Moonshine for finite groups

Madeline Dawsey and Ken Ono
(Emory University)

History of Finite Simple Groups

- (1832) Galois finds A_n $(n \geq 5)$ and $\mathrm{PSL}_2(\mathbb{F}_p)$ $(p \geq 5)$.
- (1861-1873) Mathieu finds $M_{11}, M_{12}, M_{22}, M_{23}$ and M_{24}.
- (1893) Cole classifies all simple groups with order ≤ 660.
- (1890s-1972)
  Brauer, Burnside, Feit, Frobenius, Dickson, Hall, Thompson,.....
- (1972-1983: Gorenstein Program: "The Classification")
  Aschbacher, Fischer, Glauberman, Gorenstein, Greiss, Tits,.....

The Monster

Conjecture (Fischer and Griess (1973))

There is a huge simple group $\mathbb{M}$ *with order*

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

The Monster

Conjecture (Fischer and Griess (1973))

There is a huge simple group $\mathbb{M}$ *with order*

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

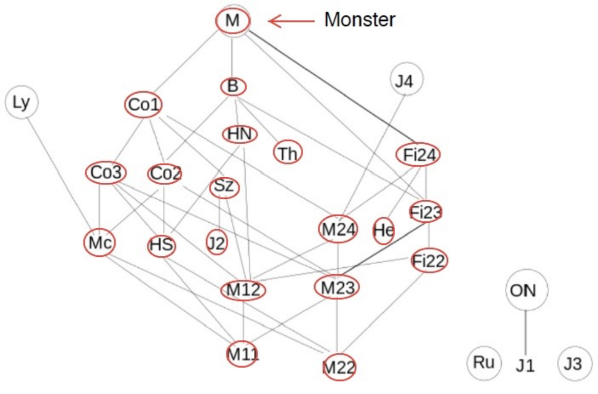
Theorem (Griess (1982))

The Monster group $\mathbb{M}$ *exists.*

Classification of Finite Simple Groups

Theorem ("The Classification" (1983))

Finite simple groups live in natural infinite families, apart from 26 **sporadic groups**.



Modular Curves

Modular Curves

Facts

1. $\mathrm{SL}_2(\mathbb{Z})$ *acts on the upper-half complex plane* $\mathbb{H}$ *by*

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \qquad \longleftrightarrow \qquad \gamma\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

Modular Curves

Facts

1. $\mathrm{SL}_2(\mathbb{Z})$ *acts on the upper-half complex plane* $\mathbb{H}$ *by*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \qquad \longleftrightarrow \qquad \gamma\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

2. *For* **congruence subgroups** $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, *number theorists are interested in the quotients*

$$Y(\Gamma) := \Gamma\backslash\mathbb{H}.$$

Modular Curves

Facts

1. $\mathrm{SL}_2(\mathbb{Z})$ *acts on the upper-half complex plane* $\mathbb{H}$ *by*

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \qquad \longleftrightarrow \qquad \gamma\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

2. *For* **congruence subgroups** $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, *number theorists are interested in the quotients*

$$Y(\Gamma) := \Gamma \backslash \mathbb{H}.$$

3. *These may be compactified by "adding cusps" to obtain compact Riemann surfaces, the* **modular curves** $X(\Gamma)$.

Example: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is generated by

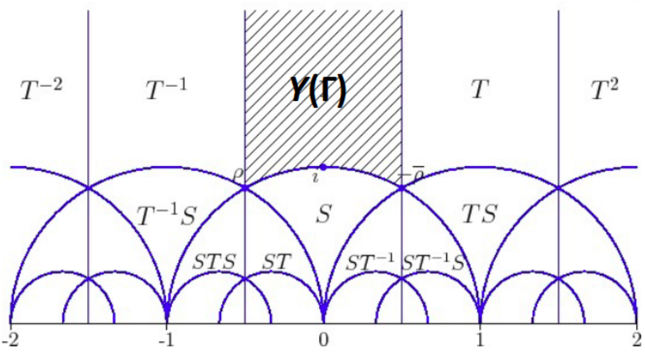
$$T\tau \mapsto \tau + 1 \quad \text{and} \quad S\tau \mapsto \frac{-1}{\tau}.$$

EXAMPLE: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

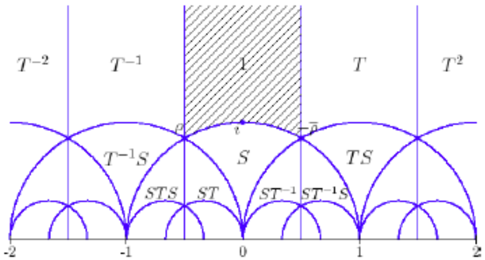
The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is generated by

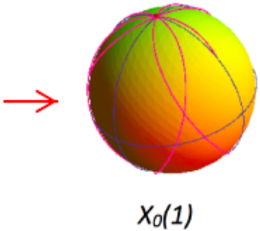
$$T\tau \mapsto \tau + 1 \quad \text{and} \quad S\tau \mapsto \frac{-1}{\tau}.$$

Therefore, $Y(\Gamma)$ can be represented by:

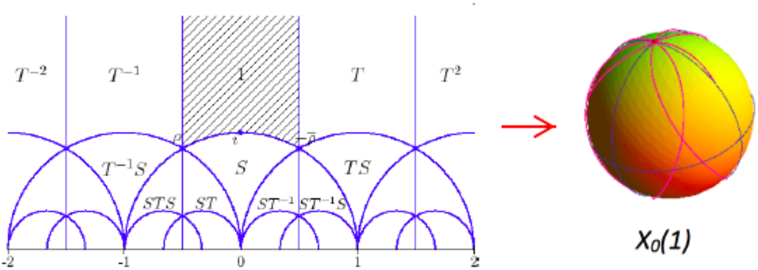


Example: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$





Example: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$



$X_0(1)$

Important Fact

*$X_0(1)$ has **genus 0**, which implies that its field of modular functions is $\mathbb{C}(j(\tau))$ with a **Hauptmodul** $j(\tau)$.*

Modular functions

Definition

A meromorphic function $f : \mathbb{H} \mapsto \mathbb{C}$ is a Γ-**modular function** if for every $\gamma \in \Gamma$ we have

$$f(\gamma\tau) = f(\tau).$$

Modular functions

Definition

A meromorphic function $f : \mathbb{H} \mapsto \mathbb{C}$ is a Γ-**modular function** if for every $\gamma \in \Gamma$ we have

$$f(\gamma\tau) = f(\tau).$$

Example ($\Gamma = \mathrm{SL}_2(\mathbb{Z})$)

The **Hauptmodul** is Klein's j-function ($q := e^{2\pi i\tau}$)

$$\begin{aligned} J(\tau) :=& j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n)q^n \\ &= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \ldots \end{aligned}$$

Glimpse of Monstrous Moonshine

Theorem (Ogg (1974))

The modular curve $X_0(p)^+$ *has* **genus 0** *if and only if* $p \mid \#\mathbb{M}$.

Glimpse of Monstrous Moonshine

Theorem (Ogg (1974))

The modular curve $X_0(p)^+$ has **genus 0** *if and only if $p \mid \#\mathbb{M}$.*

Question

What does having **genus 0** *have to do with the Monster $\mathbb{M}$?*

Glimpses of Monstrous Moonshine

John McKay observed that

$$196884 = 1 + 196883$$

Glimpses of Monstrous Moonshine

John McKay observed that

$$196884 = 1 + 196883$$

John Thompson observed that

$$\underbrace{\begin{array}{r} 196884 \\ 21493760 \\ 864299970 \end{array}}_{\text{Coefficients of } j(\tau)} \begin{array}{c} = \\ = \\ = \end{array} \underbrace{\begin{array}{l} 1 + 196883 \\ 1 + 196883 + 21296876 \\ 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \end{array}}_{\text{Dimensions of irreducible representations of the Monster } \mathbb{M}}$$

The Monster characters

The character table for $\mathbb{M}$ (ordered by size) gives dimensions:

$$\begin{aligned}
\chi_1(e) &= 1 \\
\chi_2(e) &= 196883 \\
\chi_3(e) &= 21296876 \\
\chi_4(e) &= 842609326 \\
&\vdots \\
\chi_{194}(e) &= 258823477531055064045234375.
\end{aligned}$$

Thompson's Conjecture

Conjecture (Thompson)

There is a "nice" infinite-dimensional graded module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *for which* $\dim(V_n^\natural) = c(n)$.

Thompson's Conjecture

Conjecture (Thompson)

There is a "nice" infinite-dimensional graded module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *for which* $\dim(V_n^\natural) = c(n)$.

Remark

1. *Existence of such an infinite graded module is trivial.*

Thompson's Conjecture

Conjecture (Thompson)

There is a "nice" infinite-dimensional graded module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *for which* $\dim(V_n^\natural) = c(n)$.

Remark

1. *Existence of such an infinite graded module is trivial.*
2. *We can use the trivial representation which has* $\dim \chi_1 = 1$.

Thompson's Conjecture

Conjecture (Thompson)

There is a "nice" infinite-dimensional graded module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *for which* $\dim(V_n^\natural) = c(n)$.

Remark

1. *Existence of such an infinite graded module is trivial.*
2. *We can use the trivial representation which has* $\dim \chi_1 = 1$.
3. *Using* **too many** *trivial representations is not "nice".*

Web of Numerology?

Web of Numerology?

Definition (Thompson)

Assuming the conjecture, if $g \in \mathbb{M}$, then define the **McKay–Thompson series**

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{Tr}(g|V_n^{\natural})q^n.$$

Web of Numerology?

Definition (Thompson)

Assuming the conjecture, if $g \in \mathbb{M}$, then define the **McKay–Thompson series**

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{Tr}(g|V_n^{\natural})q^n.$$

Question

Is there a $V^{\natural}$ *for which* **all** *of the* $T_g(\tau)$ *are simultaneously nice?*

Monstrous Moonshine Conjecture

Conjecture (Conway and Norton, 1979)

For each $g \in \mathbb{M}$ there is an explicit genus 0 congruence subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the **Hauptmodul**.

Monstrous Moonshine Conjecture

Conjecture (Conway and Norton, 1979)

For each $g \in \mathbb{M}$ there is an explicit genus 0 congruence subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the **Hauptmodul**.

Theorem (Frenkel–Lepowsky–Meurman (1980s))

If it exists, then the moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ is a specific vertex operator algebra whose automorphism group is $\mathbb{M}$.

Monstrous Moonshine Conjecture

Conjecture (Conway and Norton, 1979)

For each $g \in \mathbb{M}$ there is an explicit genus 0 congruence subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the **Hauptmodul**.

Theorem (Frenkel–Lepowsky–Meurman (1980s))

If it exists, then the moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *is a specific vertex operator algebra whose automorphism group is* $\mathbb{M}$.

Theorem (Borcherds (1998 Fields Medal))

The Monstrous Moonshine Conjecture is true.

Aftermath

Inspired by string theory, further moonshines have been found:

- Mathieu (Gannon)
- Umbral (Cheng, Duncan, Harvey, and Duncan, O, Griffin)
- Thompson (Griffin and Mertens)
- Pariah (Duncan, O, Mertens)
- to name a few...

Witten's Problem

Question (Black Hole States)

Consider the monstrous moonshine expressions

$$\begin{aligned} 196884 &= 1 + 196883 \\ 21493760 &= 1 + 196883 + 21296876 \\ 864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \\ &\vdots \\ c(n) &= \sum_{i=1}^{194} \mathbf{m}_i(n)\chi_i(e) \end{aligned}$$

Witten's Problem

Question (Black Hole States)

Consider the monstrous moonshine expressions

$$\begin{aligned} 196884 &= 1 + 196883 \\ 21493760 &= 1 + 196883 + 21296876 \\ 864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \\ &\vdots \\ c(n) &= \sum_{i=1}^{194} \mathbf{m}_i(n)\chi_i(e) \end{aligned}$$

How many '1's, '196883's, etc. show up in these expressions?

Some Proportions

n	$\delta\left(\mathbf{m}_1(n)\right)$	$\delta\left(\mathbf{m}_2(n)\right)$	$\cdots$	$\delta\left(\mathbf{m}_{194}(n)\right)$
1	1/2	1/2	$\cdots$	0

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
1	$1/2$	$1/2$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
1	$1/2$	$1/2$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$
60	$2.699\ldots \times 10^{-9}$	$2.732\ldots \times 10^{-8}$	$\cdots$	$0.04419\ldots$

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
1	$1/2$	$1/2$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$
60	$2.699\ldots \times 10^{-9}$	$2.732\ldots \times 10^{-8}$	$\cdots$	$0.04419\ldots$
80	$4.809\ldots \times 10^{-14}$	$7.537\ldots \times 10^{-13}$	$\cdots$	$0.04428\ldots$
100	$4.427\ldots \times 10^{-18}$	$1.077\ldots \times 10^{-16}$	$\cdots$	$0.04428\ldots$
120	$1.377\ldots \times 10^{-21}$	$5.501\ldots \times 10^{-20}$	$\cdots$	$0.04428\ldots$
140	$1.156\ldots \times 10^{-24}$	$1.260\ldots \times 10^{-22}$	$\cdots$	$0.04428\ldots$
160	$2.621\ldots \times 10^{-27}$	$3.443\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
180	$1.877\ldots \times 10^{-28}$	$3.371\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
200	$1.715\ldots \times 10^{-28}$	$3.369\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
220	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
240	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$

Distribution of Monstrous Moonshine

Theorem (Duncan, Griffin, O (2015))

If $1 \leq i \leq 194$, *then as* $n \to +\infty$ *we have*

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|n|}}$$

Distribution of Monstrous Moonshine

Theorem (Duncan, Griffin, O (2015))

If $1 \leq i \leq 194$, *then as* $n \to +\infty$ *we have*

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|n|}}$$

Corollary (Duncan, Griffin, O)

The Moonshine module is asymptotically **regular**.

Distribution of Monstrous Moonshine

Theorem (Duncan, Griffin, O (2015))

If $1 \leq i \leq 194$, *then as* $n \to +\infty$ *we have*

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|n|}}$$

Corollary (Duncan, Griffin, O)

The Moonshine module is asymptotically **regular**.
In other words, we have

$$\delta\left(\mathbf{m}_i\right) := \lim_{n \to +\infty} \frac{\mathbf{m}_i(n)}{\sum_{i=1}^{194} \mathbf{m}_i(n)} = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)}.$$

Natural Question

Question

How ubiquitous is moonshine if we **relax** *some conditions?*

Natural Question

Question

How ubiquitous is moonshine if we **relax** *some conditions?*

Definition

A finite group G admits **weak moonshine**

Natural Question

Question

How ubiquitous is moonshine if we **relax** *some conditions?*

Definition

A finite group G admits **weak moonshine** if there is an infinite dimensional graded G-module

$$V_G := \oplus_n V_G(n)$$

Natural Question

Question

How ubiquitous is moonshine if we **relax** *some conditions?*

Definition

A finite group G admits **weak moonshine** if there is an infinite dimensional graded G-module

$$V_G := \oplus_n V_G(n)$$

such that **for all** $g \in G$ the McKay-Thompson series

$$T_g(\tau) := \sum_n \mathrm{Tr}(g|V_G(n))q^n$$

is a weakly holomorphic modular function.

Finite groups enjoy weak moonshine

Theorem (Dehority, Gonzalez, Vafa, Van Peski (2017))

All finite groups admit asymptotically regular weak moonshine.

Finite groups enjoy weak moonshine

Theorem (Dehority, Gonzalez, Vafa, Van Peski (2017))

All finite groups admit asymptotically regular weak moonshine.

Variants

1. *The $T_g(\tau)$ can be required to be weakly holomorphic modular forms or mock modular forms.*

Finite groups enjoy weak moonshine

Theorem (Dehority, Gonzalez, Vafa, Van Peski (2017))

All finite groups admit asymptotically regular weak moonshine.

Variants

1. *The $T_g(\tau)$ can be required to be weakly holomorphic modular forms or mock modular forms.*
2. *We can require that each $T_g(\tau)$ is modular on $\Gamma_0(\mathrm{ord}_G(g))$.*

Finite groups enjoy weak moonshine

Theorem (Dehority, Gonzalez, Vafa, Van Peski (2017))

All finite groups admit asymptotically regular weak moonshine.

Variants

1. *The* $T_g(\tau)$ *can be required to be weakly holomorphic modular forms or mock modular forms.*
2. *We can require that each* $T_g(\tau)$ *is modular on* $\Gamma_0(\mathrm{ord}_G(g))$.
3. *In very special cases there are analytic "group compatibility" relations between* $T_g(\tau)$ *and* $T_{g^p}(\tau)$.

Example: Moonshine for D_4 and Q_8

$\mathbf{D_4}$	$\{1\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2 s\}$	$\{rs, r^3 s\}$
$\mathbf{Q_8}$	$\{1\}$	$\{-1\}$	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	−1	1	−1
χ_3	1	1	−1	−1	1
χ_4	1	1	1	−1	−1
χ_5	2	−2	0	0	0

Example: Moonshine for D_4 and Q_8

$\mathbf{D_4}$	$\{1\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2 s\}$	$\{rs, r^3 s\}$
$\mathbf{Q_8}$	$\{1\}$	$\{-1\}$	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	−1	1	−1
χ_3	1	1	−1	−1	1
χ_4	1	1	1	−1	−1
χ_5	2	−2	0	0	0

- The MT series are Hauptmoduln $J_N(\tau)$ for $\Gamma_0(N)$:

$$T(C_1;\tau) = J_1(\tau)$$
$$T(C_2;\tau) = T(C_4;\tau) = T(C_5;\tau) = J_2(\tau)$$
$$T(C_3;\tau) = J_4(\tau)$$

Example of D_4 and Q_8 continued

- If $1 \leq i \leq 5$ and $n \geq -1$, then let

$$m_i(n) = \#\{\text{mult. of } \rho_i \text{ in } V_G(n)\}.$$

Example of D_4 and Q_8 continued

- If $1 \leq i \leq 5$ and $n \geq -1$, then let

$$m_i(n) = \#\{\text{mult. of } \rho_i \text{ in } V_G(n)\}.$$

- The **multiplicity generating functions** are:

$$\mathcal{M}_i(\tau) := \sum_n m_i(n) q^n.$$

Example of D_4 and Q_8 continued

- If $1 \leq i \leq 5$ and $n \geq -1$, then let

$$m_i(n) = \#\{\text{mult. of } \rho_i \text{ in } V_G(n)\}.$$

- The **multiplicity generating functions** are:

$$\mathcal{M}_i(\tau) := \sum_n m_i(n) q^n.$$

$$\mathcal{M}_1(\tau) = q^{-1} + 24788q + 2685440q^2 + 108044482q^3 + O\left(q^4\right),$$
$$\mathcal{M}_2(\tau) = 24640q + 2686464q^2 + 108038912q^3 + O\left(q^4\right),$$
$$\mathcal{M}_3(\tau) = 24640q + 2686464q^2 + 108038912q^3 + O\left(q^4\right),$$
$$\mathcal{M}_4(\tau) = 24512q + 2687488q^2 + 108033280q^3 + O\left(q^4\right),$$
$$\mathcal{M}_5(\tau) = 49152q + 5373952q^2 + 216072192q^3 + O\left(q^4\right).$$

Example of D_4 and Q_8 continued

This weak moonshine is **asymptotically regular.**

Example of D_4 and Q_8 continued

This weak moonshine is **asymptotically regular.**
To illustrate this, for $1 \leq i \leq 5$ we let

$$\delta_i(n) := \frac{m_i(n)}{m_1(n) + m_2(n) + m_3(n) + m_4(n) + m_5(n)}.$$

Example of D_4 and Q_8 continued

This weak moonshine is **asymptotically regular.**
To illustrate this, for $1 \leq i \leq 5$ we let

$$\delta_i(n) := \frac{m_i(n)}{m_1(n) + m_2(n) + m_3(n) + m_4(n) + m_5(n)}.$$

n	$\delta_1(n)$	$\delta_2(n) = \delta_3(n)$	$\delta_4(n)$	$\delta_5(n)$
1	$0.16779\ldots$	$0.16678\ldots$	$0.16592\ldots$	$0.33271\ldots$
2	$0.16659\ldots$	$0.16665\ldots$	$0.16671\ldots$	$0.33337\ldots$
3	$0.16666\ldots$	$0.16666\ldots$	$0.16665\ldots$	$0.33332\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	$1/6$	$1/6$	$1/6$	$1/3$

Natural Problem

Fact

(1) As the D_4 and Q_8 example illustrates, nonisomorphic groups with identical character tables have the same weak moonshine.

Natural Problem

Fact

(1) As the D_4 and Q_8 example illustrates, nonisomorphic groups with identical character tables have the same weak moonshine.

(2) There are infinitely many **Brauer pairs**, *non-isomorphic groups with isomorphic character tables with common power maps on conjugacy classes ($T_g(\tau)$ on $\Gamma_0(\mathrm{ord}_G(g))$ structure).*

Natural Problem

Fact

(1) As the D_4 and Q_8 example illustrates, nonisomorphic groups with identical character tables have the same weak moonshine.

(2) There are infinitely many **Brauer pairs**, *non-isomorphic groups with isomorphic character tables with common power maps on conjugacy classes ($T_g(\tau)$ on $\Gamma_0(\mathrm{ord}_G(g))$ structure).*

Problems

(1) Can weak moonshine be refined to distinguish groups?

Natural Problem

Fact

(1) As the D_4 and Q_8 example illustrates, nonisomorphic groups with identical character tables have the same weak moonshine.

(2) There are infinitely many **Brauer pairs**, *non-isomorphic groups with isomorphic character tables with common power maps on conjugacy classes ($T_g(\tau)$ on $\Gamma_0(\mathrm{ord}_G(g))$ structure).*

Problems

(1) Can weak moonshine be refined to distinguish groups?

(2) If so, does this procedure have "uniformly bounded" length?

Main Takeaways

Theorem 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^{+}$, then weak moonshine for G extends to **width** *s weak moonshine.*

Main Takeaways

Theorem 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

Main Takeaways

Theorem 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

Corollary (D-O)

If $s \geq 3$, then **complete** *width s weak moonshine* **determines finite groups up to isomorphism.**

Notation

Notation

- G is a finite group

NOTATION

- G is a finite group
- Let $\rho_1, \rho_2, \ldots, \rho_t$ be the irreducible representations

$$\rho_i \colon G \to \mathrm{GL}_{d_i}(\mathbb{C}).$$

Notation

- G is a finite group
- Let $\rho_1, \rho_2, \ldots, \rho_t$ be the irreducible representations

$$\rho_i \colon G \to \mathrm{GL}_{d_i}(\mathbb{C}).$$

- Let $\chi_1, \chi_2, \ldots, \chi_t$ be the corresponding **characters**, and so

$$\chi_i(e) = d_i.$$

Notation

- G is a finite group
- Let $\rho_1, \rho_2, \ldots, \rho_t$ be the irreducible representations

$$\rho_i \colon G \to \mathrm{GL}_{d_i}(\mathbb{C}).$$

- Let $\chi_1, \chi_2, \ldots, \chi_t$ be the corresponding **characters**, and so

$$\chi_i(e) = d_i.$$

- Let $V_G = \oplus_n V_G(n)$ be a **weak moonshine** module for G.

Notation

- G is a finite group
- Let $\rho_1, \rho_2, \ldots, \rho_t$ be the irreducible representations

$$\rho_i \colon G \to \mathrm{GL}_{d_i}(\mathbb{C}).$$

- Let $\chi_1, \chi_2, \ldots, \chi_t$ be the corresponding **characters**, and so

$$\chi_i(e) = d_i.$$

- Let $V_G = \oplus_n V_G(n)$ be a **weak moonshine** module for G.

Goal

Extend V_G to "width s weak moonshine".

FROBENIUS r-CHARACTERS

FROBENIUS r-CHARACTERS

DEFINITION (FROBENIUS, 1896)

Let χ be a character of G, and for positive integers r we let

$$G^{(r)} := G \times \cdots \times G \quad (r \text{ copies}).$$

Frobenius r-characters

Definition (Frobenius, 1896)

Let χ be a character of G, and for positive integers r we let

$$G^{(r)} := G \times \cdots \times G \quad (r \text{ copies}).$$

The **Frobenius r-character** $\chi^{(r)} \colon G^{(r)} \to \mathbb{C}$ is defined by:

Frobenius r-characters

Definition (Frobenius, 1896)

Let χ be a character of G, and for positive integers r we let

$$G^{(r)} := G \times \cdots \times G \quad (r \text{ copies}).$$

The **Frobenius r-character** $\chi^{(r)} \colon G^{(r)} \to \mathbb{C}$ is defined by:
(1) If $r = 1$, then $\chi^{(1)}(g) := \chi(g)$.

Frobenius r-characters

Definition (Frobenius, 1896)

Let χ be a character of G, and for positive integers r we let

$$G^{(r)} := G \times \cdots \times G \quad (r \text{ copies}).$$

The **Frobenius r-character** $\chi^{(r)} \colon G^{(r)} \to \mathbb{C}$ is defined by:
(1) If $r = 1$, then $\chi^{(1)}(g) := \chi(g)$.
(2) If $r = 2$, then $\chi^{(2)}(g_1, g_2) := \chi(g_1)\chi(g_2) - \chi(g_1 g_2)$.

Frobenius r-characters

Definition (Frobenius, 1896)

Let χ be a character of G, and for positive integers r we let

$$G^{(r)} := G \times \cdots \times G \quad (r \text{ copies}).$$

The **Frobenius r-character** $\chi^{(r)} \colon G^{(r)} \to \mathbb{C}$ is defined by:

(1) If $r = 1$, then $\chi^{(1)}(g) := \chi(g)$.

(2) If $r = 2$, then $\chi^{(2)}(g_1, g_2) := \chi(g_1)\chi(g_2) - \chi(g_1 g_2)$.

(3) If $r \geq 3$, then $\chi^{(r)}(g_1, g_2, \ldots, g_r)$ is defined by

$$\chi^{(r)}(g_1, \ldots, g_r) := \chi(g_1)\,\chi^{(r-1)}(g_2, \ldots, g_r) - \chi^{(r-1)}(g_1 g_2, \ldots, g_r) - \cdots - \chi^{(r-1)}(g_2, \ldots, g_1 g_r).$$

Basic Facts about r-characters

Basic Facts about r-characters

Facts (Vanishing)

(1) If $\dim(\chi) = 1$, *then* $\chi^{(2)}(g_1, g_2) = \chi(g_1)\chi(g_2) - \chi(g_1 g_2) = 0$.

Basic Facts about r-characters

Facts (Vanishing)

(1) If $\dim(\chi) = 1$, *then* $\chi^{(2)}(g_1, g_2) = \chi(g_1)\chi(g_2) - \chi(g_1 g_2) = 0$.

(2) More generally, if $r > \dim \chi$, *then for all* $\underline{g} \in G^{(r)}$ *we have*

$$\chi^{(r)}(\underline{g}) = 0.$$

Basic Facts about r-characters

Facts (Vanishing)

(1) If $\dim(\chi) = 1$, *then* $\chi^{(2)}(g_1, g_2) = \chi(g_1)\chi(g_2) - \chi(g_1 g_2) = 0$.

(2) More generally, if $r > \dim \chi$, *then for all* $\underline{g} \in G^{(r)}$ *we have*

$$\chi^{(r)}(\underline{g}) = 0.$$

Fact (Expansion as 1-characters)

If $r \geq 2$, *then* $\chi^{(r)}(g_1, \ldots, g_r)$ *is a* **signed** *sum over* S_r *action on*

$$\chi(g_1), \chi(g_2), \ldots, \chi(g_r).$$

Deep Theorem about r-characters

Deep Theorem about r-characters

Theorem (Hoehnke and Johnson, 1992, 1998)

A finite group is determined (up to isomorphism) by its 1, 2 and 3-characters.

Deep Theorem about r-characters

Theorem (Hoehnke and Johnson, 1992, 1998)

A finite group is determined (up to isomorphism) by its 1, 2 and 3-characters.

Remarks

(1) Group theorists were surprised by this bound.

Deep Theorem about r-characters

Theorem (Hoehnke and Johnson, 1992, 1998)

A finite group is determined (up to isomorphism) by its 1, 2 and 3-characters.

Remarks

(1) Group theorists were surprised by this bound.

(2) Previous research focused on the entire "group determinant".

Deep Theorem about r-characters

Theorem (Hoehnke and Johnson, 1992, 1998)

A finite group is determined (up to isomorphism) by its 1, 2 and 3-characters.

Remarks

(1) Group theorists were surprised by this bound.

(2) Previous research focused on the entire "group determinant".

(3) Infinitely many nonisomorphic groups share 1 and 2-character tables.

Width s Weak Moonshine

Width s Weak Moonshine

Definition

G has **width $s \geq 1$ weak moonshine** if the following hold:

Width s Weak Moonshine

Definition

G has **width** $s \geq 1$ **weak moonshine** if the following hold:

1. G satisfies weak moonshine with

$$V_G := \bigoplus_{n \gg -\infty} V_G(n).$$

Width s Weak Moonshine

Definition

G has **width** $s \geq 1$ **weak moonshine** if the following hold:

1. G satisfies weak moonshine with
$$V_G := \bigoplus_{n \gg -\infty} V_G(n).$$
2. If $1 \leq r \leq s$ and $\underline{g} \in G^{(r)}$,

Width s Weak Moonshine

Definition

G has **width** $s \geq 1$ **weak moonshine** if the following hold:

1. G satisfies weak moonshine with

$$V_G := \bigoplus_{n \gg -\infty} V_G(n).$$

2. If $1 \leq r \leq s$ and $\underline{g} \in G^{(r)}$, then the *McKay-Thompson series*

$$T\left(r, \underline{g}; \tau\right) := \sum_{n \gg -\infty} \mathrm{Frob}_r\left(\underline{g}; n\right) q^n$$

Width s Weak Moonshine

Definition

G has **width** $s \geq 1$ **weak moonshine** if the following hold:

1. G satisfies weak moonshine with

$$V_G := \bigoplus_{n \gg -\infty} V_G(n).$$

2. If $1 \leq r \leq s$ and $\underline{g} \in G^{(r)}$, then the *McKay-Thompson series*

$$T\left(r, \underline{g}; \tau\right) := \sum_{n \gg -\infty} \mathrm{Frob}_r\left(\underline{g}; n\right) q^n$$

is a weakly holomorphic modular function.

Computing $\mathrm{Frob}_r(\overline{g}; n)$

Lemma

If the $m_i(n)$ are the multiplicities of ρ_i in $V_G(n)$, then

$$\mathrm{Frob}_{r}\left(\underline{g}; n\right) = \mathrm{Tr}\left(\underline{g} | V_G^{(r)}(n)\right) := \sum_{1 \leq i \leq t} m_i(n) \chi_i^{(r)}\left(\underline{g}\right).$$

WIDTH s MOONSHINE

THEOREM 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

WIDTH s MOONSHINE

THEOREM 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

COROLLARY (D-O)

If $s \geq 3$, then **complete** *width s weak moonshine* **determines finite groups up to isomorphism.**

PROOF OF THEOREM 1

Proof of Theorem 1

- Start with weak moonshine for $V_G = \oplus_n V_G(n)$.

Proof of Theorem 1

- Start with weak moonshine for $V_G = \oplus_n V_G(n)$.
- The McKay-Thompson series $T(1, g; \tau)$ are modular functions.

Proof of Theorem 1

- Start with weak moonshine for $V_G = \oplus_n V_G(n)$.
- The McKay-Thompson series $T(1, g; \tau)$ are modular functions.
- The Schur orthogonality relations assert that

$$\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = |G|\delta_{ij}.$$

Proof of Theorem 1

- Start with weak moonshine for $V_G = \oplus_n V_G(n)$.
- The McKay-Thompson series $T(1, g; \tau)$ are modular functions.
- The Schur orthogonality relations assert that

$$\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = |G|\delta_{ij}.$$

- This implies that

$$\mathcal{M}_i(\tau) := \sum_{n \gg -\infty} m_i(n) q^n = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} T(1, g; \tau).$$

Proof of Theorem 1

- Start with weak moonshine for $V_G = \oplus_n V_G(n)$.
- The McKay-Thompson series $T(1, g; \tau)$ are modular functions.
- The Schur orthogonality relations assert that

$$\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = |G|\delta_{ij}.$$

- This implies that

$$\mathcal{M}_i(\tau) := \sum_{n \gg -\infty} m_i(n) q^n = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} T(1, g; \tau).$$

- Each $\mathcal{M}_i(\tau)$ is a weakly holomorphic modular function.

Proof of Theorem 1 continued

- For each $r \geq 2$, we have that

$$T\left(r, \underline{g}; \tau\right) = \sum_{n \gg -\infty} \mathrm{Frob}_r\left(\underline{g}; n\right) q^n = \sum_{n \gg -\infty} \sum_{1 \leq i \leq t} m_i(n) \chi_i^{(r)}\left(\underline{g}\right) q^n.$$

PROOF OF THEOREM 1 CONTINUED

- For each $r \geq 2$, we have that

$$T\left(r, \underline{g}; \tau\right) = \sum_{n \gg -\infty} \mathrm{Frob}_r\left(\underline{g}; n\right) q^n = \sum_{n \gg -\infty} \sum_{1 \leq i \leq t} m_i(n) \chi_i^{(r)}\left(\underline{g}\right) q^n.$$

- But then we have

$$T\left(r, \underline{g}; \tau\right) = \sum_{1 \leq i \leq t} \chi_i^{(r)}\left(\underline{g}\right) \mathcal{M}_i(\tau).$$

Proof of Theorem 1 continued

- For each $r \geq 2$, we have that

$$T\left(r, \underline{g}; \tau\right) = \sum_{n \gg -\infty} \mathrm{Frob}_r\left(\underline{g}; n\right) q^n = \sum_{n \gg -\infty} \sum_{1 \leq i \leq t} m_i(n) \chi_i^{(r)}\left(\underline{g}\right) q^n.$$

- But then we have

$$T\left(r, \underline{g}; \tau\right) = \sum_{1 \leq i \leq t} \chi_i^{(r)}\left(\underline{g}\right) \mathcal{M}_i(\tau).$$

- The higher dimensional McKay-Thompson series are modular functions because the $\mathcal{M}_i(\tau)$ are. □

Higher Dimensional MT Series

Question

What information do the higher dimensional MT series

$$\left\{ T(r, \underline{g}; \tau) \ : \ \underline{g} \in G^{(r)} \right\}$$

encode about structure of the "seed" module V_G?

Higher Dimensional MT Series

Question

What information do the higher dimensional MT series

$$\left\{T(r,\underline{g};\tau) \;:\; \underline{g}\in G^{(r)}\right\}$$

encode about structure of the "seed" module V_G?

Answer

The r-dimensional MT series know the part of V_G assembled from the characters with $\dim\chi_i \geq r$.

Higher Dimensional MT Series

Theorem 2 (D-O)

If width s weak moonshine holds for G, $1 \le r \le s$ and $\dim\chi_i \ge r$, then the χ_i multiplicity generating function satisfies

$$\mathcal{M}_i(\tau) := \sum_{n \gg -\infty} m_i(n) q^n$$

$$= \frac{(\dim\chi_i)^{r-1}}{r!|G|^r (\dim\chi_i - 1)\cdots(\dim\chi_i - (r-1))} \sum_{\underline{g} \in G^{(r)}} \overline{\chi_i^{(r)}(\underline{g})} T(r, \underline{g}; \tau).$$

Higher Dimensional MT Series

Theorem 2 (D-O)

If width s weak moonshine holds for G, $1 \leq r \leq s$ and $\dim\chi_i \geq r$, then the χ_i multiplicity generating function satisfies

$$\mathcal{M}_i(\tau) := \sum_{n \gg -\infty} m_i(n) q^n$$

$$= \frac{(\dim\chi_i)^{r-1}}{r!|G|^r (\dim\chi_i - 1) \cdots (\dim\chi_i - (r-1))} \sum_{\underline{g} \in G^{(r)}} \overline{\chi_i^{(r)}(\underline{g})} T(r, \underline{g}; \tau).$$

Remark

Theorem 2 follows from new orthogonality relations for Frobenius r-characters.

Orthogonality of Frobenius r-characters

Orthogonality of Frobenius r-characters

Theorem (Frobenius, Johnson, D-O)

We have that

$$\sum_{\underline{g}\in G^{(r)}} \chi_i^{(r)}(\underline{g})\,\overline{\chi_j^{(r)}(\underline{g})}$$

$$= \frac{r!|G|^r\delta_{ij}}{(\dim\chi_i)^{r-1}}(\dim\chi_i - 1)\cdots(\dim\chi_i-(r-1)).$$

Orthogonality of Frobenius r-characters

Theorem (Frobenius, Johnson, D-O)

We have that

$$\sum_{\underline{g}\in G^{(r)}} \chi_i^{(r)}(\underline{g})\,\overline{\chi_j^{(r)}(\underline{g})} = \frac{r!|G|^r\delta_{ij}}{(\dim\chi_i)^{r-1}}(\dim\chi_i - 1)\cdots(\dim\chi_i - (r-1)).$$

Remarks

(1) The $r = 1$ case is due to Schur.

Orthogonality of Frobenius r-characters

Theorem (Frobenius, Johnson, D-O)

We have that

$$\sum_{\underline{g}\in G^{(r)}} \chi_i^{(r)}(\underline{g})\,\overline{\chi_j^{(r)}(\underline{g})} = \frac{r!|G|^r\delta_{ij}}{(\dim\chi_i)^{r-1}}(\dim\chi_i - 1)\cdots(\dim\chi_i-(r-1)).$$

Remarks

(1) The $r=1$ case is due to Schur.
(2) The $i\neq j$ case is due to Frobenius and Johnson.

Orthogonality of Frobenius r-characters

Theorem (Frobenius, Johnson, D-O)

We have that

$$\sum_{\underline{g}\in G^{(r)}} \chi_i^{(r)}(\underline{g})\,\overline{\chi_j^{(r)}(\underline{g})}$$

$$= \frac{r!|G|^r\delta_{ij}}{(\dim\chi_i)^{r-1}}(\dim\chi_i - 1)\cdots(\dim\chi_i - (r-1)).$$

Remarks

(1) The $r = 1$ case is due to Schur.
(2) The $i \neq j$ case is due to Frobenius and Johnson.
(3) Our contribution is the $i = j$ case which gives the "norms" of Frobenius r-characters.

Schur's Lemma

Lemma (Schur's Lemma)

Let G be a finite group, and let ρ_V and ρ_W be irreducible repns

$$\rho_V : G \to \mathrm{GL}(V),$$
$$\rho_W : G \to \mathrm{GL}(W).$$

If $f : V \to W$ is a G-linear map, then f is a scalar multiple of the identity map if $V \cong W$ and $f = 0$ if $V \not\cong W$.

Key consequences

Corollary

If $h_1, h_2 \in G$, then the following are true:

1. *We have that*

$$\sum_{g \in G} \chi_i \left(g h_1 g^{-1} h_2^{-1} \right) = \frac{\chi_i \left(h_1 \right) \overline{\chi_i \left(h_2 \right)} |G|}{\dim \chi_i}.$$

Key consequences

Corollary

If $h_1, h_2 \in G$, then the following are true:

1. *We have that*

$$\sum_{g \in G} \chi_i \left(g h_1 g^{-1} h_2^{-1}\right) = \frac{\chi_i\left(h_1\right) \overline{\chi_i\left(h_2\right)} |G|}{\dim \chi_i}.$$

2. *If χ_j is an irreducible character of G, then we have that*

$$\sum_{g \in G} \chi_i\left(h_1 g\right) \overline{\chi_j\left(g h_2\right)} = \frac{\chi_i\left(h_1 h_2^{-1}\right) |G| \delta_{ij}}{\dim \chi_i}.$$

Proof of the r-character orthogonality

Proof of the r-character orthogonality

- Let $n(\sigma)$ denote the number of disjoint cycles of $\sigma \in S_r$.

Proof of the r-character orthogonality

- Let $n(\sigma)$ denote the number of disjoint cycles of $\sigma \in S_r$.
- As disjoint cycles, we let

$$\sigma = (a_1^\sigma(1), \ldots, a_1^\sigma(k_1^\sigma))(a_2^\sigma(1), \ldots, a_2^\sigma(k_2^\sigma)) \cdots \left(a_{n(\sigma)}^\sigma(1), \ldots, a_{n(\sigma)}^\sigma\left(k_{n(\sigma)}^\sigma\right)\right).$$

PROOF OF THE r-CHARACTER ORTHOGONALITY

- Let $n(\sigma)$ denote the number of disjoint cycles of $\sigma \in S_r$.
- As disjoint cycles, we let

$$\sigma = (a_1^\sigma(1), \ldots, a_1^\sigma(k_1^\sigma))(a_2^\sigma(1), \ldots, a_2^\sigma(k_2^\sigma)) \cdots (a_{n(\sigma)}^\sigma(1), \ldots, a_{n(\sigma)}^\sigma(k_{n(\sigma)}^\sigma)).$$

- The cycles have order $k_1^\sigma, k_2^\sigma, \ldots, k_{n(\sigma)}^\sigma$ and as sets

$$\{1, 2, \ldots, r\} = \{a_1^\sigma(1), \ldots, a_1^\sigma(k_1^\sigma), a_2^\sigma(1), \ldots, a_2^\sigma(k_2^\sigma), \ldots, a_{n(\sigma)}^\sigma(1), \ldots, a_{n(\sigma)}^\sigma(k_{n(\sigma)}^\sigma)\}.$$

Proof of orthogonality continued

- We abuse notation and use a for g_a, and note that

$$\chi^{(r)}(g_1,\ldots,g_r)=\sum_{\sigma\in S_r}\operatorname{sgn}(\sigma)\chi\left(a_1^\sigma(1)\cdots a_1^\sigma\left(k_1^\sigma\right)\right)\cdots\chi\left(a_{n(\sigma)}^\sigma(1)\cdots a_{n(\sigma)}^\sigma\left(k_{n(\sigma)}^\sigma\right)\right).$$

Proof of orthogonality continued

- We abuse notation and use a for g_a, and note that

$$\chi^{(r)}(g_1, \ldots, g_r) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \chi\left(a_1^{\sigma}(1) \cdots a_1^{\sigma}(k_1^{\sigma})\right) \cdots \chi\left(a_{n(\sigma)}^{\sigma}(1) \cdots a_{n(\sigma)}^{\sigma}\left(k_{n(\sigma)}^{\sigma}\right)\right).$$

- We must evaluate

$$\Omega := \sum_{\underline{g} \in G^{(r)}} \chi^{(r)}(\underline{g}) \overline{\chi^{(r)}(\underline{g})} = \sum_{\underline{g} = (g_1, \ldots, g_r) \in G^{(r)}} \chi^{(r)}(g_1, \ldots, g_r) \overline{\chi^{(r)}(g_1, \ldots, g_r)}$$

$$= \sum_{\sigma, \tau \in S_r} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \sum_{\underline{g} \in G^{(r)}} \chi\left(a_1^{\sigma}(1) \cdots a_1^{\sigma}(k_1^{\sigma})\right) \cdots \chi\left(a_{n(\sigma)}^{\sigma}(1) \cdots a_{n(\sigma)}^{\sigma}\left(k_{n(\sigma)}^{\sigma}\right)\right)$$

$$\times \overline{\chi\left(a_1^{\tau}(1) \cdots a_1^{\tau}(k_1^{\tau})\right)} \cdots \overline{\chi\left(a_{n(\tau)}^{\tau}(1) \cdots a_{n(\tau)}^{\tau}\left(k_{n(\tau)}^{\tau}\right)\right)}.$$

Proof of orthogonality continued

- By reordering we can rewrite as:

$$\Omega = \sum_{\sigma,\tau\in S_r} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \sum_{g_1,\dots,g_{r-1}\in G} \Big[\chi\left(a_1^\sigma(1)\cdots a_1^\sigma\left(k_1^\sigma\right)\right)\cdots\chi\left(a_{n(\sigma)-1}^\sigma(1)\cdots a_{n(\sigma)-1}^\sigma\left(k_{n(\sigma)-1}^\sigma\right)\right)$$

$$\times \overline{\chi\left(a_1^\tau(1)\cdots a_1^\tau\left(k_1^\tau\right)\right)}\cdots\overline{\chi\left(a_{n(\tau)-1}^\tau(1)\cdots a_{n(\tau)-1}^\tau\left(k_{n(\tau)-1}^\tau\right)\right)}\Big]$$

$$\times \sum_{g_r\in G}\chi\left(a_{n(\sigma)}^\sigma(1)\cdots a_{n(\sigma)}^\sigma\left(k_{n(\sigma)}^\sigma\right)\right)\overline{\chi\left(a_{n(\tau)}^\tau(1)\cdots a_{n(\tau)}^\tau\left(k_{n(\tau)}^\tau\right)\right)}.$$

Proof of orthogonality continued

- By reordering we can rewrite as:

$$\Omega = \sum_{\sigma,\tau\in S_r} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \sum_{g_1,\ldots,g_{r-1}\in G} \Big[\chi\left(a_1^\sigma(1)\cdots a_1^\sigma\left(k_1^\sigma\right)\right)\cdots\chi\left(a_{n(\sigma)-1}^\sigma(1)\cdots a_{n(\sigma)-1}^\sigma\left(k_{n(\sigma)-1}^\sigma\right)\right)$$
$$\times\overline{\chi\left(a_1^\tau(1)\cdots a_1^\tau\left(k_1^\tau\right)\right)}\cdots\overline{\chi\left(a_{n(\tau)-1}^\tau(1)\cdots a_{n(\tau)-1}^\tau\left(k_{n(\tau)-1}^\tau\right)\right)}\Big]$$
$$\times\sum_{g_r\in G}\chi\left(a_{n(\sigma)}^\sigma(1)\cdots a_{n(\sigma)}^\sigma\left(k_{n(\sigma)}^\sigma\right)\right)\overline{\chi\left(a_{n(\tau)}^\tau(1)\cdots a_{n(\tau)}^\tau\left(k_{n(\tau)}^\tau\right)\right)}.$$

- Apply the previous corollary to Schur's Lemma to the red sum **and repeat.**

Proof of orthogonality continued

- By reordering we can rewrite as:

$$\Omega = \sum_{\sigma,\tau\in S_r} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \sum_{g_1,\dots,g_{r-1}\in G} \Big[\chi\left(a_1^\sigma(1)\cdots a_1^\sigma\left(k_1^\sigma\right)\right)\cdots\chi\left(a_{n(\sigma)-1}^\sigma(1)\cdots a_{n(\sigma)-1}^\sigma\left(k_{n(\sigma)-1}^\sigma\right)\right)$$

$$\times\, \overline{\chi\left(a_1^\tau(1)\cdots a_1^\tau\left(k_1^\tau\right)\right)}\cdots\overline{\chi\left(a_{n(\tau)-1}^\tau(1)\cdots a_{n(\tau)-1}^\tau\left(k_{n(\tau)-1}^\tau\right)\right)}\Big]$$

$$\times \sum_{g_r\in G}\chi\left(a_{n(\sigma)}^\sigma(1)\cdots a_{n(\sigma)}^\sigma\left(k_{n(\sigma)}^\sigma\right)\right)\overline{\chi\left(a_{n(\tau)}^\tau(1)\cdots a_{n(\tau)}^\tau\left(k_{n(\tau)}^\tau\right)\right)}.$$

- Apply the previous corollary to Schur's Lemma to the red sum **and repeat.**
- Keep careful track of the steps. □

Example of D_4 and Q_8 revisited

Example of D_4 and Q_8 revisited

- The MT series for $(r^3 s, rs) \in D_4^{(2)}$ and $(-k, k) \in Q_8^{(2)}$ are

$$T\left(2, \left(r^3 s, rs\right); \tau\right) = \sum_{1 \le i \le 5} \chi_i^{(2)}\left(r^3 s, rs\right) \mathcal{M}_i(\tau) = \chi_5^{(2)}\left(r^3 s, rs\right) \mathcal{M}_5(\tau)$$
$$= 98304q + 10747904q^2 + 432144384q^3 + O\left(q^4\right).$$

$$T\left(2, (-k, k); \tau\right) = \sum_{1 \le i \le 5} \chi_i^{(2)}(-k, k) \mathcal{M}_i(\tau) = \chi_5^{(2)}(-k, k) \mathcal{M}_5(\tau)$$
$$= -98304q - 10747904q^2 - 432144384q^3 + O\left(q^4\right).$$

Example of D_4 and Q_8 revisited

- The MT series for $\left(r^3s, rs\right) \in D_4^{(2)}$ and $(-k,k) \in Q_8^{(2)}$ are

$$T\left(2, \left(r^3s, rs\right); \tau\right) = \sum_{1\le i\le 5} \chi_i^{(2)}\left(r^3s, rs\right)\mathcal{M}_i(\tau) = \chi_5^{(2)}\left(r^3s, rs\right)\mathcal{M}_5(\tau)$$
$$= 98304q + 10747904q^2 + 432144384q^3 + O\left(q^4\right).$$

$$T\left(2, (-k,k); \tau\right) = \sum_{1\le i\le 5} \chi_i^{(2)}(-k,k)\mathcal{M}_i(\tau) = \chi_5^{(2)}(-k,k)\mathcal{M}_5(\tau)$$
$$= -98304q - 10747904q^2 - 432144384q^3 + O\left(q^4\right).$$

- That they are unequal distinguishes D_4 from Q_8.

Summary

Theorem 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine.*

Summary

Theorem 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

Summary

Theorem 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

Corollary (D-O)

For $s \geq 3$, width s weak moonshine **determines groups.**

Summary

Theorem 1 (D-O)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

Corollary (D-O)

For $s \geq 3$, width s weak moonshine **determines groups.**

Theorem 2 (D-O)

If $\dim \chi_i \geq r$, then the multiplicity generating functions satisfy

$$\mathcal{M}_i(\tau) := \sum_{n \gg -\infty} m_i(n) q^n =_* \sum_{\underline{g} \in G^{(r)}} \overline{\chi_i^{(r)}(\underline{g})} T(r, \underline{g}; \tau).$$