The spectrum of the Folland–Stein operator on some Heisenberg Bieberbach manifolds

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Introduction

Let $\mathbb{H}$ be the Heisenberg group. For a lattice $N \subset \mathbb{H}$, the cpt quotient $N \backslash \mathbb{H}$ naturally has a CR structure.

In 2004, Folland determined the eigenvalues and eigenfunctions of the Kohn Laplacian on $N \backslash \mathbb{H}$.

$\rightsquigarrow$ We are interested in the quotient of $N \backslash \mathbb{H}$ by an action of a finite group.

Heisenberg group

Let $\mathbb{H}$ be the 3-dim Heisenberg group.

group multiplication:

$$(p, q, s) \cdot (p', q', s') := (p + p', q + q', s + s' + pq')$$

for $(p, q, s), (p', q', s') \in \mathbb{H}$.

CR structure: Let us consider the left-invariant complex vector field on $\mathbb{H}$,

$$V = \frac{1}{2}(Q + iP),$$

where $P = \dfrac{\partial}{\partial p}$, $Q = \dfrac{\partial}{\partial q} + p\dfrac{\partial}{\partial s}$.

Then $T^{1,0}\mathbb{H} = \mathrm{Span}_{\mathbb{C}}\{V\}$ is a CR structure, which is CR diffeomorphic to the CR structure of the boundary of $\{(z, w) \in \mathbb{C}^2 \mid \mathrm{Im}\, w > |z|^2\}$.

Heisenberg group

Take the hermitian metric which makes V, $\bar{V}$, $\frac{1}{4}\frac{\partial}{\partial s}$ orthonormal.
The Kohn Laplacian

$$\Box_b = \bar{\partial}_b^* \bar{\partial}_b : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$$

is given by $\Box_b f = \mathscr{L}_1 f$ where

$$\mathscr{L}_\alpha = \frac{1}{4}\left(-\left(P^2 + Q^2\right) + i\alpha \frac{\partial}{\partial s} \right) \quad (\alpha \in \mathbb{R}).$$

We call $\mathscr{L}_\alpha$ the Folland–Stein operator.

Heisenberg Bieberbach manifolds

$\underline{U(1) \curvearrowright \mathbb{H}}$: For $e^{i\theta} \in U(1)$, $(p, q, s) \in \mathbb{H}$,

$$e^{i\theta} \cdot (p, q, s) = (p\cos\theta - q\sin\theta, p\sin\theta + q\cos\theta,$$
$$s + \frac{1}{2}(\cos\theta\sin\theta(p^2 - q^2) + (\cos^2\theta - \sin^2\theta - 1)pq).$$

Definition

A subgroup $\Gamma \subset \mathbb{H} \rtimes U(1)$ is a Heisenberg Bieberbach group if

- Γ is discrete and torsion-free,
- $\Gamma \backslash \mathbb{H}$ is compact.

The quotient $\Gamma \backslash \mathbb{H}$ is called a Heisenberg Bieberbach manifolod.

Heisenberg Bieberbach manifolds

The action $U(1) \curvearrowright \mathbb{H}$ preserves the CR str on $\mathbb{H}$ and the metric.

$\rightsquigarrow$ H-B mfd $\Gamma\backslash\mathbb{H}$ has a natural CR str and the F-S operator $\mathscr{L}_\alpha$ can be considered a diff operator on $\Gamma\backslash\mathbb{H}$.

Example

(0) A lattice (i.e, a cocompact discrete subgroup) $N \subset \mathbb{H}$ is a Heisenberg Bieberbach group. We call $N\backslash\mathbb{H}$ a Heisenberg nilmanifold.

For $L \in \mathbb{Z}_{>0}$,

$$N_L := \mathbb{Z} \times L\mathbb{Z} \times \mathbb{Z}$$

is a lattice.

$\rightsquigarrow$ $N_L\backslash\mathbb{H}$ is a S^1-bundle over the torus $\mathbb{R}^2/(\mathbb{Z} \times L\mathbb{Z})$

Heisenberg Bieberbach manifolds

Example

(1) Let

$$\varphi := \left(0, 0, \frac{1}{2}\right) e^{i\pi} \in \mathbb{H} \rtimes U(1).$$

$\rightsquigarrow \varphi(p, q, s) = (-p, -q, s + \frac{1}{2})$: π-rotation w.r.t (p, q)

For $l \in \mathbb{Z}_{>0}$, the subgroup

$$\Gamma_{2l,\pi} = \langle N_{2l}, \varphi \rangle$$

is a Heisenberg Bieberbach group. Note that $\Gamma_{2l,\pi}/N_{2l} = \mathbb{Z}/2\mathbb{Z}$.

$\rightsquigarrow N_{2l}\backslash\mathbb{H} \longrightarrow \Gamma_{2l,\pi}\backslash\mathbb{H}$ is a 2-fold covering.

Heisenberg Bieberbach manifolds

Example

(2) Let $N'_{2l} = \sqrt{2l}\mathbb{Z} \times \sqrt{2l}\mathbb{Z} \times \mathbb{Z}$ ($l \in \mathbb{Z}_{>0}$),

$$\psi := \left(0, 0, \frac{1}{4}\right) e^{i\frac{\pi}{2}} \in \mathbb{H} \rtimes U(1).$$

$\rightsquigarrow \psi(p, q, s) = (-q, p, s - pq + \frac{1}{4})$: $\frac{\pi}{2}$-rotation w.r.t $(p.q)$

The subgroup

$$\Gamma'_{2l,\frac{\pi}{2}} = \langle N'_{2l}, \psi \rangle$$

is a Heisenberg Bieberbach group. Note that $\Gamma'_{2l,\frac{\pi}{2}}/N'_{2l} = \mathbb{Z}/4\mathbb{Z}$.

$\rightsquigarrow N'_{2l}\backslash\mathbb{H} \longrightarrow \Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$ is a 4-fold covering.

Harmonic analysis on a H-nilmanifold $N_L \backslash \mathbb{H}$

Recall $N_L = \mathbb{Z} \times L\mathbb{Z} \times \mathbb{Z}$.

Since $(0,0,1) \in N_L$ and $f \in L^2(N_L \backslash \mathbb{H})$ is N_L-invariant, we have

$f(p,q,s+1) = f(p,q,s)$.

$\rightsquigarrow L^2(N_L \backslash \mathbb{H}) = \bigoplus_{n \in \mathbb{Z}} \mathscr{H}_n$, where $\mathscr{H}_n = \{f(p,q,s) = e^{2\pi i n s} g(p,q)\}$

$\underline{n = 0}$

$g(p,q) \in \mathscr{H}_0$ is a function on the torus $\mathbb{R}^2/(\mathbb{Z} \times L\mathbb{Z})$, and so

$$\mathscr{H}_0 = \bigoplus_{(\mu,\nu) \in \mathbb{Z} \times L^{-1}\mathbb{Z}} \mathbb{C} e^{2\pi i(\mu p + \nu q)}.$$

Harmonic analysis on a H-nilmanifold $N_L\backslash\mathbb{H}$

$n \neq 0$

For $a \in \mathbb{Z}/|n|\mathbb{Z}$, $b \in \mathbb{Z}/L\mathbb{Z}$, we define the Weil-Brezin transform $W_n^{a,b} : L^2(\mathbb{R}) \longrightarrow \mathscr{H}_n^{a,b} := \mathrm{Im}\, W_n^{a,b} \subset \mathscr{H}_n$ by

$$(W_n^{a,b} g)(p,q,s) = e^{2\pi ins} \sum_{k\in\mathbb{Z}} g\left(p + k + \frac{j}{|n|} + \frac{u}{Ln}\right) e^{2\pi in\left(k+\frac{j}{|n|}+\frac{u}{Ln}\right)q},$$

where $0 \leq j \leq |n|-1$ with $a \equiv j \bmod |n|$ and $0 \leq u \leq L-1$ with $b \equiv u \bmod L$.

Proposition

$$L^2(N_L\backslash\mathbb{H}) = \bigoplus_{(\mu,\nu)\in\mathbb{Z}\times L^{-1}\mathbb{Z}} \mathbb{C}e^{2\pi i(\mu p+\nu q)} \oplus \bigoplus_{n\neq 0, a\in\mathbb{Z}/|n|\mathbb{Z}, b\in\mathbb{Z}/L\mathbb{Z}} \mathscr{H}_n^{a,b}$$

Eigenvalues of the Folland–Stein operator on $N_L\backslash\mathbb{H}$

F-S operator: $\mathscr{L}_\alpha = \frac{1}{4}\left(-(P^2+Q^2) + i\alpha\frac{\partial}{\partial s}\right)$

$$\mathscr{H}_n^{a,b} \overset{W_n^{a,b}}{\longleftrightarrow} L^2(\mathbb{R})$$
$$\mathscr{L}_\alpha \longleftrightarrow \frac{1}{4}\left(-\left(\frac{\partial^2}{\partial x^2} - 4\pi^2 n^2 x^2\right) - 2\pi n\alpha\right)$$

Theorem (Folland'04)

The eigenvalues of $\mathscr{L}_\alpha$ on $N_L\backslash\mathbb{H}$ are

$$\Lambda_\alpha(n,\lambda) := \frac{\pi|n|}{2}(2\lambda + 1 - \alpha\,\mathrm{sgn}\,n) \quad (\lambda\in\mathbb{Z}_{\geq 0}, n\in\mathbb{Z}\setminus\{0\})$$

and $\pi^2|\xi|^2$ ($\xi\in\mathbb{Z}\times L^{-1}\mathbb{Z}$). The multiplicity of $\Lambda_\alpha(n,\lambda)$ is $L|n|$. In particular, the function $W_n^{a,b}F_{n,\lambda}$ is an eigenfunction with $\Lambda_\alpha(n,\lambda)$, where $F_{n,\lambda}(x) = H_\lambda(\sqrt{2\pi|n|}x)e^{-\pi|n|x^2}$ (H_λ: Hermite polynomial)

Eigenvalues of the Folland–Stein operator on $\Gamma_{2l,\pi}\backslash\mathbb{H}$

Recall from Example (1) that

$$\Gamma_{2l,\pi} = \langle N_{2l}, \varphi \rangle, \quad \varphi = \left(0, 0, \frac{1}{2}\right) e^{i\pi}$$

and $N_{2l}\backslash\mathbb{H} \longrightarrow \Gamma_{2l,\pi}\backslash\mathbb{H}$ is a 2-fold covering.

For any $\Lambda_\alpha(n,\lambda)$-eigfct $G \in \mathrm{Span}_{\mathbb{C}} \{ W_n^{a,b} F_{n,\lambda} \}_{a,b}$ on $N_{2l}\backslash\mathbb{H}$,

$$G \text{ is an also eigenfunction on } \Gamma_{2l,\pi}\backslash\mathbb{H} \iff \varphi^* G = G.$$

Proposition (S.)

For $n \neq 0$ and $\lambda \in \mathbb{Z}_{\geq 0}$, we have

(i) $\varphi^*(W_n^{a,0} F_{n,\lambda}) = e^{\pi i(n+\lambda)} W_n^{-a,0} F_{n,\lambda}$.

(ii) *For $b \neq 0$, $\varphi^*(W_n^{a,b} F_{n,\lambda}) = e^{\pi i(n+\lambda)} W_n^{-a-1,-b} F_{n,\lambda}$ if $n > 0$, and $\varphi^* W_n^{a,b} F_{n,\lambda} = e^{\pi i(n+\lambda)} (W_n^{-a+1,-b} F_{n,\lambda})$ if $n < 0$.*

Eigenvalues of the Folland–Stein operator on $\Gamma_{2l,\pi}\backslash\mathbb{H}$

If an $\Lambda_\alpha(n,\lambda)$-eigenfunction $G \in \mathrm{Span}_{\mathbb{C}}\{W_n^{a,b}F_{n,\lambda}\}_{a,b}$ on $N_{2l}\backslash\mathbb{H}$ is written as

$$G = \sum_{a\in\mathbb{Z}/|n|\mathbb{Z},\, b\in\mathbb{Z}/2l\mathbb{Z}} c^{a,b}W_n^{a,b}F_{n,\lambda} \quad (c^{a,b}\in\mathbb{C}),$$

we have the following.

Corollary

Let $\lambda \in \mathbb{Z}_{\geq 0}$*.*

(i) *For* $n>0$, $\varphi^*G = G$ *if and only if* $e^{\pi i(n+\lambda)}c^{a,0} = c^{-a,0}$ *and* $e^{\pi i(n+\lambda)}c^{a,b} = c^{-a-1,-b}$ *for all* $a\in\mathbb{Z}/|n|\mathbb{Z}$, $0\neq b\in\mathbb{Z}/2l\mathbb{Z}$.

(ii) *For* $n<0$, $\varphi^*G = G$ *if and only if* $e^{\pi i(n+\lambda)}c^{a,0} = c^{-a,0}$ *and* $e^{\pi i(n+\lambda)}c^{a,b} = c^{-a+1,-b}$ *for all* $a\in\mathbb{Z}/|n|\mathbb{Z}$, $0\neq b\in\mathbb{Z}/2l\mathbb{Z}$.

Eigenvalues of the Folland–Stein operator on $\Gamma_{2l,\pi}\backslash\mathbb{H}$

Let $\mathscr{H}^{\varphi}_{n,\lambda} = \{ G \in \mathrm{Span}_{\mathbb{C}} \{ W^{a,b}_n F_{n,\lambda} \}_{a,b} \mid \varphi^* G = G \}$

Theorem (S.)

$$\dim \mathscr{H}^{\varphi}_{n,\lambda} = \begin{cases} l|n| + 1 & \text{if } |n| + \lambda \text{ is even,} \\ l|n| - 1 & \text{if } |n| + \lambda \text{ is odd.} \end{cases}$$

Since $\dim \mathrm{Span}_{\mathbb{C}} \{ W^{a,b}_n F_{n,\lambda} \}_{a,b} = 2l|n|$, the dim of each eigenspace on $\Gamma_{2l,\pi}\backslash\mathbb{H}$ is about half of the dim of the corresponding eigenspaces on $N_{2l}\backslash\mathbb{H}$

Eigenvalues of the Folland–Stein operator on $\Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$

Recall from Example (2) that

$$\Gamma'_{2l,\frac{\pi}{2}} = \langle N'_{2l}, \psi \rangle, \quad \psi = \left(0, 0, \frac{1}{4}\right) e^{i\frac{\pi}{2}},$$

and $N'_{2l}\backslash\mathbb{H} \longrightarrow \Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$ is a 4-fold covering.
Since $S_{2l}(N'_{2l}) = N_{2l}$, where S_{2l} is the isom $(p, q, s) \mapsto \left(\frac{p}{\sqrt{2l}}, \sqrt{2l}q, s\right)$,
we have the decomposition

$$L^2(N'_{2l}\backslash\mathbb{H}) = \bigoplus_{\mu,\nu} \mathbb{C}e^{2\pi in(\mu p+\nu q)} \oplus \bigoplus_{n\neq 0, a\in\mathbb{Z}/|n|\mathbb{Z}, b\in\mathbb{Z}/2l\mathbb{Z}} \widetilde{\mathscr{H}_n^{a,b}},$$

where $\widetilde{\mathscr{H}_n^{a,b}} = S_{2l}^*(\mathscr{H}_n^{a,b})$.

Eigenvalues of the Folland–Stein operator on $\Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$

Moreover, the eigenvalues of $\mathscr{L}_\alpha$ on $N'_{2l}\backslash\mathbb{H}$ are also

$$\Lambda_\alpha(n,\lambda) = \frac{\pi|n|}{2}(2\lambda + 1 - \alpha\,\mathrm{sgn}\,n) \quad (\lambda \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\setminus\{0\})$$

and $\pi^2|\xi|^2$ ($\xi \in \sqrt{2l}^{-1}\mathbb{Z}\times\sqrt{2l}^{-1}\mathbb{Z}\times\mathbb{Z}$).

Proposition

Let

$$F_{n,\lambda,l}(x) = H_\lambda(2\sqrt{l\pi|n|}x)e^{-2l\pi|n|x^2}.$$

The function $\widetilde{W_n^{a,b}}F_{n,\lambda,l} := (W_n^{a,b}F_{n,\lambda,l})\circ S_{2l} \in \widetilde{\mathscr{H}_n^{a,b}}$ *is an eigenfunction of* $\mathscr{L}_\alpha$ *on* $N'_{2l}\backslash\mathbb{H}$ *with the eigenvalue* $\Lambda_\alpha(n,\lambda)$.

Eigenvalues of the Folland–Stein operator on $\Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$

Proposition (S.)

$$\psi^*(\widetilde{W_n^{j,u}}F_{n,\lambda,l}) = \frac{1}{\sqrt{2ln}}e^{\frac{\pi}{2}i(n+\lambda)} \sum_{\substack{0\leq j'\leq n-1 \\ 0\leq u'\leq 2l-1}} e^{-4ln\pi i\left(\frac{j'}{n}+\frac{u'}{2ln}\right)\left(\frac{j}{n}+\frac{u}{2ln}\right)} \widetilde{W_n^{j',u'}}F_{n,\lambda,l}$$

for all $0 \leq j \leq n-1,\ 0 \leq u \leq 2l-1$ *if* $n > 0$, *and*

$$\psi^*(\widetilde{W_n^{j,u}}F_{n,\lambda,l}) = \frac{1}{\sqrt{2l|n|}}e^{\frac{\pi}{2}i(n+3\lambda)} \sum_{\substack{0\leq j'\leq |n|-1 \\ 0\leq u'\leq 2l-1}} e^{-4ln\pi i\left(\frac{j'}{|n|}+\frac{u'}{2ln}\right)\left(\frac{j}{|n|}+\frac{u}{2ln}\right)} \widetilde{W_n^{j',u'}}F_{n,\lambda,l}$$

for all $0 \leq j \leq |n|-1,\ 0 \leq u \leq 2l-1$ *if* $n < 0$.

Eigenvalues of the Folland–Stein operator on $\Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$

If an $\Lambda_\alpha(n,\lambda)$-eigfct $\widetilde{G} \in \mathrm{Span}_{\mathbb{C}}\left\{\widetilde{W_n^{j,u}}F_{n,\lambda,l}\right\}_{j,u}$ on $N'_{2l}\backslash\mathbb{H}$ is written as

$$\widetilde{G} = \sum_{\substack{0\le j\le |n|-1\\ 0\le u\le 2l-1}} c^{j,u}\widetilde{W_n^{j,u}}F_{n,\lambda,l} \quad (c^{j,u}\in\mathbb{C}),$$

we have the following.

Corollary

(i) *For $n>0$, $\psi^*\widetilde{G} = \widetilde{G}$ if and only if*

$$c^{j',u'} = \frac{1}{\sqrt{2ln}} e^{\frac{\pi}{2}i(n+\lambda)} \sum_{\substack{0\le j\le n-1\\ 0\le u\le 2l-1}} e^{-4ln\pi i\left(\frac{j'}{n}+\frac{u'}{2ln}\right)\left(\frac{j}{n}+\frac{u}{2ln}\right)} c^{j,u}$$

for all $0\le j'\le n-1,\ 0\le u'\le 2l-1$.

Eigenvalues of the Folland–Stein operator on $\Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$

Corollary

(ii) *For $n < 0$, $\psi^*\widetilde{G} = \widetilde{G}$ if and only if*

$$c^{j',u'} = \frac{1}{\sqrt{2l|n|}} e^{\frac{\pi}{2}i(n+3\lambda)} \sum_{\substack{0\le j\le |n|-1\\ 0\le u\le 2l-1}} e^{-4ln\pi i\left(\frac{j'}{|n|}+\frac{u'}{2ln}\right)\left(\frac{j}{|n|}+\frac{u}{2ln}\right)} c^{j,u}$$

for all $0 \le j' \le |n|-1,\ 0 \le u' \le 2l-1.$

Eigenvalues of the Folland–Stein operator on $\Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$

Let $\widetilde{\mathscr{H}}^{\psi}_{n,\lambda} = \{ \widetilde{G} \in \mathrm{Span}_{\mathbb{C}} \{ \widetilde{W}^{j,u}_n F_{n,\lambda,l} \}_{j,u} \mid \psi^* \widetilde{G} = \widetilde{G} \}$

Theorem (S.)

(i) *If $|n|$ is even or l is even, we have*

$$\dim \widetilde{\mathscr{H}}^{\psi}_{n,\lambda} = \begin{cases} \frac{l|n|}{2} + 1 & \text{if } |n| + \lambda \equiv 0 \mod 4, \\ \frac{l|n|}{2} & \text{if } |n| + \lambda \equiv 1, 2 \mod 4, \\ \frac{l|n|}{2} - 1 & \text{if } |n| + \lambda \equiv 3 \mod 4. \end{cases}$$

(ii) *If $|n|$ is odd and l is odd, we have*

$$\dim \widetilde{\mathscr{H}}^{\psi}_{n,\lambda} = \begin{cases} \frac{l|n|}{2} + \frac{1}{2} & \text{if } |n| + \lambda \equiv 0, 2 \mod 4, \\ \frac{l|n|}{2} - \frac{1}{2} & \text{if } |n| + \lambda \equiv 1, 3 \mod 4. \end{cases}$$

Weyl's law

Corollary

$M = \Gamma_{2l,\pi}\backslash\mathbb{H}$ or $\Gamma'_{2l,\frac{\pi}{2}}\backslash\mathbb{H}$. Let $N_M(t)$ be the number of positive eigenvalues of $\mathscr{L}_\alpha$ on M (counted with multiplicity) which are less than or equal to $t > 0$. For $-1 \leq \alpha \leq 1$, we have

$$\lim_{t\to\infty} \frac{N_M(t)}{t^2} = A_\alpha \operatorname{vol}(M),$$

where

$$A_\alpha = \begin{cases} \dfrac{1}{\pi^2}\displaystyle\int_{-\infty}^{\infty} \frac{x}{\sinh x} e^{-\alpha x}\, dx & \text{if } -1 < \alpha < 1, \\ \dfrac{1}{2\pi^2}\displaystyle\int_{-\infty}^{\infty} \left(\frac{x}{\sinh x}\right)^2 dx & \text{if } \alpha = \pm 1. \end{cases}$$

Thank you very much!