Pre-Kähler structures and finite-nondegeneracy

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Outline

- General Theory : Pre-Kähler $\leftrightarrow$ CR hypersurface interaction
  - What are Pre-Kähler structures? How do they arise?
  - Basic properties
  - Straightenability and finite-nondegeneracy
  - Pre-Kähler $\leftrightarrow$ pre-Sasakian correspondence

- Case study in dimension 4.
  - local equivalence problem
  - local invariants

Submanifolds in pseudo-Kähler structures

A *(pseudo-)Kähler structure* (M, ω, J) consists of

- M — a $2n$-dimensional manifold
- $J : TM \to TM$ — an integrable almost complex structure, i.e.,
  - $J^2 = -\mathrm{Id}$ and $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$
- ω — a closed real *nondegenerate* 2-form of type $(1,1)$, i.e.,
  - $\omega(v, w) = \omega(Jv, Jw)$

Its (pseudo-)Riemannian metric is $g(v, w) = \omega(v, Jw)$.

A complex submanifold $\iota : M' \hookrightarrow M$ inherits the structure

$$(M', \iota^*\omega, J')$$

where $J \circ \iota_* = \iota_* \circ J'$, which need not be (pseudo-)Kähler.

Submanifold structure examples

Example 1: If $M = \mathbb{C}^2$, $M' = \mathbb{C}$, $\iota(z_1) = (z_1, 0)$, and

$$\omega = \mathrm{i}\, \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} + \mathrm{i}\, \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_1},$$

then $\iota^* \omega = 0$.

Example 2: If $M = \mathbb{C}^4$, $M' = \mathbb{C}^3$, $\iota(z_1, z_2, z_3) = (z_1, z_2, z_3, \frac{1}{2} z_1^2)$, and

$$\omega = 2\mathrm{Re}\big(\mathrm{i}\, \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} + \mathrm{i}\, \mathrm{d}z_3 \wedge \mathrm{d}\overline{z_4}\big)$$

then

$$\iota^* \omega = \mathrm{i}\left(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_1} + \overline{z_1}\mathrm{d}z_3 \wedge \mathrm{d}\overline{z_1} + z_1 \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_3}\right),$$

and

$$X \lrcorner\, \omega = \omega(X, \cdot) = 0 \quad \text{where} \quad X = x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_3}.$$

Claim: $(\mathbb{C}^3, \iota^* \omega)$ is homogeneous with a finite-dimensional symmetry group.

Pre-Kähler structure definition

A *pre-Kähler structure* (M, ω, J) consists of

- M — a $2n$-dimensional manifold
- $J : TM \to TM$ — an integrable almost complex structure, i.e.,
  - $J^2 = -\mathrm{Id}$ and $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$
- ω — a closed real 2-form (i.e., *pre-symplectic*[a]) of type $(1,1)$, i.e.,
  - $\omega(v, w) = \omega(Jv, Jw)$

Basic questions

Given a pre-Kähler structure (M, ω, J):

- Are its symmetry algebras finite dimensional?
- Is it *locally straightenable*, i.e., locally a product structure

$$M \supset U \cong M' \times V$$

for some lower dimensional (M', ω', J') and $V \subset \mathbb{C}^{n'}$?

[a] c,f. Liberman–Marle *Symplectic geometry and anlytical mechanics*

Basic properties: submanifold realizations

Proposition

Every $2n$-dimensional pre-Kähler structure with rank $2r$ form ω, locally arises on a complex submanifold in a $2(2n-r)$-dimensional psuedo-Kähler manifold.

Key idea: Writing ω at a point in local coordinates

$$\omega|_0 = \mathrm{i} \sum_{j,k=1}^{r} H_{j,\bar{k}} \mathrm{d}z_j \wedge \overline{\mathrm{d}z_k}\Big|_0 ,$$

it is a pullback of

$$\omega + \sum_{j=1}^{n-r} \mathrm{Re}(\mathrm{i}\, \mathrm{d}z_{r+j} \wedge \overline{\mathrm{d}Z_j})$$

on $\mathbb{C}^n \times \mathbb{C}^{n-r}$ (locally).

Basic properties: kernel integrability

Lemma

The kernel of a constant rank closed $(1,1)$ form ω is complex and integrable.

One approach:

- Write ω locally as $\sum H_{j,\bar{k}} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$ for some coefficients $H = (H_{j,\bar{k}})$.
- Changing coframes to $(\theta^1, \ldots, \theta^n) = Udz$ transforms $H \mapsto U^* H U$. So $\exists (\theta^j)$ s.t. $\omega = \sum_{j=1}^{\operatorname{rank}\omega/2} \pm \theta^j \wedge \overline{\theta^j}$, and hence

$$(\ker \omega)^{\perp} = \left\{ \theta^1, \overline{\theta^1}, \ldots, \theta^{\operatorname{rank}\omega/2}, \overline{\theta^{\operatorname{rank}\omega/2}} \right\}. \tag{1}$$

- Apply

$$0 = d\omega = 2 \sum_{i=1}^{\operatorname{rank}\omega/2} \pm \operatorname{Re}(\mathrm{d}\theta^i \wedge \overline{\theta^i}),$$

to conclude (1) is a differential ideal.

The Freeman filtration

Set $\mathcal{K}_{-1} := T^{1,0}M$ and $\mathcal{K}_0 := \mathcal{K}_{-1} \cap \ker(\omega)$. Recursively define

$$\cdots \subset \mathcal{K}_{j+1} \subset \mathcal{K}_j \subset \cdots \subset \mathcal{K}_{-1} \tag{*}$$

as

$$(\mathcal{K}_{j+1})_p = \{x \in (\mathcal{K}_j)_p \mid \mathrm{ad}_x^{(j)} = 0\},$$

where $\mathrm{ad}_x^{(j)} : (\mathcal{K}_{-1})_p/(\mathcal{K}_0)_p \to (\mathcal{K}_{j-1})_p/(\mathcal{K}_j)_p$ is given by

$$\mathrm{ad}_x^{(j)}\big(y + (\mathcal{K}_0)_p\big) := [X, \overline{Y}] \pmod{(\mathcal{K}_j)_p \oplus T_p^{0,1}M}$$

$\forall\, x \in (\mathcal{K}_j)_p, y \in T_p^{1,0}M$, where $X \in \Gamma\big((\mathcal{K}_j)_p\big)$ and $Y \in \Gamma(T_p^{1,0}M)$ extending x, y.

We call (*) the *Freeman filtration* of (M, ω, J).

Proposition

The Freeman filtration (*) is well defined almost everyhwere.

Finite nondegeneracy

From now on, assume (M, ω, J) admits a Freeman filtration.

Definition

Let ℓ be the smallest integer such that

$$\mathcal{K}_{\ell-1} = \mathcal{K}_{\ell}.$$

If $\mathcal{K}_{\ell} = 0$ then (M, ω, J) is *(uniformly) ℓ-nondegenerate* and (*finitely nondegenerate*).
Otherwise (M, ω, J) is *holomorphically degenerate*,

As in CR geometry, there are more general definitions of finite-nondegeneracy and holomorphic degeneracy that can be defined without using the Freeman filtration.

Finite type criterion

From CR geometry:

Theorem (Stanton, '96)

The algebra of holomorphic infinitesimal symmetries at a point in an analytic real hypersurface is finite dimensional if and only if its CR structure at that point is not holomorphically degenerate.

Pre-Kähler analogue:

Theorem (Makhmali–S.)

For a real-analytic pre-Kähler structure admitting a Freeman filtration, the following are equivalent:

- *It is finitely-nondegenerate.*
- *Its symmetry algebras are all finite-dimensional.*
- *It is not locally straightenable anywhere.*

From potentials to hypersurfaces

From the local $\overline{\partial}$-Poincaré Lemma: working locally with $U \subset M$, $\exists\, \rho : U \to \mathbb{R}$ such that

$$\partial\overline{\partial}\rho = \frac{\mathrm{i}}{2}\,\omega|_U$$

We call ρ a (local) *potential* of ω. It is defined up to

$$\rho' = \rho + \mathrm{Re}(f)$$

for holomorphic functions f.[b]
Each potential determines a hypersurface and CR symmetry

$$M_\rho := \{(w,z) \in \mathbb{C} \times U \,|\, \mathrm{Re}(w) = \rho(z)\} \quad \text{and} \quad X_\rho := \left.\frac{\partial}{\partial \mathrm{Im}(w)}\right|_{M_\rho} \in \Gamma(TM_\rho).$$

The structure on M_ρ defined by its abstract CR structure plus the symmetry X_ρ does not depend on the potential.
proof: $(w,z) \mapsto (w - f(z), z)$ transforms (M_ρ, X_ρ) to $(M_{\rho'}, X_{\rho'})$.

[b] Poincaré, Sur les propriétés du potentiel et sur les fonctions abéliennes,1898

For

- local potential $\rho : U \subset M \to \mathbb{R}$ of (M, ω, J),
- (local) biholomorphism $\varphi : U \subset M \to \varphi(U) \subset M$ preserving ω,

φ transforms ρ to another potential of ω becasue

$$\omega = \varphi^* \omega = -2\,\mathrm{i}\,\varphi^* \partial\overline{\partial}\rho = -2\,\mathrm{i}\,\partial\overline{\partial}\varphi^* \rho.$$

Hence

$$\rho(z) = \rho\left(\varphi^{-1}(z)\right) - \mathrm{Re}(f(z)),$$

for some holomorphic function $f \in \Omega(U)$.
The corresponding transformation $(w, z) \mapsto (w - f(z), \varphi(z))$ (locally) preserves (M_ρ, X_ρ).

Lemma

Infinitesimal symmetries of (M, ω, J) naturally embed into holomorphic infinitesimal symmetries of $M_\rho \subset \mathbb{C}^{n+1}$.

From rigid CR hypersurfaces to pre-Kähler structures

Theorem (Baouendi–Rothschild–Treves, '85)

A hypersurface-type CR manifold S with an infinitesimal symmetry X transversal to its CR distribution at a point p can be locally realized as a real hypersurface of the form

$$\{(w,z) \in U \subset \mathbb{C} \times \mathbb{C}^n \,|\, \mathrm{Re}(w) = \rho(z)\}$$

with X represented by $\frac{\partial}{\partial \mathrm{Im}(w)}$.

$(S, X) \longrightarrow (\mathbb{C}^n, -2\,\mathrm{i}\,\partial\overline{\partial}\rho)$ inverts $(\mathbb{C}^n, -2\,\mathrm{i}\,\partial\overline{\partial}\rho) \longrightarrow (M_\rho, X_\rho)$.

Theorem (Makhmali–S.)

There is a one-to-one correspondence between germs of pre-Kähler structures and germs of hypersurface-type CR structures equipped with distinguished transversal symmetries.

Nondegeneracy branching in dimension 4

A 4-dimensional pre-Kähler manifold is either

- (pseudo-)Kähler with $\operatorname{rank}(\omega) = 4$,
- 2-nondegenerate with $\operatorname{rank}(\omega) = 2$,
- (locally) a product of 2-d. Kähler with $\mathbb{C}$, or
- $\omega = 0$.

CR structures associated with 4-d. 2-nondegenerate pre-Kähler structures are (locally) 5-d. 2-nondegenerate CR hypersurfaces.

The latter have been thoroughly studied (by [Ebenfelt, ‘01], [Kaup–Zaitsev, ‘06], [Fels–Kaup, ‘08], [Isaev–Zaitsev, ‘13], [Medori–Spiro, ‘14], [Merker–Pochiola, ‘20], [Kolař–Kossovskiy, ‘22], and others).

2-nondegeneracy in dimension 4

From now on, let (M, ω, J) be 2-nondegenerate and $\dim M = 4$.

Since ω is rank 2, there exist type $(1,0)$ forms (θ^1, θ^2) spanning $(T^{1,0}M)^*$ such that

$$\omega = \frac{\mathrm{i}}{2}\theta^1 \wedge \overline{\theta^1},$$

which is unique up to an action of

$$G_1 := \left\{ A \in \mathrm{GL}(2, \mathbb{C}) \,\middle|\, A = \begin{pmatrix} e^{a\,\mathrm{i}} & 0 \\ b_1 + \mathrm{i}\, b_2 & c_1 + \mathrm{i}\, c_2 \end{pmatrix} \right\}.$$

Call such coframes $(\theta^1, \theta^2, \overline{\theta^1}, \overline{\theta^2})$ 1-*adapted*.

2-adapted coframes

2-nondegeneracy expressed in coframes implies

$$d\theta^1 \equiv \lambda\, \overline{\theta^1} \wedge \theta^2 \mod \{\theta^1\}$$

And the action of G_1 can further normalize $(\theta^1, \theta^2, \overline{\theta^1}, \overline{\theta^2})$ to make $\lambda = 1$. Such *2-adapted coframes* are unique up to a

$$G_2 = \left\{ A \in \mathrm{GL}(2, \mathbb{C}) \,\middle|\, A = \begin{pmatrix} e^{a\,\mathrm{i}} & 0 \\ b_1 + \mathrm{i}\, b_2 & e^{2a\,\mathrm{i}} \end{pmatrix} \right\} \subset G_1.$$

action.
Hence,

$$\begin{aligned} d\,\theta^1 =& \overline{\theta^1} \wedge \theta^2 + A^1_{12}\theta^1 \wedge \theta^2 + A^1_{1\bar{1}}\theta^1 \wedge \overline{\theta^1} + A^1_{1\bar{2}}\theta^1 \wedge \overline{\theta^2} + A^1_{2\bar{2}}\theta^2 \wedge \overline{\theta^2} \\ d\theta^2 =& A^2_{12}\theta^1 \wedge \theta^2 + A^2_{i\bar{j}}\theta^i \wedge \overline{\theta^j}. \end{aligned}$$

2-adapted differential relations

The system of equations

$$0 = \mathrm{d}\omega = \frac{\mathrm{i}}{2}\left(\mathrm{d}\theta^1 \wedge \overline{\theta^1} + \theta^1 \wedge \mathrm{d}\overline{\theta^1}\right), \quad \mathrm{d}^2\theta^1 = 0, \quad \mathrm{d}^2\theta^2 = 0,$$

implies

$$A^1_{1\bar{2}} = \overline{A^1_{12}}, \quad A^1_{\bar{1}\bar{2}} = 0, \quad A^2_{2\bar{2}} = 2\overline{A^1_{12}}$$

and

$$A^2_{2,\bar{1};\bar{2}} - 2A^1_{1,\bar{1};\bar{2}} = 3A^2_{1,\bar{2}} - \bar{A}^1_{1,2}\left(A^2_{2,\bar{1}} - 2A^1_{1,\bar{1}}\right).$$

3-adapted coframes

The G_2 coframe transformation

$$\theta^1 \mapsto \theta^1, \quad \theta^2 \to \theta^2 + \tfrac{1}{3}(A^2_{2\bar{1}} - 2A^1_{1\bar{1}})\theta^1$$

yields *3-adapted coframes*, achieving

$$\begin{aligned} d\theta^1 &= \overline{\theta^1} \wedge \theta^2 + B^1_{12}\theta^1 \wedge \theta^2 + B^1_{1\bar{1}}\theta^1 \wedge \overline{\theta^1} - \overline{B^1_{12}}\theta^1 \wedge \overline{\theta^2} \\ d\theta^2 &= B^2_{12}\theta^1 \wedge \theta^2 + B^2_{1\bar{1}}\theta^1 \wedge \overline{\theta^1} + 2B^1_{1\bar{1}}\theta^2 \wedge \overline{\theta^1} - 2\overline{B^1_{12}}\theta^2 \wedge \overline{\theta^2}. \end{aligned}$$

for some B^k_{ij}. There is no $\theta^1 \wedge \overline{\theta^2}$ term in $d\theta^2$!
3-adapted coframes are unique up to a $U(1)$ action of

$$G_3 = \left\{ A \in \mathrm{GL}(2, \mathbb{C}) \;\middle|\; A = \begin{pmatrix} e^{a\,i} & 0 \\ 0 & e^{2a\,i} \end{pmatrix} \right\} \subset G_2.$$

The $U(1)$-principal principal bundle

The bundle of 3-adapted coframes is a $U(1)$-principal bundle $\mathcal{G}$.
The 3-adapted coframes lift to canonical 1-forms $(\theta^1, \theta^2, \overline{\theta^1}, \overline{\theta^2})$ on $\mathcal{G}$.
and there exists a $\mathfrak{u}(1)$-valued 1-form ψ whose restriction to fibers of $G \to M$ coincides with the $U(1)$ Maurer–Cartan form, and

$$
\begin{aligned}
\mathrm{d}\theta^1 &= -\,\mathrm{i}\,\psi \wedge \theta^1 + \overline{\theta^1} \wedge \theta^2 - (C^1_{12}\theta^2 + C^1_{1\bar{1}}\overline{\theta^1} - \overline{C^1_{12}}\,\overline{\theta^2}) \wedge \theta^1 \\
\mathrm{d}\theta^2 &= -\,2\,\mathrm{i}\,\psi \wedge \theta^2 + C^2_{12}\theta^1 \wedge \theta^2 + C^2_{1\bar{1}}\theta^1 \wedge \overline{\theta^1} - 2(C^1_{1\bar{1}}\overline{\theta^1} - \overline{C^1_{12}}\,\overline{\theta^2}) \wedge \theta^2.
\end{aligned}
$$

for some C^k_{ij}. Normalizing

$$
\psi \to \psi + x_1\theta^1 + \overline{x_1}\,\overline{\theta^1} + x_2\theta^2 + \overline{x_2}\,\overline{\theta^2}
$$

we can moreover obtain

$$
\begin{aligned}
\mathrm{d}\theta^1 &= -\,\mathrm{i}\,\psi \wedge \theta^1 + \overline{\theta^1} \wedge \theta^2 \\
\mathrm{d}\theta^2 &= -2\,\mathrm{i}\,\psi \wedge \theta^2 + T_1\theta^1 \wedge \overline{\theta^1} + T_2\theta^1 \wedge \theta^2.
\end{aligned}
$$

The Cartan connection (1 of 2)

Theorem (Makhmali–S.)

Any 2-nondegenerate pre-Kähler structure (ω, J) *on* M *canonically defines a Cartan geometry* $(\mathcal{G} \to M, \varphi)$ *of type* $(\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R}), \mathrm{U}(1))$ *where the Cartan connection and its curvature are*

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ \overline{\theta^1} & -\mathrm{i}\,\psi & \overline{\theta^2} \\ \theta^1 & \theta^2 & \mathrm{i}\,\psi \end{pmatrix}, \quad \mathrm{d}\varphi + \varphi \wedge \varphi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{i}\,\Psi & \overline{\Theta^2} \\ 0 & \Theta^2 & \mathrm{i}\,\Psi \end{pmatrix},$$

which

$$\Theta^2 = T_1 \theta^1 \wedge \overline{\theta^1} + T_2 \theta^1 \wedge \theta^2, \quad \Psi = \mathrm{i}\,\overline{T_2}\theta^1 \wedge \overline{\theta^2} - \mathrm{i}\,T_2 \overline{\theta^1} \wedge \theta^2 + T_3 \theta^1 \wedge \overline{\theta^1}.$$

for some functions T_1, T_2, T_3 *on* $\mathcal{G}$*. Conversely, any such Cartan geometry defines a unique pre-Kähler structure on* M*.*

The Cartan connection (2 of 2)

Theorem (continued)

Bianchi identities yield

$$T_3 = \frac{\mathrm{i}}{2}\left(T_{2;\bar{1}} - T_{1;2}\right) = -\overline{T_3}.$$

(M, ω, J) *is the locally flat pre-Kähler if and only if*

$$\mathbf{T}_1 := T_1\overline{T_1}, \quad \mathbf{T}_2 := T_2\overline{T_2}. \tag{2}$$

vanish, in which case it is given by the local potential

$$\rho(z) := \frac{|z_1|^2 + \mathrm{Re}(z_1^2\overline{z_2})}{1 - |z_2|^2}.$$

Thank you very much!