The nontrivial zeros of $\zeta(s)$ appear to be "distributed like" the eigenvalues of random Hermitian matrices.

Relation to our work

Theorem (Griffin, O, Rolen, Zagier)

GUE holds for Riemann's $\zeta(s)$ in **derivative aspect**.

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1. *The $J_{\gamma}^{d,n}(X)$ model the zeros of the nth derivative $\Xi^{(n)}(X)$.*

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$$\lim_{n\to+\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

4. *The zeros of the $\{H_d(X)\}$ and the eigenvalues in GUE both satisfy Wigner's Semicircle Distribution.* □

Computing derivatives Is not Easy

Theorem (Pustylnikov (2001), Coffey (2009))

As $n \to +\infty$, we have

$$\xi^{(2n)}(1/2) = \frac{(2n)(2n-1)(2n-2)^{\frac{-1}{4}}}{2^{2n-2}\ln(2n-2)^{\frac{1}{4}}}\left[\ln\left(\frac{2n-2}{\pi}\right) - \ln\ln\left(\frac{2n-2}{\pi}\right) + o(1)\right]^{2n-\frac{3}{2}} \times \exp\left(-\frac{2n-2}{\ln(2n-2)}\right).$$

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Remarks

1. *Derivatives essentially drop to 0 for "small" n before exhibiting* **exponential growth.**
2. *This* **is insufficient** *for approximating* $J_\gamma^{d,n}(X)$.

First 10 Taylor coefficients of $\Xi(x)$

m	$\hat{b}_m$
0	6.214 009 727 353 926 (–2)
1	7.178 732 598 482 949 (–4)
2	2.314 725 338 818 463 (–5)
3	1.170 499 895 698 397 (–6)
4	7.859 696 022 958 770 (–8)
5	6.474 442 660 924 152 (–9)
6	6.248 509 280 628 118 (–10)
7	6.857 113 566 031 334 (–11)
8	8.379 562 856 498 463 (–12)
9	1.122 895 900 525 652 (–12)
10	1.630 766 572 462 173 (–13)

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3. *Let* $L = L(n) \approx \log\left(\frac{n}{\log n}\right)$ *be the unique positive solution of the equation* $n = L \cdot (\pi e^L + \frac{3}{4})$.

Arbitrary precision asymptotics

Theorem (Griffin, O, Rolen, Zagier)

To **all orders**, *as* $n \to +\infty$, *there are* $b_k \in \mathbb{Q}(L)$ *such that*

$$F(n) \sim \sqrt{2\pi}\,\frac{L^{n+1}}{\sqrt{(1+L)n-\frac{3}{4}L^2}}\,e^{L/4-n/L+3/4}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right),$$

where $b_1 = \frac{2L^4+9L^3+16L^2+6L+2}{24\,(L+1)^3}$.

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Remarks

1. *Using two terms (i.e.* b_1*) suffices for our RH application.*
2. **Analysis** + **Computer** $\Longrightarrow$ *hyperbolicity for* $d \leq 10^{20}$.

Example: $\widehat{\gamma}(n)$:= two-term approximation

n	$\widehat{\gamma}(n)$	$\gamma(n)$	$\gamma(n)/\widehat{\gamma}(n)$
10	$\approx$ $1.6313374394 \times 10^{-17}$	$\approx$ $1.6323380490 \times 10^{-17}$	$\approx$ 1.000613367
100	$\approx$ $6.5776471904 \times 10^{-205}$	$\approx$ $6.5777263785 \times 10^{-205}$	$\approx$ 1.000012038
1000	$\approx$ $3.8760333086 \times 10^{-2567}$	$\approx$ $3.8760340890 \times 10^{-2567}$	$\approx$ 1.000000201
10000	$\approx$ $3.5219798669 \times 10^{-32265}$	$\approx$ $3.5219798773 \times 10^{-32265}$	$\approx$ 1.000000002
100000	$\approx$ $6.3953905598 \times 10^{-397097}$	$\approx$ $6.3953905601 \times 10^{-397097}$	$\approx$ 1.000000000

How do these asymptotics imply Theorem 1?

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Theorem 1 is an example of a **general phenomenon**!

Hyperbolic Polynomials in Mathematics

Remark

Hyperbolicity of "generating polynomials" is studied in enumerative combinatorics in connection with **log-concavity**

$$a(n)^2 \geq a(n-1)a(n+1).$$

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$$a(n)^2 \geq a(n-1)a(n+1).$$

- *Group theory (lattice subgroup enumeration)*
- *Graph theory*
- *Symmetric functions*
- *Additive number theory (partitions)*
- ...

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for each j for real sequences $\{A(n)\}$ and $\{\delta(n)\} \to 0$.

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What do we mean?

For fixed d and $0 \leq j \leq d$, as $n \to +\infty$ we have

$$\log\left(\frac{a(n+j)}{a(n)}\right)$$
$$= A(n)j - \delta(n)^2 j^2 + \sum_{i=0}^{d} \mathrm{o}_{i,d}(\delta(n)^i) j^i + \mathrm{O}_d\left(\delta(n)^{d+1}\right).$$

General Theorem

Definition

If $a(n)$ has appropriate growth, then the **renormalized Jensen polynomials** are defined by

$$\widehat{J}_a^{d,n}(X) := \frac{1}{a(n) \cdot \delta(n)^d} \cdot J_a^{d,n}\left(\frac{\delta(n)X - 1}{\exp(A(n))}\right).$$

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General Theorem (Griffin, O, Rolen, Zagier)

Suppose that $a(n)$ has **appropriate growth**.

General Theorem

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If $a(n)$ has appropriate growth, then the **renormalized Jensen polynomials** are defined by

$$\widehat{J}_a^{d,n}(X) := \frac{1}{a(n)\cdot\delta(n)^d}\cdot J_a^{d,n}\left(\frac{\delta(n)X-1}{\exp(A(n))}\right).$$

General Theorem (Griffin, O, Rolen, Zagier)

Suppose that $a(n)$ has **appropriate growth**.
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Example

We have that $p(4) = 5$ because the partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Log concavity of $p(n)$

Example

The roots of the quadratic $J_p^{2,n}(X)$ are

$$\frac{-p(n+1) \pm \sqrt{p(n+1)^2 - p(n)p(n+2)}}{p(n+2)}.$$

It is **hyperbolic** if and only if $p(n+1)^2 > p(n)p(n+2)$.

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Theorem (Nicolas (1978), DeSalvo and Pak (2013))

If $n \geq 25$*, then* $J_p^{2,n}(X)$ *is hyperbolic.*

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Table 1. Conjectured minimal values of $N(d)$

d	1	2	3	4	5	6	7	8	9
$N(d)$	1	25	94	206	381	610	908	1269	1701

Our result

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1. *The proof can be refined case-by-case to prove the minimality of the claimed $N(d)$ (Larson, Wagner).*

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Remarks

1. *The proof can be refined case-by-case to prove the minimality of the claimed $N(d)$ (Larson, Wagner).*
2. *This is a consequence of the* **General Theorem**.

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A **weight** k **weakly holomorphic modular form** is a function f on $\mathbb{H}$ satisfying:

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$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau).$$

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Example (Partition Generating Function)

We have the weight $-1/2$ modular form

$$f(\tau) = \sum_{n=0}^{\infty} p(n) e^{2\pi i \tau (n - \frac{1}{24})}.$$

Jensen polynomials for modular forms

Theorem 3 (Griffin, O, Rolen, Zagier)

Let f be a weakly holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with real coefficients and a pole at $i\infty$. Then for each degree $d \geq 1$

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Remark (Partition Number Example)

Thm 3 separates the roots despite the fact for large n we have

$$J_p^{d,n}(X) \approx p(n+j) \cdot (X+1)^d.$$

Natural Questions

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Is there an even more general theorem?

Hermite Polynomial Generating Function

Lemma (Generating Function)

We have that

$$e^{-t^2+Xt} =: \sum_{d=0}^{\infty} H_d(X) \cdot \frac{t^d}{d!} = 1 + X \cdot t + (X^2 - 2) \cdot \frac{t^2}{2} + \ldots$$

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Remark

The rough idea of the proof is to show for large fixed n that

$$\sum_{d=0}^{\infty} \widehat{J}_a^{d,n}(X) \cdot \frac{t^d}{d!} \approx e^{-t^2+Xt} = e^{-t^2} \cdot e^{Xt}.$$

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How does the **shape** *of $F(t)$ impact "limiting polynomials"?*

More General Theorem

Most General Theorem (Griffin, O, Rolen, Zagier)

If $a(n)$ *has* **appropriate growth** *for the power series*

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$$\lim_{n \to +\infty} \frac{1}{a(n) \cdot \delta(n)^d} \cdot J_a^{d,n}\left(\frac{\delta(n)\,X - 1}{E(n)}\right) = d! \sum_{k=0}^{d} (-1)^{d-k} c_{d-k} \cdot \frac{X^k}{k!}.$$

Sketch of the Proof

- By definition, we have that

$$\frac{1}{a(n)\cdot\delta(n)^d}\cdot J_{\alpha}^{d,n}\left(\frac{\delta(n)X-1}{E(n)}\right)$$
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- By hypothesis, as $n\to+\infty$ we have

$$\frac{a(n+j)}{a(n)}E(n)^{-j} = \sum_{i=0}^{d}\left(c_i+o_{i,d}(1)\right)j^i\,\delta(n)^i + O_d\left(\delta(n)^{d+1}\right)$$

Sketch of the Proof continued

- Therefore, as $n \to +\infty$ the bracketed expression satisfies

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- Substituting in for the bracketed expression gives

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Remark (Limit Polynomials)

If $a : \mathbb{N} \mapsto \mathbb{R}$ *is appropriate for* $F(t)$*, then*

$$F(-t) \cdot e^{Xt} = \sum_{d=0}^{\infty} \widehat{H}_d(X) \cdot \frac{t^d}{d!}.$$

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A sequence with appropriate growth for $F(t) = e^{-t^2}$ **has type** $Z : \mathbb{N} \to \mathbb{R}^+$ if $J_a^{d,n}(X)$ is hyperbolic for $n \geq Z(d)$.

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3. **Have heuristics** *for $Z(d)$ for modular form coefficients.*

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Speculation (Griffin, O, Rolen, Zagier)

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Evidence

If we let $\widehat{Z}(d) := 10d^2 \log d$, then we have

d	$N(d)$	$\widehat{Z}(d)$	$N(d)/\widehat{Z}(d)$
1	1	≈ 1	≈ 1.00
2	25	≈ 27.72	≈ 0.90
4	206	≈ 221.80	≈ 0.93
8	1269	≈ 1330.84	≈ 0.95
16	6917	≈ 7097.82	≈ 0.97
32	35627	≈ 35489.13	≈ 1.00

Our Results

General Theorem (Griffin, O, Rolen, Zagier)

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Applications

Hermite Distributions

1. Jensen-Pólya criterion for RH for $1 \leq d \leq 10^{20}$ **and all** n.
2. Jensen-Pólya criterion for RH for any d **for all** large n.
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+ general theory including Bernoulli and Eulerian distributions.