Graded rings of modular forms of rational weight

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Motivation

We write

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}); ad - bc = 1 \right\}.$$

For any positive integer N, we denote by $\Gamma(N)$ the principal congruence subgroup of $SL_2(\mathbb{Z})$ of level N defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \begin{array}{c} a \equiv d \equiv 1 \bmod N, \\ b \equiv c \equiv 0 \bmod N \end{array} \right\}.$$

Around 2000, I learned from Eiichi Bannai (a specialist of algebraic combinatorics and group theory)

Theorem

There are two algebraically independent modular forms x_1 and x_2 of weight $1/5$ of $\Gamma(5)$ such that all the modular forms of weight $\ell/5$ of $\Gamma(5)$ are generated by them, including those of usual integral weight.

Starting point

I was very much inpressed by his result.
If we consider only integral weight, then for example the dimension of modular forms of weight 1 is given by

$$\dim A_1(\Gamma(5)) = 6.$$

So the graded ring $A(\Gamma(5))$ of all the modular forms of integral weight should have at least 6 generators and they should have plenty of relations.
According to E. Bannai, we have

$$\begin{aligned} A_{1/5}(\Gamma(5)) &= \mathbb{C}x_1 + \mathbb{C}x_2 \\ A_1(\Gamma(5)) &= \mathbb{C}x_1^5 + \mathbb{C}x_1^4x_2 + \mathbb{C}x_1^3x_2^2 + \mathbb{C}x_1^2x_2^3 + \mathbb{C}x_1x_2^4 + \mathbb{C}x_2^5. \end{aligned}$$

Problem

What happens for general N?(construction+graded ring structure)

What is a rational weight?

Let H be the upper half plane and Γ a discrete group of $SL_2(\mathbb{Z})$.

- A $\mathbb{C}^{\times}$-valued function $J(\gamma, \tau)$ of $\Gamma \times H$ is called an automorphy factor if $J(\gamma, \tau)^{\pm 1}$ is holomorphic with respect to τ, and satisfies the following cocycle condition.

$$J(\gamma_1\gamma_2, \tau) = J(\gamma_1, \gamma_2\tau)J(\gamma_2, \tau).$$

  For any holomorphic function $f(\tau)$, we may define action of Γ by

$$f|_J[\gamma] = f(\gamma\tau)J(\gamma, \tau)^{-1}.$$

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- Fix a real number $r \in \mathbb{R}$. If

$$|J(\gamma,\tau)| = |c\tau + d|^r$$

  for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then J is said to be an automorphy factor of weight r.

Modular forms of weight J

- Automorphy factor of real weight r can be written as
$$J(\gamma,\tau) = v(\gamma)(c\tau+d)^r$$
where $(c\tau+d)^r$ means that we fix a certain branch (usually by taking $-\pi < \arg(c\tau+d) \le \pi$) and $v(\gamma)$ is a complex number with $|v(\gamma)| = 1$ which depends only on γ. We call $v(\gamma)$ a multiplier. So automorphy factors of weight r depends on the choice of the multiplier and cannot be prolonged to a bigger group in general.

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- A holomorphic function $f(\tau)$ of H is called a modular form of weight J (or of weight r for short) if it satisfies

$$f|_J[\gamma] = f \qquad \text{for all } \gamma \in \Gamma$$

and f is holomorphic at each cusp of Γ. This last condition is a little complicated to explain and we omit it here. (See my book "Topics on modular forms").

Why rational weight?

For Siegel modular forms of degree bigger than 1 (or for any algebraic group of real rank bigger than 1), there exists no automorphy factor of rational weight except for integral weight or half-integral weight. (Due to P. Deligne and R. Hill.)

There is no good Hecke theory for rational weight of modular forms of $SL_2(\mathbb{R})$, or no good theory of Fourier coefficients except for integral or half integral weight. There are theories of modular forms (or automorphic representations) of n-th covering groups of $SL_2(\mathbb{R})$, but natural theory can be developped only when the base field contains the n-th roots of unity.

But maybe for modular varieties and/or representations of reflection groups, they have some interest. Also it might be interesting from the theory of combinatorics or some congruences of Fourier coefficients.

Modular forms of weight $(N-3)/2N$

For any $m = (m', m') \in \mathbb{Q}^2$, we define theta constants by

$$\theta_m(\tau) = \sum_{p \in \mathbb{Z}} exp\left(2\pi i \left(\frac{1}{2}(p+m')^2\tau + (p+m')m''\right)\right).$$

The Dedekind η function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{2\pi i \tau}.$$

We assume that N is odd and $N \geq 5$. For $s \in \mathbb{Z}$ with $1 \leq s \leq (N-1)/2$, we put

$$\begin{aligned} x_s &= \theta_{\left(\frac{2s-1}{2N}, \frac{1}{2}\right)}(N\tau)/\eta(\tau)^{3/N}. \\ &= q^{s(s-1)/2N}\sum_{p \in \mathbb{Z}}(-1)^p \cdot q^{(Np^2+(2s-1)p)/2} / \prod_{n=1}^{\infty}(1-q^n)^{3/N} \end{aligned}$$

First theorem

Theorem

(1) The functions $x_1, \ldots, x_{(N-1)/2}$ *are linearly independent modular forms of* $\Gamma(N)$ *of weight* $(N-3)/2N$ *with the same multiplier.*

(2) If we denote by $j(\gamma, \tau)$ *the automorphy factor associated to these, then we have* $j(\gamma, \tau)^N = (c\tau + d)^{(N-3)/2}$*. So this is the usual integral weight.*

(3) This automorphy factor can be prolonged to an automorphy factor of $SL_2(\mathbb{Z})$*, and the multiplier is the same as that of* $\eta^{3(N^2-1)/N}$ *and written by the Dedekind sum.*

(4) We write $\Theta_1 = \{x_1, \ldots, x_{(N-1)/2}\}$ *and denote by* $\mathcal{T}_1$ *the vector space over* $\mathbb{C}$ *spanned by* Θ_1*. Then* $\mathcal{T}_1$ *is invariant by the action of* $SL_2(\mathbb{Z})$*.*

Natural problems

Let $j(\gamma, \tau)$ be the automorphy factor of weight $(N-3)/2N$ of $\Gamma(N)$ (or $SL_2(\mathbb{Z})$) defined above. We put

$$\kappa = \frac{N-3}{2N}.$$

For any non-negative integer ℓ, denote by $A_{\ell\kappa}(\Gamma(N))$ the space of modular forms of $\Gamma(N)$ of weight $j(\gamma, \tau)^{\ell}$. Put

$$A^{(\kappa)}(\Gamma(N)) := \sum_{\ell=1}^{\infty} A_{\ell\kappa}(\Gamma(N)).$$

We have $\overline{\Gamma(N)\backslash H} = \mathrm{Proj}(A^{(\kappa)}(\Gamma(N)))$.

Problem

(1) Is the graded ring $A^{(\kappa)}(\Gamma(N))$ generated by Θ_1?
(2) If so, what are the fundamental relations of Θ_1?

Dimensions

Let $\mathcal{L}$ be the line bundle corresponding to $j(\gamma,\tau)$. We have

Theorem (Application of Riemann Roch)

$$\dim A_{\ell\kappa}(\Gamma(N)) = \frac{(\ell(N-3)-2(N-6))N^2}{48}\prod_{p|N}\left(1-\frac{1}{p^2}\right).$$

$$+\dim H^1(\overline{\Gamma(N)\backslash H},\ell\mathcal{L}).$$

In particular, if $\ell > 4(N-6)/(N-3)$, then we have $H^1(\ell\mathcal{L})=0$.

$$g = 1+\frac{N^2(N-6)}{24}\prod_{p|N}\left(1-\frac{1}{p^2}\right).$$

$$deg(div(x_s)) = \frac{\ell(N-3)N^2}{48}\prod_{p|N}\left(1-\frac{1}{p^2}\right).$$

Examples for small N in Abhand. Hamburg 2000

The case $N = 5$ (due to E. Bannai). The genus is 0. The forms x_1 and x_2 are algebraically independent and we have

$$A^{(1/5)}(\Gamma(5)) = \mathbb{C}[x_1, x_2].$$

The case $N = 7$. The genus is 3. We have

$$A^{(2/7)}(\Gamma(7)) = \mathbb{C}[x_1, x_2, x_3]$$

and the fundamental relation is

$$x_1^3 x_3 + x_2 x_3^3 - x_1 x_2^3 = 0.$$

This relation also gives the equation of the modular variety $\overline{\Gamma(7)\backslash H}$ embedded in $P^2(\mathbb{C})$.

This non-singular quartic equation is essentially due to F. Klein.

Example for small N: continue

The case $N = 9$. The genus is 10. We have

$$A^{(1/3)}(\Gamma(9)) = \mathbb{C}[x_1, x_2, x_3, x_4].$$

The fundamental relations are given by

$$\begin{aligned} x_1x_3^2 - x_3x_4^2 - x_4x_1^2 &= 0 \\ x_2^3 + x_1x_4^2 - x_4x_3^2 - x_3x_1^2 &= 0. \end{aligned}$$

The same equation of the modular variety is also obtained by
Y. Kopeliovich and J. R. Quine (Ramanujan J. 1998).
The above results are all obtained by a kind of brute force, but
To go further, we need some tools.
After explaining that, we will give the similar results for $N = 11$ and $N = 13$, and give a counter example for $N = 23$.

Mumford theorem for normal generation

For any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, we define $\mathcal{R}(\mathcal{F},\mathcal{G})$ and $\mathcal{S}(\mathcal{F},\mathcal{G})$ by the following exact sequence

$$0 \to \mathcal{R}(\mathcal{F},\mathcal{G}) \to \Gamma(\mathcal{F}) \otimes \Gamma(\mathcal{G}) \to \Gamma(\mathcal{F} \otimes \mathcal{G}) \to \mathcal{S}(\mathcal{F},\mathcal{G}) \to 0$$

Theorem (Mumford 1970)

Let X be a projective variety over algebraically closed field k. Assume that L is an ample invertible sheaf on X and $\Gamma(L)$ has no base points. Suppose $\mathcal{F}$ is a coherent sheaf on X such that

$$H^i(\mathcal{F} \otimes L^{-i}) = \{0\} \text{ for any } i \geq 1.$$

Then we have
(a) $H^i(\mathcal{F} \otimes L^j) = \{0\}$ *if* $i + j \geq 0$, $i \geq 1$,
(b) $\mathcal{S}(\mathcal{F} \otimes L^i, L) = \{0\}$ *for any* $i \geq 0$.

Fujita Theorem for relations:

Corollary (Mumford)

For a smooth curve C of genus g, if $\det(L) \geq 2g+1$ *(too big for us!), then* $\Gamma(\ell L)$ *is generated by* $\Gamma(L)$ *(simply generated).*

Theorem (Fujita 1977)

Let L, M be line bundles on an irreducible reduced curve C over algebraically closed field and let D be an effective divisor such that $H^1(L-D) = \{0\}$. Also assume that $|D|$ has no base points and $\mathcal{S}(M, L-D) = \{0\}$. Then the natural map

$$\mathcal{R}(M, L) \otimes \Gamma(D) \to \mathcal{R}(M, L+D)$$

is surjective.

Application to our case

Theorem (Corollary to Mumford and Fujita)

Let $\mathcal{L}$ is a line bundle of a compact Riemann surface.
We assume that
(i) $\mathcal{L}$ is ample and effective,
(ii) $\Gamma(\mathcal{L})$ is base point free (i.e. global sections have no common zero)
(iii) $H^1(2\mathcal{L}) = 0$.

Then we have
(1) The natural map $\Gamma(\mathcal{L}) \otimes \Gamma(\ell\mathcal{L}) \to \Gamma((\ell+1)\mathcal{L})$ is surjective for $\ell \geq 3$.

(2) Assume that $\Gamma(\mathcal{L}) \otimes \Gamma(2\mathcal{L}) \to \Gamma(3\mathcal{L})$ is surjective. Then, any relations among elements in $\Gamma(\mathcal{L}) \otimes \Gamma(\ell\mathcal{L}) \to \Gamma((\ell+1)\mathcal{L})$ for $\ell \geq 4$ come from relations in $\Gamma(\mathcal{L}) \otimes \Gamma(3\mathcal{L}) \subset \Gamma(4\mathcal{L})$.

How to use?

The invertible sheaf (or equivalently a line bundle) corresponding to $j(\gamma,\tau)$ is base point free.

The reason is as follows.

(i) Any theta constant has an infinite product expansion obtained as a corollary to

$$\sum_{p\in\mathbb{Z}} x^{p^2} z^p = \prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}).$$

The Dedekind eta also. So x_r has nowhere vanishing on H.

(ii) Any cusp of $\Gamma(N)$ is equivalent to $i\infty$ by an element of $SL_2(\mathbb{Z})$. To examine the behaviour at vaious cusps, we must see the behaviour of $\mathcal{T}_1 = \mathbb{C}x_1 + \cdots \mathbb{C}x_{(N-1)/2}$ under the action of $SL_2(\mathbb{Z})$. But we know that $\mathcal{T}_1$ is stable by $SL_2(\mathbb{Z})$. This space contains x_1 which does not vanish at $i\infty$. So at every cusp, there is a global section which does not vanish at the cusp.

How to show other conditions?

By Riemann Roch, the condition $H^1(2\mathcal{L})=0$ is equivalent to the condition

$$\dim A_{2\kappa}(\Gamma(N)) = \frac{N^2}{8}\prod_{p|N}\left(1-\frac{1}{p^2}\right).$$

Prove this in some way.

We show the following surjectivity

$$\Gamma(\mathcal{L})\otimes\Gamma(\mathcal{L})\to\Gamma(2\mathcal{L}),$$
$$\Gamma(\mathcal{L})\otimes\Gamma(2\mathcal{L})\to\Gamma(3\mathcal{L})$$

by direct calculations.

The case $N = 11$ and $N = 13$

We denote by Θ_ℓ the set of monomials in $x_1, \ldots, x_{(N-1)/2}$ of degree ℓ.
Let $\mathcal{T}_\ell$ be the vector space spanned by Θ_ℓ.
In this case, by the result of Riemann Roch, we have $H^1(\ell\mathcal{L}) = 0$ for all $\ell \geq 3$. We write $N = p$ to emphasize it is a prime. Then for $\kappa = (p-3)/2p$, we have

$$\dim A_{2\kappa}(\Gamma(p)) = \#(\Theta_2) + \dim H^1(2\mathcal{L}) = \frac{p^2-1}{8} + \dim H^1(2\mathcal{L}).$$

$$\dim A_{3\kappa}(\Gamma(p)) = \#(\Theta_3) = \frac{(p-3)(p-1)(p+1)}{48}$$

$$\begin{aligned} \dim A_{4\kappa}(\Gamma(p)) &= \frac{(p-1)p(p+1)}{24} \\ &= \#(\Theta_4) - \frac{(p-5)(p-3)(p-1)(p+1)}{384}, \end{aligned}$$

$$\#(\Theta_4) = \frac{(p-1)(p+1)(p+3)(p+5)}{384}.$$

Proofs

- We see easily that whole elements of Θ_1, Θ_2 and Θ_3 are linearly independent.
- We have $\mathcal{T}_3 = A_{3\kappa}(\Gamma(N))$ and $\mathcal{T}_4 = A_{4\kappa}(\Gamma(N))$. The former is obvious by dimension formula. For the latter, we count linearly independent forms in Θ_4.
- We can show that $\mathcal{T}_2 = A_{2\kappa}(\Gamma(N))$ in the following way. If $g \in A_{2\kappa}(\Gamma(N))$, then $x_1 g \in A_{3\kappa}(\Gamma(N)) = \mathcal{T}_3$. So there are polynomials A and B such that

$$x_1 g = x_1 A(x_1, \ldots, x_{(N-1)/2}) + B(x_2, \ldots, x_{(N-1)/2}).$$

  Since $x_i(x_1 g) = x_1(x_i g) \in x_1\mathcal{T}_3$, we have $x_i B \in x_1\mathcal{T}_3$ for all i. Direct calculations show that there is no such B except for 0. Hence $g = A(x_1, \ldots, x_{(N-1)/2N}) \in A_{2\kappa}(\Gamma(N))$. In the same way, we can see $A_\kappa(\Gamma(N)) = \mathcal{T}_1$.

$N = 11$. There are 15 relations of order 4

Put $x = x_1$, $y = x_2$, $z = x_3$, $v = x_4$, $w = x_5$.

$$\begin{aligned}
x^3v + v^3w - y^3z &= 0, \\
xz^3 - vy^3 - w^3y &= 0, \\
v^3z - z^3y + x^3w &= 0, \\
-y^3x + x^3z + vw^3 &= 0, \\
z^3w + w^3x - v^3y &= 0, \\
z^2vw + x^2yw - y^2zv &= 0, \\
-x^2yv - zvw^2 + xyz^2 &= 0, \\
-w^2yz + v^2yx - x^2wz &= 0, \\
xzv^2 - y^2zw - xvw^2 &= 0, \\
xy^2w - z^2xv + v^2yw &= 0, \\
-xyvw + z^2wy + z^2x^2 - y^3z &= 0, \\
-wvyz - x^2zv + x^2w^2 + xz^3 &= 0, \\
-vyzx + w^2yx + w^2v^2 + x^3w &= 0, \\
-yzxw - v^2zw + v^2y^2 + w^3v &= 0, \\
xzvw - y^2xv + y^2z^2 - yv^3 &= 0.
\end{aligned}$$

$N = 13$: There are 35 relations

$x = x_1$, $y = x_2$, $z = x_3$, $v = x_4$, $w = x_5$, $u = x_6$.

$$
\begin{aligned}
x^2zw - xyv^2 + zvu^2 &= 0\\
x^2zw - yz^2v - xywu + z^2w^2 &= 0\\
xyzw - y^2v^2 - xv^2u + yvw^2 &= 0\\
z^2vu - yvw^2 + xwu^2 &= 0\\
xzv^2 - y^2zw - ywu^2 &= 0\\
x^3u - xzv^2 + xzwu + vw^2u &= 0\\
x^3u - z^3v + v^3w &= 0\\
x^2yu - yzv^2 + zvw^2 &= 0\\
x^3z - xy^3 + wu^3 &= 0\\
x^2yu - xz^2w + yzwu + wu^3 &= 0\\
xy^2u - yz^2w + zv^2u &= 0\\
xy^2u - xyvw + x^2u^2 + zwu^2 &= 0\\
x^2zu - y^2vw + z^2wu &= 0\\
xv^3 - y^3u - yw^3 &= 0\\
xv^3 - y^2vw + xyu^2 - xvwu &= 0\\
x^3v - x^2z^2 + xyzu - z^3w + v^2w^2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
xyzu - xzvw + yvwu &= 0 \\
x^3v - y^3z + w^3u &= 0 \\
xzvw - yv^3 - y^2u^2 + v^2w^2 - w^3u &= 0 \\
yv^3 - z^3w - xu^3 &= 0 \\
x^2yv - xyz^2 + vwu^2 &= 0 \\
y^2zu - yzvw + zw^3 - vwu^2 &= 0 \\
zw^3 - v^3u - yu^3 &= 0 \\
xy^2v - y^2z^2 - yzu^2 + zvwu &= 0 \\
x^2vu - xyw^2 + yzu^2 &= 0 \\
xz^3 - y^3v - zu^3 &= 0 \\
x^2zv - y^3v - xyvu + yz^2u &= 0 \\
xv^2w - yz^2u - xw^2u &= 0 \\
x^3w - yz^3 + vw^3 &= 0 \\
xyzv - yz^3 - xzw^2 + yv^2w &= 0 \\
xzw^2 - y^2vu - xvu^2 &= 0 \\
x^2yw - y^2zv + v^2wu &= 0 \\
xzvu - yzw^2 + v^2wu - w^2u^2 &= 0 \\
xy^2w - xz^2v + zw^2u &= 0 \\
x^2v^2 - xy^2w - x^2wu + yzvu - zw^2u &= 0
\end{aligned}
$$

How to prove relations?

In general for Siegel modular forms (or maybe for a modular form on Tube domain), if enough many Fourier coefficients vanishes, then the form itself vanishes. Precise evaluation is not very easy (e.g. results of C. Poor and D. Yuan).

But in case of one variable, this is very easy.
For the proof of the relations, use the following criterion:
For $f \in A_k(SL_2(\mathbb{Z}))$, if $f = o(q^{k/12})$, then $f = 0$. For a modular form f of Γ, take the norm of Γ to $SL_2(\mathbb{Z})$ by

$$\prod_{\gamma \in \Gamma \backslash SL_2(\mathbb{Z})} f|_k[\gamma].$$

Our case: If $f \in A_{\ell\kappa}(\Gamma(N))$ satisfies $f = o(q^b)$ for some

$$b > \ell(N-3)/24 \times N^3 \prod_{p|N}(1-p^{-2})$$

then we have

$N = 11$

Theorem

We have

$$A^{(4/11)}(\Gamma(11)) = \mathbb{C}[x_1, \ldots, x_5]$$

The fundamental relations are give by the above 15 *relations.*

$$\sum_{\ell=0}^{\infty} \dim A_{4\ell/11}(\Gamma(11)) t^{4\ell/11} = \frac{1 + 3t^{4/11} + 6t^{8/11} + 10t^{12/11}}{(1 - t^{4/11})^2}.$$

The map

$$\overline{\Gamma(11)\backslash H} \ni p \to (x_1(p) : \cdots : x_5(p)) \in \mathbb{P}^4$$

is injective and induces an isomorphic to the image.
(The genus of the curve is 26*.)*

$N = 13$

Theorem

We have

$$A^{(5/13)}(\Gamma(13)) = \mathbb{C}[x_1, \ldots, x_6].$$

The fundamental relations are given by the above 35 *relations.*

$$\sum_{\ell=0}^{\infty} \dim A_{5\ell/13}(\Gamma(13)) t^{5\ell/13} = \frac{1 + 4t^{5/13} + 10t^{10/13} + 20t^{15/13}}{(1 - t^{5/13})^2}.$$

The map

$$\overline{\Gamma(13)\backslash H} \ni p \to (x_1(p) : \cdots : x_6(p)) \in \mathbb{P}^5$$

is injective and induces an isomorphic to the image.
(The genus of the curve is 50*.)*

Counter example for $N = 23$

When $N = p$ is a prime, then we have

$$\dim \mathcal{T}_3 \leq \binom{\frac{p-1}{2}+2}{3} = \frac{(p-3)(p-1)(p+1)}{48}.$$

On the other hand, we have

$$\dim A_{3(p-3)/2p}(\Gamma(p)) = \frac{(p-3)(p-1)(p+1)}{48} + \dim H^1(3\mathcal{L}).$$

So if there is a relation between Θ_3, then the dimension of cT_3 is less than $\dim A_{3(p-3)/2p}(\Gamma(p))$.
But we can show that there are relations of degree 3. For example,

$$x_1x_3^2 - x_1^2x_4 + x_3x_7x_8 - x_4x_5x_9 - x_2x_6x_9 + x_2x_4x_{10} + x_6x_8x_{11} - x_7x_{10}x_{11} = 0.$$

Further problem

For a positive integer d, consider a sub graded ring of $A^{(\kappa)}(\Gamma(N))$ defined by

$$A^{(\kappa,d)}(\Gamma(N)) = \sum_{\ell=1}^{\infty} A_{d\ell\kappa}(\Gamma(N)).$$

Problem

Does there exist some d such that

$$A^{(\kappa,d)}(\Gamma(N)) \subset \mathbb{C}[x_1, \ldots, x_{(N-1)/2N}]?$$

Or is it true that

$$A_{\ell\kappa}(\Gamma(N)) \subset \mathbb{C}[x_1, \ldots, x_{(N-1)/2N}]?$$

for all big enough ℓ?