On arithmetic Chern-Simons-Kim invariants for any number rings

Hikaru Hirano
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Dictionary in Arithmetic Topology

3-dim topology $\Leftrightarrow$ number theory

closed 3-manifold M	$\overline{Spec\ O_k}$ $= Spec\ O_k \cup \{\text{infinite primes}\}$
knot $K\ :\ S^1 \to M$	maximal ideal $\mathfrak{p} = Spec\ (O_k/\mathfrak{p}) \to Spec\ (O_k)$
link $L = K_1 \cup K_2 \cup \cdots \cup K_r$	finite set of maximal ideals $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r\}$
3-manifold group $\pi_1(M)$	maximal unramified Galois group $\pi_1(\overline{Spec\ O_k})$ $= \mathrm{Gal}(k^{un}/k)$ k^{un} : max extension unramified over all finite & infinite primes

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Chern-Simons gauge theory

M : closed oriented 3-manifold
G : compact Lie group
$E \to M$: trivial G-bundle on M
$\mathcal{A}$: the space of connections on E
For $A \in \mathcal{A}$, we set
$CS(A) = \frac{1}{4\pi} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$

$\cdots$ Chern-Simons functional

$Z(M) = \int_{\mathcal{A}} exp(2\pi i m CS(A)) \mathcal{D}A \ (m \in \mathbb{Z})$

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Remark

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=Chern-Simons gauge theory with finite gauge group

M : closed oriented 3-manifold
$[M] \in H_3(M, \mathbb{Z})$: fundamental class
G : finite group
BG : classifying space for G
$c \in H^3(BG, U(1))$ fixed
ρ : $\pi_1(M) \to G$ group hom $\Longleftrightarrow f_\rho : M \to BG$
For $\rho \in \mathrm{Hom}(\pi_1(M), G)$, we set
$CS_c(\rho) = < f_\rho^* c, [M] > \in U(1)$

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$$Z_c(M) = \sum_{\rho:\pi_1(M)\to G} CS_c(\rho)$$

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$$CS_c(\rho) \stackrel{def}{=} inv(j_3(\rho^*(c)))$$

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Here $j_i : H^i(\pi_1(X), \mathbb{Z}/n\mathbb{Z}) \to H^i(X, \mathbb{Z}/n\mathbb{Z})$ is the natural homomorphism induced by Hockschild-Serre spectral sequence

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$X = Spec\ O_K$
$inv\ :\ H^3(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ "fundamental class isomorphism"
G: finite group
$c \in H^3(G, \mathbb{Z}/n\mathbb{Z})$ fixed
For $\rho \in \mathrm{Hom}(\pi_1(X), G)$, we set

$$H^3(G, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho^*} H^3(\pi_1(X), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j_3} H^3(X, \mathbb{Z}/n\mathbb{Z})$$

$$CS_c(\rho) \stackrel{def}{=} inv(j_3(\rho^*(c)))$$

$\cdots$ Arithmetic Chern-Simons-Kim functional

Here $j_i : H^i(\pi_1(X), \mathbb{Z}/n\mathbb{Z}) \to H^i(X, \mathbb{Z}/n\mathbb{Z})$ is the natural homomorphism induced by Hockschild-Serre spectral sequence

$$H^p(\pi_1(X), H^q(\widetilde{X}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow H^{p+q}(X, \mathbb{Z}/n\mathbb{Z})$$

Arithmetic Chern-Simons-Kim invariant

$$Z_c(X) = \sum_{\rho:\pi_1(X)\to G} exp(\frac{2\pi i}{n} CS_c(\rho))$$

$\cdots$ Arithmetic Chern-Simons-Kim invariant

$Z_c(X)$ is an arithmetic analogue of Dijkgraff-Witten invariant.

Minhyong-Kim

Known results about C-S-K invariants

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M.Kim , D.Kim , H.Chung , J.Park , H.Yoo
computation of $CS_c(\rho)$ for the cases
K : imainary quadratic field
$G = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
c , ρ : specific

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T.Chinburg , F.Bleher , R.Greenberg , M Kakde ;
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Let $G = \mathbb{Z}/n\mathbb{Z}$, $c = id \cup \beta(id) \in H^3(G, \mathbb{Z}/n\mathbb{Z})$
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$\beta : H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ Bockstein map
$\rho : \pi_1(X) \to \mathbb{Z}/n\mathbb{Z}$
$L = K(v^{\frac{1}{n}})$:the unramified kummer extension corresponding to ρ
s.t. $\exists I \in Div\ K\ nI = -div(v)$

Proposition (Eric Ahlqvist , Magnus Carlson)

$$CS_c(\rho) = 0 \Leftrightarrow \left(\frac{L/K}{I}\right) \in \mathrm{Gal}(L/K)\ is\ trivial$$

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3-dim topology $\Leftrightarrow$ number theory

3-dim topology	number theory
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Can we define C-S-K invariant for any number field K?

Remark

If K is not totally imaginary, one doesn't have the fundamental class isomorphism
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I extended C-S-K invariant for any number rings by using modified étale cohomologies.

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I showed the non-vanishing criterion of Ahlqvist and Carlson for any number field.

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As a corollary of 2rd result, I calculated C-S-K invariants for
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In the course to show Main Results, I showed the following technical results.

(Tech 1)
I introduced a certain Galois category $FEt_{\overline{X}}$ so that

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1st Technical Result

We say $(Y, M) \in Ob\ \overline{X}_{et}$ is finite étale over $\overline{X}$ if
$Y \to X \in Ob\ X_{et}$ is finite étale and $M = Y_\infty$
$FEt_{\overline{X}}$: the full subcategory of $\overline{X}_{et}$ whose objects are finite étale
$\overline{\eta} : Spec\ \overline{K} \to X$ geometric point

Proposition

$FEt_{\overline{X}}$ is a Galois category and the functor

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So we have $\pi_1(\overline{X}) \stackrel{def}{=} \pi_1(FEt_{\overline{X}}, F_{\overline{\eta}}) = \mathrm{Gal}(K^{un}/K)$

Remark

$$\pi_1^{ab}(\overline{X}) = \mathrm{Gal}(K^{un}_{ab}/K) = Cl_K$$

1st Technical Result

We say $(Y, M) \in Ob\ \overline{X}_{et}$ is finite étale over $\overline{X}$ if
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$FEt_{\overline{X}}$: the full subcategory of $\overline{X}_{et}$ whose objects are finite étale

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The cohomologies of a sheaf on Artin-Verdier Site $\overline{X}_{et}$ is called modified etale cohomologies.

Let $\overline{Y} \in FEt_{\overline{X}}$ be a Galois object , $S \in Sh(\overline{X}_{et})$

Proposition

There is a spectral sequence

$$H^p(Gal(\overline{Y}/\overline{X}), H^q(\overline{Y}, S|\overline{Y})) \Rightarrow H^{p+q}(\overline{X}, S)$$

$S = \mathbb{Z}/n\mathbb{Z}$: constant sheaf on $\overline{X}_{et}$

$(\overline{Y}_i \to \overline{X}, \overline{Y}_i \to \overline{Y}_j)$:inverse system of finite Galois objects .

$\widetilde{\overline{X}} = \underset{\underset{i}{\leftarrow}}{lim} \overline{Y}_i$

On passing to the inverse limit , we have the following.

Corollary

There is a spectral sequence

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Definition

K : number field $\mu_n \subset K$
$X = Spec(O_K)$, $\overline{X} = X \cup X_\infty$
inv : $H^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ "fundamental class isomorphism"
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$\cdots$ Arithmetic Chern-Simons functional

Here $j_i : H^i(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \to H^i(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ is the natural homomorphism induced by Hockschild-Serre spectral sequence

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$$Z_c(\overline{X}) = \sum_{\rho:\pi_1(\overline{X})\to G} exp(\frac{2\pi i}{n} CS_c(\rho))$$

$\cdots$ Arithmetic Chern-Simons-Kim invariant

Remark

If K is totally imaginary , one can obtain $\pi_1(\overline{X}) = \pi_1(X)$ and $H^i(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = H^i(X, \mathbb{Z}/n\mathbb{Z})$. So this is indeed an extension of Kim's invariant.

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3-dim topology $\Leftrightarrow$ number theory

M : closed oriented 3-manifold Dijkgraaf-Witten functional $CS_c(\rho) = < f_\rho^* c, [M] >$	$\overline{X} = \overline{Spec\ O_K}$, K : any number field Arithmetic C-S-K functional $CS_c(\rho) = inv(j_3(\rho^*(c)))$
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$id \in H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$: identity map
$\beta : H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ Bockstein map
$\rho : \pi_1(\overline{X}) \to \mathbb{Z}/n\mathbb{Z}$
$L = K(v^{\frac{1}{n}})$:the kummer extension unramified over all primes including infinite ones corresponding to ρ
s.t. $\exists I \in Div\ K\ nI = -div(v)$

Proposition

$$CS_c(\rho) = 0 \Leftrightarrow \left(\frac{L/K}{I}\right) \in \mathrm{Gal}(L/K)\ is\ trivial$$

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$L = K(v^{\frac{1}{n}})$:the kummer extension unramified over all primes including infinite ones corresponding to ρ
s.t. $\exists I \in Div\ K\ nI = -div(v)$

Proposition

$$CS_c(\rho) = 0 \Leftrightarrow \left(\frac{L/K}{I}\right) \in \mathrm{Gal}(L/K)\ \textit{is trivial}$$

2nd Main Result

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3rd Main Result

$K=\mathbb{Q}(\sqrt{p_1p_2\cdots p_r})$, $p_i\equiv 1(4)$.
ϵ : the fundamental unit of K, Assume that $Norm_{K/\mathbb{Q}}(\epsilon)=-1$
$G=\mathbb{Z}/2\mathbb{Z}$, $c=id\cup\beta(id)\in H^3(G,\mathbb{Z}/n\mathbb{Z})$
$T\overset{def}{=}\{(x_1,x_2,\cdots,x_r)\in(\mathbb{Z}/n\mathbb{Z})^r\,|\,\sum_i x_i=0\}$

$e_{ij}\overset{def}{=}(0,\cdots,0,\overset{i-th}{1},0,\cdots,0,\overset{j-th}{1},0,\cdots,0)\ \in T\ (1\le i<j\le r)$
By Gauss genus theory , we can identify $\rho:\pi_1(\overline{X})\to G$ with
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(c.f. T.Ono ; Algebraic Number Theory)

Definition (mod 2 arithmetic linking number)

$$(-1)^{lk_2(p_i,p_j)}=\left(\frac{p_i}{p_j}\right)\ (LegendleSymbol)$$

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Theorem

For $\rho : T \to \mathbb{Z}/2\mathbb{Z}$,

$$CS_c(\rho) = \sum_{i<j} \rho(e_{ij}) lk_2(p_i, p_j) \in \mathbb{Z}/2\mathbb{Z}$$

So we have ,

$$Z_c(\overline{X}) = \sum_{\rho: T \to \mathbb{Z}/2\mathbb{Z}} \left(\prod_{i<j} \left(\frac{p_i}{p_j} \right)^{\rho(e_{ij})} \right)$$

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Examples

1. Vanishing case

$K = \mathbb{Q}(\sqrt{5 \cdot 29 \cdot 37})$ $c = id \cup \beta(id)$

enumeration of $\rho : T \to \mathbb{Z}/2\mathbb{Z}$:

$\rho_0(1,1,0) = 0$, $\rho_0(0,1,1) = 0$, $\rho_0(1,0,1) = 0$

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2. Non-vanishing case

$K = \mathbb{Q}(\sqrt{5 \cdot 13 \cdot 73})$ $c = id \cup \beta(id)$

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enumeration of $\rho : T \to \mathbb{Z}/2\mathbb{Z}$:
$\rho_0(1,1,0) = 0$, $\rho_0(0,1,1) = 0$, $\rho_0(1,0,1) = 0$
$\rho_1(1,1,0) = 1$, $\rho_1(0,1,1) = 0$, $\rho_1(1,0,1) = 1$
$\rho_2(1,1,0) = 0$, $\rho_2(0,1,1) = 1$, $\rho_2(1,0,1) = 1$
$\rho_3(1,1,0) = 1$, $\rho_3(0,1,1) = 1$, $\rho_3(1,0,1) = 0$
$lk_2(5,29) = 0$, $lk_2(29,37) = 1$, $lk_2(37,5) = 1$
$CS_c(\rho_0) = 0$, $CS_c(\rho_1) = 1$, $CS_c(\rho_2) = 0$, $CS_c(\rho_3) = 1$
$Z_c(\overline{X}) = 0$

2. Non-vanishing case
$K = \mathbb{Q}(\sqrt{5 \cdot 13 \cdot 73})$ $c = id \cup \beta(id)$
$lk_2(5,13) = lk_2(13,73) = lk_2(73,5) = 1$
$CS_c(\rho_0) = CS_c(\rho_1) = CS_c(\rho_2) = CS_c(\rho_3) = 0$
$Z_c(\overline{X}) = 4$