Methods of Tanaka theory in the local geometry of k-nondegenerate CR structures of hypersurface type

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Very brief review of standard Tanaka theory for filtered structures

Let D be a bracket generating distribution on a manifold M.

Setting $D^1 := D$ and $D^{j+1} = D^j + [D, D^j]$, one gets the filtration

$$D(q) = D^1(q) \subset D^2(q) \subset \cdots \subset D^j(q) \subset \cdots$$

of the tangent space T_qM.

The associated grading:

$$\mathfrak{g}_-(q) = \underbrace{D^1(q)}_{\mathfrak{g}_{-1}(q)} \oplus \underbrace{D^2(q)/D^1(q)}_{\mathfrak{g}_{-2}(q)} \oplus \ldots \oplus \underbrace{D^\mu(q)/D^{\mu-1}(q)}_{\mathfrak{g}_{-\mu}(q)}$$

is endowed with the structure of a $\mathbb{Z}_-$-graded nilpotent Lie algebra, called the Tanaka symbol of the distribution at q.

Fix a $\mathbb{Z}_-$-graded nilpotent Lie algebra $\mathfrak{g}_- = \bigoplus_{i=-\mu}^{-1} \mathfrak{g}_i$, generated by $\mathfrak{g}_{-1}$ and assume that

$$\mathfrak{g}_-(q) \sim \mathfrak{g}_-, \quad \forall q.$$

One says that D is of (constant) type $\mathfrak{g}_-$.

To a distribution of type $\mathfrak{g}_-$ one can assign a principal $\mathrm{Aut}(\mathfrak{g}_-)$-bundle $P^0(\mathfrak{g}_-)$ over M whose fiber over q consists of all graded Lie algebra isomorphisms from $\mathfrak{g}_-$ to $\mathfrak{g}_-(q)$,

$$P^0(\mathfrak{g}_-) = \{(q, \varphi) : \varphi \in \mathrm{Iso}\big(\mathfrak{g}_-, \mathfrak{g}_-(q)\big)\}.$$

Additional structures on D can be encoded in the choice of a subgroup $G_0 \subset \mathrm{Aut}(\mathfrak{g}_-)$ with Lie algebra $\mathfrak{g}_0$, leading to a G_0-reduction P^0 of the bundle $P^0(\mathfrak{g}_-)$.

The bundle P^0 is called the Tanaka structure of type $\mathfrak{g}_- \oplus \mathfrak{g}_0$.

Universal algebraic prolongation

The **universal prolongation of the symbol** $\mathfrak{g}_- \oplus \mathfrak{g}_0$, is the maximal non-degenerate $\mathbb{Z}$-graded Lie algebra containing $\mathfrak{g}_- \oplus \mathfrak{g}_0$ as its non-positive part.

Notation: $\mathfrak{u}(\mathfrak{g}_- \oplus \mathfrak{g}_0) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$

Non-degenericity means that $\mathrm{ad}\, x|_{\mathfrak{g}_-} \neq 0$ for any non-zero $x \in \mathfrak{g}^i(\mathfrak{m})$ with $i \geq 0$.

The components $\mathfrak{g}_k$ with $k > 0$ can be constructed recursively, e.g.:

$$\mathfrak{g}_k = \left\{ \begin{array}{l} f \in \bigoplus_{i<0} \mathrm{Hom}(\mathfrak{g}_i, \mathfrak{g}_{i+k}) : \\ f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)], \forall v_1, v_2 \in \mathfrak{m} \end{array} \right\}$$

The latter recursive formula can be use to define the algebraic prolongation if $\mathfrak{g}_0$ is just a subspace of $\mathfrak{der}(\mathfrak{g}_-)$ (not a Lie subalgebra) or in the case of $\mathfrak{g}_- = \mathfrak{g}_{-1}$, if it is a subspace of $\mathrm{Hom}(\mathfrak{g}_{-1}, W)$ for a vector space W.

Tanaka's Main Theorem on prolongation

Theorem (Tanaka, 1970)

Assume that $\dim \mathfrak{u}(\mathfrak{g}_- \oplus \mathfrak{g}_0) < \infty$.

1. *To any structure of type* $\mathfrak{g}_- \oplus \mathfrak{g}_0$ *one can assign the canonical frame on a bundle over* M *of dimension equal to* $\dim \mathfrak{u}(\mathfrak{g}_- \oplus \mathfrak{g}_0)$;
2. *Dimension of algebra of infinitesimal symmetries of a structure of type* $\mathfrak{g}_- \oplus \mathfrak{g}_0$ *is not greater than* $\dim \mathfrak{u}(\mathfrak{g}_- \oplus \mathfrak{g}_0)$;
3. *This upper bound is sharp. Moreover, if* $\mathfrak{g}_0$ *contains the grading element, then there is a unique structure of type* $\mathfrak{g}_- \oplus \mathfrak{g}_0$, *up to a local equivalence, for which this bound is achieved and this structure is locally equivalent to a natural left invariant structure on the simply connected Lie group with the Lie algebra* $\mathfrak{g}_-$.

Tanaka's main Theorem of prolongation: continued

In more detail, to a Tanaka structure of type $\mathfrak{g}_- \oplus \mathfrak{g}_0$ one can assign in a canonical way (choosing a normalization condition on each step) a sequence of bundles

$$M \leftarrow P^0 \leftarrow P^1 \leftarrow P^2 \leftarrow \cdots \leftarrow P^k,$$

where $k = \#$of negatively graded components $+$ $\#$of nonnegatively graded components in $\mathfrak{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0)$ such that

1. P^0 is the principal bundle over M with the structure group having Lie algebra $\mathfrak{g}^0$;
2. P^i is the affine bundle over P^{i-1} with fibers being affine spaces over the linear space $\mathfrak{g}_i$ for any $i = 1, \dots k$;
3. P^k is endowed with the canonical frame.

CR-structures of hypersurface type

An (abstract) CR structure of hypersurface type is a triple (M, D, J), where M is an odd dimensional (real) manifold, D is a corank 1 distribution on M, and

$J : D(q) \to D(q)$, $J^2 = -\mathrm{Id}$.

so that if $H \subset \mathbb{C}D$ is the $\mathbf{i}$-eigenspace of J then H is involutive, $[H, H] \subset H$.

Such structures naturally appear on any real hypersurface M in $\mathbb{C}^N$: $D = TM \cap \mathbf{i}TM$, J is induced by the multiplication by $\mathbf{i}$.

Levi kernel of CR structure

Let $\mathfrak{L}$ be the Levi form on H, $\mathfrak{L}(X,Y) = \mathbf{i}[X,\overline{Y}] \mod \mathbb{C}D$.

Levi kernel $K = \ker\mathfrak{L}$.

Levi nondegenerate case, i.e. when $K = 0$, is very well understood (Cartan, Tanaka, Chern-Moser).
From the point of view of Tanaka theory: assume that $\mathfrak{L}$ is of signature (p,q)

1. D is contact $\Rightarrow \mathfrak{g}_-$ is $2(p+q)+1$-dimensional Heisenberg algebra $\Rightarrow \mathrm{Aut}(\mathfrak{g}_-) = \mathrm{CSP}(2(p+q))$;
2. Reduction had to preserve the Levi form $\Rightarrow \mathfrak{g}_0 = \mathfrak{cu}(p,q)$;
3. $\mathfrak{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0) = \bigoplus_{i=-2}^{2} \mathfrak{g}_i \cong \mathfrak{su}(p+1,q+1)$-parabolic geometry.

Freeman Sequence. Levi k-nondegenerate structures

Uniformly Levi degenerate case is when K is non-trivial at every point. We also assume that K is a subdistribution of H.

Set $K_{-1} := H$, $K_0 := K$,
$K_1 := \{[v \in K_0 : [v, \overline{K}_{-1}] \subset K_0 \oplus \overline{K}_{-1}]\}$,and by induction
$K_j := \{[v \in K_{j-1} : [v, \overline{K}_{-1}] \subset K_{j-1} \oplus \overline{K}_{-1}]\}$

We assume that K_j is a subdistibution of K_{j-1}.
$K_{-1} \supset K_0 \supset K_1 \supset \ldots$-*the Freeman sequence*

Definition (uniform finite nondegeneracy)

The structure H is *uniformly finitely nondegenerate* if the jth step of the iterative construction above yields a regular vector distribution K_j for all j, and if there exists an integer l such that $K_l = 0$. Such structures are also called *uniformly k-nondegenerate*, where $k = \min\{j \mid K_{j-1} = 0\}$.

Modification of Tanaka theory for k-nondegenerate CR structures of hypersurface type

Ansatz:

- *To preserve k-nondegeneracy on the graded level it is inevitable to assign weight j to elements of K_j/K_{j+1}, $0 \le j \le k-2$.*
- *The CR structure induces the natural bigrading of the contact algebra so that the K_j/K_{j+1}, $0 \le j \le k-2$ are naturally related to certain subspaces of the $(j, \pm(j+2))$-components of this bigrading, so that the analog of the Tanaka universal prolongation is aligned with those subspaces, in particular $(k-1, k+1)$-bigraded component mustr vanish.*

All constructions below are done at given point $p \in M$.

The natural filtration on $\mathbb{C}TM$: $K_{k-2} \oplus \overline{K_{k-2}} \subset \dots K_0 \oplus \overline{K_0} \subset \mathbb{C}D \subset \mathbb{C}TM$

Associated grading:

$$\bigoplus_{j=0}^{k-2} \underbrace{(K_j/K_{j+1} \oplus \overline{K}_j/\overline{K}_{j+1})}_{\text{weight } j} \oplus \underbrace{\mathbb{C}D/(K_0 \oplus \overline{K_0})}_{\mathfrak{g}_{-1}(=H/K \oplus \overline{H}/\overline{K_0})} \oplus \underbrace{\mathbb{C}TM/\mathbb{C}D}_{\mathfrak{g}_{-2}},$$

where $\mathfrak{g}_- := \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ is the Heisenberg algebra.

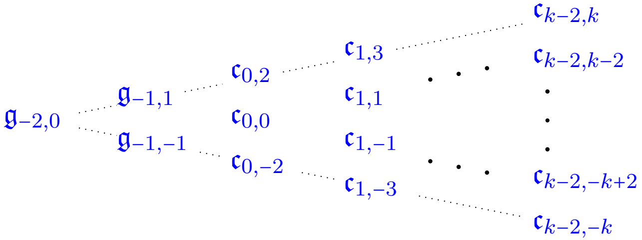
Natural bigrading on $\mathfrak{g}_-$:

$\mathfrak{g}_{-1,-1} := \overline{K}_{-1}/\overline{K}_0, \quad \mathfrak{g}_{-1,1} := K_{-1}/K_0, \quad \mathfrak{g}_{-2,0} := g_{-2}$ (here $K_{-1} = H$);

The bigraded contact algebra

Let $\mathfrak{c} = \mathfrak{g}_- + \bigoplus_{j=0}^{\infty} \mathfrak{c}_j$ be the universal prolongation of $\mathfrak{g}_-$.
$\mathfrak{c}$ is called the *contact algebra*;
For example, $\mathfrak{c}_0 = \mathfrak{csp}(\mathfrak{g}_{-1})$-the *conformal symplectic Lie algebra*.
The bigrading on $\mathfrak{g}_-$ induces the bigrading on $\mathfrak{c}$:

$$\mathfrak{c} = \mathfrak{g}_- \oplus \bigoplus_{j \geq 0} \bigoplus_{l=0}^{j+2} \mathfrak{c}_{j,j+2-2l}(p)$$



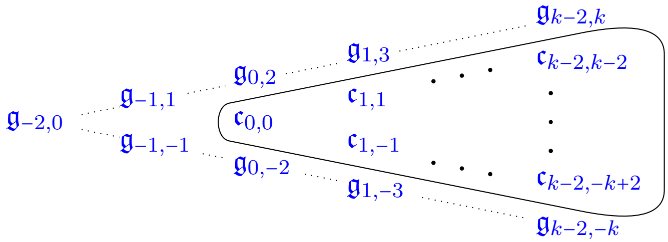
$$\overline{\mathfrak{c}_{j,k}} = c_{j,-k}.$$

CR symbol

$\forall v \in (K_j/K_{j+1})_p$ the map $\mathrm{ad}_v : (\overline{K}_{-1}/\overline{K}_0)_p \to (\overline{K}_{j-1}/\overline{K}_j)_p$ is well defined and identifies the space $(K_j/K_{j+1})_p$ with a subspace $\mathfrak{g}_{j,j+2}(p) \subset \mathfrak{c}_{j,j+2}(p)$ and $\mathfrak{g}_{j,-(j+2)}(p) := \overline{\mathfrak{g}_{j,-(j+2)}(p)} \subset \mathfrak{c}_{j,-(j+2)}(p)$.
A subspace $\mathfrak{g}(p)$ in the contact algebra $\mathfrak{c}(p)$,

$$\mathfrak{g}(p) = \mathfrak{g}_-(p) \oplus \bigoplus_{0\le j\le k-2} (\mathfrak{g}_{j,j+2}(p) \oplus \mathfrak{g}_{j,-j-2}(p))$$

is called the *CR symbol at the point* p of the k-nondegenerate hypersurface-type CR structure $\mathcal{H} = K_{-1}$.



Assumption: *CR symbols at different points are isomorphic.*

Nested Levi foliations

The distributions $(K_j \oplus \overline{K_j}) \cap TM$ are involutive for all $j \geq 0$, and thus generate nested foliations $F_0, F_1, \ldots, F_{k-2}$ of M.

By the Frobenius Theorem, locally the leaf spaces $N_0 := M/F_0, \ldots, N_{k-2} := M/F_{k-2}$ are smooth manifolds and the quotient maps define the nested sequence of fiber bundles:

$$N_{k-1} := M \xrightarrow{q_{k-2}} N_{k-2} \xrightarrow{q_{k-3}} \cdots \xrightarrow{q_1} N_1 \xrightarrow{q_0} N_0.$$

Symbol Families

Let $\kappa_j := q_j \circ \cdots \circ q_{k-2}$ and let $L_j(p_0)$ be the leaf of the foliation F_j passing through $p_0 \in M$. H/K_l for $0 \le l < j$ descends through κ_j to a well-defined bundle on N_j.
In particular, H/K_0 descends through κ_1 to a well-defined bundle on N_1 $\Rightarrow$ For any $p \in L_1(p_0)$ there is a canonical identification –through the map $(\kappa_j)_*$ – of the symbol $\mathfrak{g}(p)$ with a subspace $\rho_{p_0}(p)$ of $\mathfrak{c}(p_0)$,

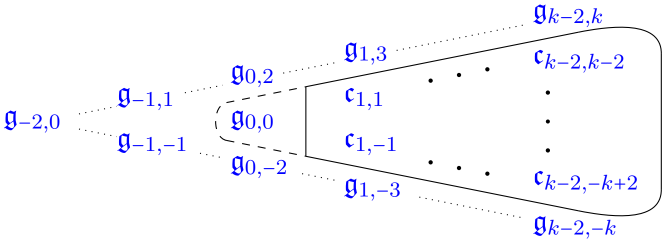
$$\rho_{p_0}(p) = \mathfrak{g}_-(p) \oplus \bigoplus_{0 \le j \le k-2} \left(\rho_{p_0}(p)_{j,j+2} \oplus \rho_{p_0}(p)_{j,-j-2}\right),$$

where $\rho_{p_0}(p)_{j,\pm(j+2)}(p) \subset \mathfrak{c}_{j,\pm(j+2)}, \ \forall 0 \le j \le k-2.$
The family $\mathscr{S}_{p_0} := \{\rho_{p_0}(p) \,|\, p \in L_1(p_0)\}$ is called the *CR symbol family of the k-nondegenerate CR structure H at p_0*.
For $k = 2$, $L_1(p_0)$ is a point , so the symbol family is just one symbol, $\mathscr{S}_{p_0} = \mathfrak{g}(p_0)$.

(0,0)-component to form the CR symbol

Assumption: Symbol families at different points are isomorphic (to a given (modeled) symbol family $\mathscr{S}$). ⇒ (The image of) each symbol family is locally homogeneous (in the corresponding Grassmannian)

Let $\mathfrak{g}_{0,0}(p_0)$ be the Lie algebra of infinitesimal symmetries of (the image of) the symbol family $\mathscr{S}_{p_0}$.
E.g. if the family $\mathscr{S}_{p_0}$ consists of one point $\mathscr{S}_{p_0} = \mathfrak{g}(p_0)$, then
$\mathfrak{g}_{0,0}(p_0) = \{x \in \mathfrak{c}_{0,0}(p_0) : [x, \mathfrak{g}_{j,\pm(j+2)}(p_0)] \subset \mathfrak{g}_{j,\pm(j+2)}(p_0), 0 \le j \le k-2\}$.



The step 0 bundle associated with CR structure

For p's from the same leaf of F_1, the contact algebras $\mathfrak{c}(p)$ and the symbol families $\mathscr{S}_p$ are canonically identified– through the pushforward of the canonical projection $\kappa_1 : M \to N_1$. Therefore, for every $p_1 \in N_1$ one can speak about the contact algebra $c(p_1)$ and the symbol family $\mathscr{S}_{p_1}$.

Let $(M, \mathcal{H})$ be, as before, a k-nondegenerate CR manifold of hypersurface with a constant symbol family model by $\mathscr{S}$. Define P^0 as the bundle $b_0 : P^0 \to N_1$ whose fiber $P^0(p_1) := b_0^{-1}(p)$ over a point $p_1 \in N_1$ satisfies

$$P^0(p_1) := \left\{ \psi : \mathfrak{g}_- \to \mathfrak{g}_-(p_1) \;\middle|\; \begin{array}{l} \psi \text{ is a bi-graded Lie algebra isomor-} \\ \text{phism inducing an isomorphism be-} \\ \text{tween the families } \mathscr{S} \text{ and } \mathscr{S}(p_1) \end{array} \right\}$$

Modified CR symbols

$$P^0 \xrightarrow{b_0} N_1 \xrightarrow{q_0} N_0, \quad \mathrm{pr} := q_0 \circ b_0.$$

Fix $\psi_0 \in P^0$. Let $\psi : (-\varepsilon, \varepsilon) \to P^0_{\mathrm{pr}(\psi_0)}$ be a curve in $P^0_{\mathrm{pr}(\psi_0)} := \mathrm{pr}^{-1}\big(\mathrm{pr}(\psi_0)\big)$ with $\psi(0) = \psi_0$,
and define $\theta_0 : T_{\psi_0} P^0_{\mathrm{pr}(\psi_0)} \to \mathfrak{c}_0 \big(= \mathfrak{csp}(\mathfrak{g}_{-1})\big)$ by

$$\theta_0\big(\psi'(0)\big) := \big((q_0)_* \circ \psi_0\big)^{-1} \circ \big((q_0)_* \circ \psi\big)'(0).$$

Let $\mathfrak{g}_0^{\mathrm{mod}}(\psi_0) := \theta_0 \left(T_{\psi_0} P^0_{\mathrm{pr}(\psi_0)}\right) \subset \mathfrak{c}_0$.
Given $p \in M$ such that $q_1 \circ \cdots \circ q_{k-2}(p) = b_0(\psi_0)$, the *modified CR symbol* of the structure H attached to the pair $\Pi = (p, \psi_0)$ is

$$\mathfrak{g}^{\mathrm{mod}}(p, \psi_0) := \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{mod}}(\psi_0) \oplus \bigoplus_{1 \le j \le k-2} \underbrace{\big(\mathfrak{g}_{j,j+2}(p) \oplus \mathfrak{g}_{j,-j-2}(p)\big)}_{(\mathfrak{c}_{j,j+2} \oplus \mathfrak{c}_{j,-j-2}) \text{ via } \psi_0}.$$

Universal CR prolongation of the modified symbol of the k-nondegenerate CR structure

Let $\pi_{i,j} : \mathfrak{c} \to \mathfrak{c}_{i,j}$ be the canonical projection. Fix $\Pi = (p, \psi_0)$ such that $q_{k-2} \circ \cdots \circ q_1(p) = b_0(\psi_0)$, i.e., $\Pi \in M \times_{N_1} P^0$.
Define $\mathfrak{g}_\ell^{\mathrm{mod}}(\Pi)$ with $\ell > 0$ recursively as follows:

$$\mathfrak{g}_\ell^{\mathrm{mod}}(\Pi) = \left\{ \begin{array}{l} f \in \bigoplus_{i<0} \mathrm{Hom}(\mathfrak{g}_i^{\mathrm{mod}}(\Pi), \mathfrak{g}_{i+\ell}^{\mathrm{mod}}(\Pi)) : \\ f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)], \forall v_1, v_2 \in \mathfrak{g}_-, \\ \pi_{\ell,\pm(\ell+2)}(f) \in \mathfrak{g}_{\ell,\pm(\ell+2)}(p), \ \pi_{\ell,\pm(\ell+2)}([f, \mathfrak{g}_0^{\mathrm{mod}}]) \in \mathfrak{g}_{\ell,\pm(\ell+2)}(p) \end{array} \right\}$$

In particular, since $\mathfrak{g}_{k-1,k+1}(p) = 0$, we require that
$\pi_{k-1,\pm(k+1)}(f) = 0, \ \pi_{k-1,\pm(k+1)}([f, \mathfrak{g}_0^{\mathrm{mod}}]) = 0 \quad \forall f \in \mathfrak{g}_{k-1}^{\mathrm{mod}}(\Pi)$.
The k-nondegenerate universal prolongation of the modified CR symbol $\mathfrak{g}^{\mathrm{mod}}(\psi_0)$ at Π is

$$\mathfrak{u}\big(\mathfrak{g}^{\mathrm{mod}}(\Pi), k\big) := \bigoplus_{\ell \in Z} \mathfrak{g}_\ell^{\mathrm{mod}}(\Pi).$$

Canonical absolute parallelisms

A point $\Pi_0 \in M \times_{N_1} P^0$ is *regular* (w.r.t. the Tanaka prolongation) if the maps $\Pi \mapsto \dim \mathfrak{g}_\ell^{\mathrm{mod}}(\Pi)$ are constant in a neighborhood of Π_0 for all $\ell \geq 0$ and of *finite type* if there is ℓ such that $\mathfrak{g}_\ell^{\mathrm{mod}}(\Pi) = 0$.

Theorem (S. Marini, D. Sykes and I.Z., 2024, in preparation)

1. *Given a k-nondegenerate hypersurface-type CR structure such that there exists a point $\Pi_0 = (p, \psi_0) \in M \times_{N_1} P^0$, which is regular and of finite type there exists a bundle over a neighborhood $\mathcal{O}$ of ψ_0 in P^0 of dimension equal to $\dim \mathfrak{u}(\mathfrak{g}^{\mathrm{mod}}(\Pi_0), k)$ that admits a canonical absolute parallelism.*
2. *The dimension of the algebra of infinitesimal symmetries of a k-nondegenerate, hypersurface-type CR structure of the previous item is not greater than $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{\mathrm{mod}}(\Pi_0), k)$.*

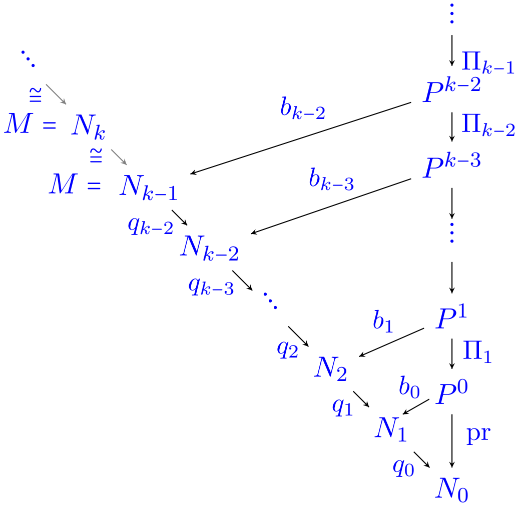


Figure: Natural foliation quotients and bundles arising from uniformly k-nondegenerate structures.

Reduction to the level sets of modified symbols

A *level set of* $\mathfrak{g}^{\text{mod}}$ is

$$R_X := \left\{\Pi = (p,\psi) \in M \times_{N_1} P^0 \mid g^{\text{mod}}(\Pi) = \text{a fixed subspace X}\right\}.$$

If there is a nonempty level set of positive codimension we can make a reduction of the bundle $M \times_{N_1} P^0$ to it.
Suppose that the level set is projected onto M. Applying θ_0 to a tangent space at $\Pi = (p,\psi)$ to the intersection of the fibers of $p \times P^0$ and the level set R_X gives a subspace $\mathfrak{g}^{\text{red}}(\Pi) \subset \mathfrak{g}^{\text{mod}}(\Pi)$.
For homogeneous models this can be iterated to obtain a constant maximally *reduced modified symbol* $\mathfrak{g}^{\text{red}}$ with level set $P^{0,\text{red}}$. Let $g^{0,\text{red}}$ be the nonpositively weighted part of $\mathfrak{g}^{\text{red}}$.

Proposition

$\mathfrak{g}^{0,\text{red}}$ *is a Lie algebra.*

- For $k = 2$, $\mathfrak{g}^{\text{red}} = \mathfrak{g}^{0,\text{red}}$ and the 2-nondegenerate Tanaka prolongation $\mathfrak{u}(\mathfrak{g}^{\text{red}}, 2)$ is always a Lie algebra.
- For $k > 2$ the the 2-nondegenerate Tanaka prolongation $\mathfrak{u}(\mathfrak{g}^{\text{red}}, k)$ is not necessary a Lie algebra.

Absolute parallelism for 2-nondegenerate structure with given reduced modified symbol

Theorem (combining C.Porter and I.Z. (2021) and D. Sykes and I.Z. (2023))

Assume $\dim u(\mathfrak{g}^{0,\mathrm{red}}, 2) < \infty$. *Then the following three statements hold.*

1. *Given a* 2*-nondegenerate, hypersurface-type CR structure such that the corresponding bundle* P^0 *admits a subbundle* $P^{0,\mathrm{red}}$ *with the constant reduced modified symbol* $\mathfrak{g}^{0,\mathrm{red}}$*, there exists a bundle over* $P^{0,\mathrm{red}}$ *of dimension equal to* $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}}, 2)$ *that admits a canonical absolute parallelism;*
2. *The dimension of the algebra of infinitesimal symmetries of a* 2*-nondegenerate, hypersurface-type CR structure of item (1) is not greater than* $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}}, 2)$.
3. *Any* 2*-nondegenerate CR structure with the constant reduced modified symbol* $\mathfrak{g}^{0,\mathrm{red}}$ *whose algebra of infinitesimal symmetries has dimension equal to* $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}}, 2)$ *is locally equivalent to the flat CR structure with reduced modified symbol* $\mathfrak{g}^{0,\mathrm{red}}$*, i.e. a special CR structure on the homogeneous space corresponding to the pairs of the Lie algebras* $(\mathfrak{R}\mathfrak{g}^{0,\mathrm{red}}, \mathfrak{R}\mathfrak{g}^{\mathrm{red}}_{0,0})$.

Reduction to the level set of the prolongation (heuristics)

Given a k-nondegenerate hypersurface-type CR structure with constant symbol family such that there exists a point $\Pi_0 = (p, \psi_0) \in M \times_{N_1} P^0$, which is regular and of finite type let $\widehat{P}$ be the bundle of of dimension $\dim \mathfrak{u}(\mathfrak{g}^{\mathrm{mod}}(\Pi_0), k)$, where the canonical frame is constructed.
Assuming that a homogeneous model with properties above exists, the reduction to the level sets of the k-nondegenerate universal prolongations $u(\mathfrak{g}^{\mathrm{mod}}(\Pi), k)$, repeated several time, if necessary, should lead either to the canonical absolute parallelism on the bundle of minimal possible dimension in this class and to the Lie algebra isomorphic to the infinitesimal symmetry algebra of the maximally symmetric homogeneous models in this class or to the proof that such homogeneous models do not exist.

For 2-nondegenrate case, the minimal dimension of M is 5, $\dim_{\mathbb{C}} \mathfrak{g}_{-1,1} = 1$, so there is only one symbol and any the modified (=reduced modified) symbol must coincide with $\mathfrak{c}^0 := \mathfrak{g}_- \oplus \mathfrak{c}_0$.
$\mathfrak{u}(\mathfrak{c}^0, 2) \cong \mathfrak{so}(5) \cong B_2, \quad \mathfrak{Ru}(\mathfrak{c}^0, 2) \cong \mathfrak{so}(3, 2)$ recovering the existence of

canonical absolute parallelism part of Isaev-Zaitsev (2013), Merker-Pocchiola (2013), Medori-Spiro (2014).

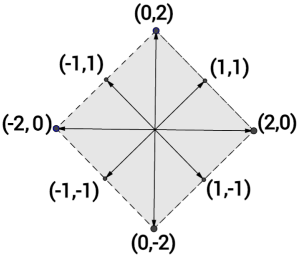


Figure 1

The classical universal Tanaka prolongation in this case is equal to the whole (infinite dimensional) contact algebra, and the extra-condition for our 2-nondegenerate universal prolongation are $\mathfrak{g}_{1,\pm 3} = 0$.

Classification of 2-non-degenerate CR symbols with one dimensional Levi kernel

The space of symbols of 2-nondegenerate CR structures, up to an isomorphism $\cong$ the space of pairs

(a real line ℓ of nondegenerate Hermitian forms on $\mathfrak{g}_{-1,1}$, a complex line of self-adjoint anti-linear operators A on $\mathfrak{g}_{-1,1}$),

up to the natural action of $GL(\mathfrak{g}_{-1,1})$, where

$$A(y) = \mathrm{ad}_v(\bar{y}), \quad v \in \mathfrak{g}_{0,2},\ y \in \mathfrak{g}_{-1,1}$$

Classification of all such pairs David Sykes and I.Z. (2020) as an analog of Jordan normal form in the spirit of the classical Kronecker theory of matrix pencils.

Matrix Representations of Reduced Modified Symbols (RMS) with 1-dim Levi kernel

An RMS $\mathfrak{g}^{0,\mathrm{red}}$ is represented by a tuple $(H_\ell, A, \Omega, \mathscr{A})$:
H_ℓ, A and Ω are $(n-1)\times(n-1)$ matrices, $H_\ell A = (H_\ell A)^T$, and $\mathscr{A}$ is a matrix algebra

$$\mathscr{A} \subset \left\{ \alpha \;\middle|\; \begin{array}{l} \alpha A H_\ell^{-1} + A H_\ell^{-1} \alpha^T = \eta A H_\ell^{-1} \text{ and} \\ \alpha^T H_\ell \overline{A} + H_\ell \overline{A} \alpha = \eta' H_\ell \overline{A} \text{ for some } \eta, \eta' \in \mathbb{C} \end{array} \right\}.$$

such that

$$\mathfrak{g}_0^{\mathrm{red}} = \left\langle \begin{pmatrix} \Omega & A \\ 0 & -H_\ell{}^{-1}\Omega^T H_\ell \end{pmatrix}, \begin{pmatrix} -\overline{H_\ell}^{-1}\Omega^* \overline{H_\ell} & 0 \\ \overline{A} & \overline{\Omega} \end{pmatrix}, \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -H_\ell{}^{-1}\alpha^T H_\ell \end{pmatrix} \middle| \alpha \in \mathscr{A} \right\}, I \right\rangle.$$

Flat structure classification in $\mathbb{C}^4$ (D. Sykes, 2022, Indiana UMJ)

label	H_ℓ (reduced Levi form)	A (Levi kernel adjoint op.)	Ω (obstruction to bigrading)	matrices spanning $\mathscr{A}$	symmetry group dimension
Type I	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	0	8
Type II	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}$	0	8
Type III	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & 3a \end{pmatrix}$	9
Type IV.A ($\epsilon = 1$) Type IV.B ($\epsilon = -1$)	$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	10
Type V.A ($\epsilon = 1$) Type V.B ($\epsilon = -1$)	$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0	$\begin{pmatrix} a & b \\ -\epsilon b & a \end{pmatrix}$	15
Type VI	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	15
Type VII	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	0	$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$	16

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