A classification and invariants of CR maps between certain models of real hypersurfaces

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(based on joint works with Lamel and Reiter)

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Motivations

- Poincaré observed that there are nontrivial biholomorphic invariants of real hypersurfaces (ca. 1907).
- CR maps from the 3-sphere

$$\mathbb{S}^3 := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

  into itself are restrictions of automorphisms the unit ball (Cartan, Tanaka, Chern-Moser).
- The same is true for higher dimensional spheres: The CR automorphism group $\mathrm{Aut}(\mathbb{S}^{2n+1})$ of a sphere is completely and explicitly classified.

- The classification for CR maps from $\mathbb{S}^{2n+1}$ into $\mathbb{S}^{2n'+1}$, $n' > n$ depends on how large n' compared to n.
- *"Spherically equivalent classes"*: Two maps H and $\tilde{H}$ are equivalent if there are $\psi \in \mathrm{Aut}(\mathbb{S}^{2n'+1})$ and $\gamma \in \mathrm{Aut}(\mathbb{S}^{2n+1})$ such that

$$\tilde{H} = \psi^{-1} \circ H \circ \gamma.$$

- If $n \leq n' \leq 2n - 1$ then all CR maps from $\mathbb{S}^{2n+1}$ into $\mathbb{S}^{2n'+1}$ is equivalent to the linear map $z \mapsto (z, 0)$ ("rigidity", J. J. Faran 1980s, Webster ($n' = n + 1 \geq 4$)).
- This rigidity fails when $n' \geq 2n$.

- For $n' = 2n$, $n \geq 3$, there are exactly *two* classes, represented by the linear map and the *Whitney map* (Huang–Ji 2001)

$$\mathcal{W}(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, z_1 z_n, z_2 z_n, \ldots, z_n^2).$$

- For $n' = 2n + 1$, $n \geq 3$, every rational sphere map is equivalent to one in the D'Angelo's family (Hamada 2005).

$$F_\theta = (z_1, \ldots, z_n, \cos(\theta) z_{n+1}, \sin(\theta) z_1 z_{n+1}, \ldots, \sin(\theta) z_{n+1} z_{n+1})$$

- For $n' > 2n+1$, there are further studies (Huang, Ji, Yin, Xu, Lebl, Ebenfelt, Minor, and others) and the "HJY gap conjecture," which is related to the "SOS conjecture" about the rank of Hermitian sums of squares. The collection of maps becomes "bigger" when the target dimension increases.
- CR maps from $\mathbb{S}^3$ into $\mathbb{S}^5$ are classified into four equivalent classes (Faran 1982)

$$(z_1, z_2) \mapsto \begin{cases} \left(z_1^2, \sqrt{2} z_1 z_2, z_2^2\right) \\ \left(z_1, z_2 z_1, z_2^2\right) \\ \left(z_1^3, \sqrt{3} z_1 z_2, z_2^3\right) \\ \left(z_1, z_2, 0\right). \end{cases}$$

$\Longrightarrow$ The case of CR maps from $\mathbb{S}^3$ exhibits different behaviors. Monomial maps are better understood (D'Angelo 1980s).

- CR maps from $\mathbb{S}^3$ into $\mathbb{S}^7$ have not been classified. There are 14 "discrete" and two parametric families of inequivalent monomial maps (D'Angelo 1988).
- Degree two rational maps from $\mathbb{S}^3$ into $\mathbb{S}^m$ are classified (Lebl, Ji et al).
- We cannot hope for a full classification of rational maps from $\mathbb{S}^3$ into $\mathbb{S}^m$ for $m \geq 7$. There are a lot of maps.
- For *"highly degenerate maps"* from $\mathbb{S}^3$, there is a recent result of della Sala–Lamel–Reiter–S. (2024).

Similar considerations for hyperquadric maps (Baoundi, D'Angelo, Huang, Ebenfelt, Zaitsev, Lebl, Xiao, Gao–Ng, ...).
The study of CR maps between more general real hypersurfaces has been done by many mathematicians that are too numerous to mention in this talk.

The tube over the future light cone

Smooth boundary part of the classical domain of type IV is locally equivalent to the tube over the future light cone. In $\mathbb{C}^3$, the tube is given by

$$\mathcal{T}\colon (\Re Z_1)^2 + (\Re Z_2)^2 - (\Re Z_3)^3 = 0, \quad \Re Z_3 > 0.$$

$\mathcal{T}$ is uniformly Levi-degenerate and 2-nondegenerate. Homogeneous Levi-degenerate CR manifolds of five dimensional are completely classified (Fels–Kaup 2008). The tube is the one with the "largest" stability group.

The tube $\mathcal{T}$ is locally equivalent to

$$\mathcal{X} = \left\{ (z, \zeta, w) \in \mathbb{C}^3 \mid \Im w = \frac{|z|^2 + \Re(\bar{z}^2 \zeta)}{1 - |\zeta|^2} \right\}, \quad |\zeta| < 1.$$

CR maps from $\mathbb{H}^3$ into $\mathcal{X}$

Related to proper holomorphic maps between classical domains, which have been studied by Tsai, Tu, Mok, Xiao–Yuan, Kim–Mok–Seo, and others.
From their defining functions, we can easily construct simple CR maps from the Heisenberg hypersurface $\mathbb{H}^3$ into $\mathcal{X}$:

$$\mathcal{X} \colon \Im w = \frac{|z|^2 + \Re(\bar{z}^2 \zeta)}{1 - |\zeta|^2}.$$

- The map $(z, w) \mapsto (0, \zeta, 0)$ sends $\mathbb{H}^3$ into $\mathcal{X}$ (non CR transversally).
- $\ell : (z, w) \mapsto (z, 0, w)$ (the "linear" map).
- Consider $V = \{w\zeta + iz^2 = 0\}$. On V, $\Re(\bar{z}^2\zeta) = -|\zeta|^2 \Im w$. Plugging this into defining function for $\mathcal{X}$, we obtain $\Im w = |z|^2$. The map $(z, w) \mapsto (z, -iz^2/w, w)$, singular along $w = 0$, sends an open set of $\mathbb{H}^3$ into $\mathcal{X}$.

These (transversal) maps can be constructed from proper algebraic maps from ball into the classical domains of type IV (Mok, Xiao–Yuan, Kim–Mok–Seo, ...).

Equivalences

Let M and M' be CR manifolds and let $H\colon (M,p) \to (M',p')$ a germ of smooth CR maps. The product group

$$G := \mathrm{Aut}(M,p) \times \mathrm{Aut}(M',p')$$

acts on the space of map germs.

Definition

We say that H and $\tilde{H}$ are *equivalent* if there exist germs of local CR diffeomorphisms $\gamma\colon (M,p) \to (M,\tilde{p})$ and $\psi\colon (M',\tilde{H}(\tilde{p})) \to (M',H(p))$, such that

$$H = \psi \circ \tilde{H} \circ \gamma^{-1}.$$

This definition is similar to these for sphere and hyperquadric maps.

A classification

Theorem (Reiter–S. (2022))

Let $U \subset \mathbb{H}^3$ be an open $H\colon U \to \mathcal{X}$ a smooth CR map.

(a) *If H is CR transversal at some point $p \in U$, then H is CR transversal on U, the germs (H, q), $q \in U$, are mutually equivalent and are equivalent to exactly one of the following four pairwise inequivalent germs at the origin:*

(i) $\ell(z, w) = (z, 0, w)$,

(ii) $r_1(z, w) = \left(\dfrac{z(1 + iw)}{1 - w^2}, \dfrac{2z^2}{1 - w^2}, \dfrac{w}{1 - w^2}\right)$,

(iii) $r_{-1}(z, w) = \left(\dfrac{z(1 - iw)}{1 - w^2}, \dfrac{-2z^2}{1 - w^2}, \dfrac{w}{1 - w^2}\right)$,

(iv) $\iota(z, w) = (2z,\ 2w,\ 2w) \Big/ \left(1 + \sqrt{1 - 4w^2 - 4iz^2}\right)$.

(b) *If H is nowhere transversal, then for each q, the germ (H, q) is equivalent to the germ at the origin of a map $t_q\colon (z, w) \mapsto (0, \phi_q(z, w), 0)$ for a local CR function ϕ_q.*

Higher dimension

- CR maps from $\mathbb{S}^n$ into D_{IV}^m, with $4 \leq n \leq m-1 \leq 2n-4$ are rigid (Xiao–Yuan 2020).
- Local model for the case $m = 4$:

$$\mathcal{X} = \left\{ (z, \zeta, w) \mid \mathbb{C}^3 \mid \Im w = \frac{z\bar{z}^t + \Re\left(\overline{zz^t}\zeta\right)}{1 - |\zeta|^2} \right\}$$

  Here $z = (z_1, z_2)$ and z^t is its transpose.
- The Heisenberg hypersurface in $\mathbb{H}^5 \subset \mathbb{C}^3$:

$$\mathbb{H}^5 = \left\{ (z, w) \in \mathbb{C}^3 \mid \Im w = z\bar{z}^t \right\}.$$

Theorem (Reiter–S., 2024)

Let $U \subset \mathbb{H}^5$ and $H\colon U \to \mathcal{X} \subset \mathbb{C}^4$ a C^2-smooth CR map.

(a) *If H is CR transversal at a point, then it is transversal on U. The germs (H, q), $q \in U$, are mutually equivalent and are equivalent to the exactly one of the germs at the origin of the following maps:*

(i) $\ell(z, w) = (z, 0, w)$,

(ii) $r(z, w) = \left(\frac{z(I + iwA)}{1 - w^2}, \frac{2zAz^t}{1 - w^2}, \frac{w}{1 - w^2} \right)$, *with* $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *and* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *is the identity matrix,*

(iii) $\iota(z, w) = \frac{2}{1 + \sqrt{1 - 4w^2 - 4izz^t}} (z,\ w,\ w)$.

Here $z = (z_1, z_2) \in \mathbb{C}^2$.

(b) *H is nowhere transversal.*

Sketch of the proof

Suppose $H = (f, \phi, g)$ be a map sending the Heisenberg hypersurface into the rational model $\mathcal{X}$. We determine the map via the following steps.

- Partial normalizing the map. Get an equivalent map $\widetilde{H}$ with a simple 2-jet, namely,

$$
\begin{aligned}
\widetilde{f}(z,w) &= z + \frac{i}{2}\alpha z w + \nu w^2 + O(3),\\
\widetilde{\phi}(z,w) &= \lambda w + \alpha z^2 + \mu z w + \sigma w^2 + O(3),\\
\widetilde{g}(z,w) &= w + O(3),
\end{aligned}
\tag{1}
$$

  Then, α and λ are two important invariants of the maps which characterize the isometry and the rationality, respectively. The fact that α appears in two places is a kind of Gauss–Codazzi equation, well-known in the case of sphere maps.

Sketch of the proof

- Determining μ, ν, and σ (must be zero). Determine higher order jets (up to 4-jets). General "finite jet determination" results (of Chern–Moser (2-jets), Baouendi, Ebenfelt, Rothschild, Zaitsev, Mir, Lamel, ...) imply that 4-jets is enough. This is also clear from our calculation.
- Determining H, H_w and H_{ww} along the first Segre set $\Sigma = \{w = 0\}$. For example, we determine

$$f(z,0) = \frac{2z}{1 + \sqrt{1 - 4i\bar{\lambda}z^2}}, \quad g(z,0) = 0,$$

and an explicit formula for $\phi(z,0)$. We divide into two cases depending whether $\lambda \neq 0$ which eventually lead to rational and irrational maps.

Sketch of the proof

- Propagating along second Segre sets $(z, \bar{z}) \mapsto (z, 2iz\bar{z})$ which covers an open set. We obtain holomorphic functional relation for f, g, ϕ. For example, if $\lambda = 0$, then $\sigma = 0$ and

$$4z^3 g - 4z^2 wf + w^2 z\phi - w^2 \Upsilon(z,w)\,(w(\mu + 4i\nu) - \alpha z) = 0,$$

  where $\Upsilon := g\phi + if^2$.
- Propagating H_w and H_{ww} along the second Segre set. Obtain two more holomorphic functional relations.
- Solving for f, g, ϕ from this nonlinear system. The implicit function theorem cannot applied at the origin. But one can solve at generic point.
- Solutions are generally meromorphic. Identifying genuine solution by analyzing further jets lead to several cases.

Cartan's classical symmetric domains

There are 4 types (excluding 2 special types of complex dimension 16 and 27) of domains arising from the classification of Hermitian symmetric spaces, denoted by

$$D_I^{p,q}, \quad D_{II}^p, \quad D_{III}^p, \quad D_{IV}^m.$$

For examples, the type I is

$$D_I^{p,q} = \{Z \in \mathrm{Mat}(p,q;\mathbb{C}) \colon I_{p\times q} - ZZ^* > 0\}$$

and the type IV, also call the *Lie ball*, is

$$D_{IV}^m = \left\{Z \in \mathbb{C}^m \colon 1 - 2ZZ^* + |ZZ^t| > 0, ZZ^* < 1\right\}$$

Smooth boundary part of these classical domains are interesting models for CR geometry. E.g., boundary of $D_{IV}^m \leftrightarrow$ tube over the future light cone.

Cartan's classical symmetric domains

- Automorphisms of these domains are explicitly known.
- Moreover, their Bergman kernels are explicit and give rise to Kähler–Einstein metrics.
- For type IV domain D_{IV}^m, the Bergman kernel is

$$K(Z,Z) = C_{D_{IV}^m}\left(1 - 2ZZ^* + |ZZ^t|^2\right)^{-m}.$$

- An interesting fact for us: $K(Z,Z)^{-1/m}$ is a local defining function for smooth point of the boundary.
- Biholomorphisms are isometries of the Bergman metric which extend to relevant CR diffeomorphisms of the smooth boundary part.

Holomorphic maps of classical domains

Theorem (Tsai 1993)

Let $f : \Omega \to \Omega'$ *be a proper holomorphic map between two bounded symmetric domains such that* Ω *is irreducible and of rank* ≥ 2, *and such that* $\mathrm{rank}(\Omega') \leq \mathrm{rank}(\Omega)$. *Then,* $\mathrm{rank}(\Omega') = \mathrm{rank}(\Omega)$, *and* f *is a totally geodesic embedding.*

Theorem (Mok)

Assume $m \geq 2$.

(i) *If* $F : \mathbb{B}^n \to D_m^{IV}$ *is a holomorphic isometry, then* $n \leq m - 1$.

(ii) *There exists a non-totally geodesic holomorphic isometric embedding* $G : \mathbb{B}^{m-1} \to D_m^{IV}$ *with* $G^*\omega_{D_m^{IV}} = \lambda \omega_{\mathbb{B}^{m-1}}$ *for some* $\lambda > 0$.

Theorem (Xiao–Yuan 2020)

Assume $n \geq 4, n+1 \leq m \leq 2n-3$.

1. *(Local version) Let* F *be a holomorphic map from a connected open set* U *in* $\mathbb{C}^n$ *containing* $p \in \partial\mathbb{B}^n$ *to* $\mathbb{C}^m$. *Assume that* $F(\partial\mathbb{B}^n \cap U) \subset \partial D_m^{IV}$ *and* $F(U) \not\subset \partial D_m^{IV}$. *Then* F *extends to a holomorphic isometric embedding from* $\mathbb{B}^n$ *into* D_m^{IV} *with* $F^*(\omega_{D_m^{IV}}) = \frac{m}{n+1}\omega_{\mathbb{B}^n}$.
2. *(Global version) Any algebraic proper holomorphic map* F *from* $\mathbb{B}^n$ *to* D_m^{IV} *is an isometric embedding with* $F^*(\omega_{D_m^{IV}}) = \frac{m}{n+1}\omega_{\mathbb{B}^n}$.

Theorem (Xiao–Yuan 2020)

Let $n \geq 4$ and F be a C^2-smooth CR transversal CR map from an open piece of $\partial\mathbb{B}^n$ to an open smooth piece of ∂D^{IV}_{n+1}. Then F extends to a holomorphic isometry from $\mathbb{B}^n$ to D^{IV}_{n+1}. Furthermore, F is equivalent to either

$$\left(z_1, \cdots, z_{n-1}, \frac{\frac{1}{2}\sum_{i=1}^{n-1} z_i^2 - z_n^2 + z_n}{\sqrt{2}(1-z_n)}, \sqrt{-1}\,\frac{\frac{1}{2}\sum_{i=1}^{n-1} z_i^2 + z_n^2 - z_n}{\sqrt{2}(1-z_n)}\right); \tag{2}$$

or

$$\left(z_1, \cdots, z_{n-1}, z_n, 1 - \sqrt{1 - \sum_{j=1}^{n} z_j^2}\right). \tag{3}$$

What about the case $n \leq 3$? These two maps, when $n = 2$, can be transferred to maps from $\mathbb{H}^3$ into $\mathcal{X}$.

The case of $\mathbb{B}^2 \to D^3_{IV}$

Corollary (Reiter–S. (2022))

Let $H\colon \mathbb{B}^2 \to D^3_{IV}$ be a proper holomorphic map which extends smoothly to some boundary point $p \in \partial\mathbb{B}^2$. Then H is equivalent to exactly one of the following four pairwise inequivalent maps:

(i) $R_0(z,w) = \left(\frac{z}{\sqrt{2}}, \frac{2w^2+2w-z^2}{4(w+1)}, \frac{i\left(2w^2+2w+z^2\right)}{4(w+1)}\right)$,

(ii) $P_1(z,w) = \left(zw, \frac{z^2-w^2}{2}, \frac{i(z^2+w^2)}{2}\right)$,

(iii) $P_{-1}(z,w) = \left(z, \frac{w^2}{2}, \frac{iw^2}{2}\right)$,

(iv) $I(z,w) = \left(z, w, 1-\sqrt{1-z^2-w^2}\right)/\sqrt{2}$.

The case of $\mathbb{B}^3 \to D_{IV}^4$

Corollary (Reiter–S., 2024)

Let $H\colon \mathbb{B}^3 \to D_4^{\mathrm{IV}}$ be a proper holomorphic map which extends smoothly to some boundary point $p \in \partial\mathbb{B}^3$. Then H is equivalent to one of the following pairwise inequivalent maps:

(i)

$$R_0(z,w) = \left(\frac{z}{\sqrt{2}}, \frac{2w^2 + 2w - zz^t}{4(w+1)}, \frac{i\left(2w^2 + 2w + zz^t\right)}{4(w+1)} \right), \tag{4}$$

(ii)

$$I(z,w) = \left(z,\ w,\ 1 - \sqrt{1 - zz^t - w^2} \right) / \sqrt{2}, \tag{5}$$

(iii)

$$P(z_1,z_2,w) = \left(z_1,\ z_2 w,\ \frac{w^2 - z_2^2}{2}, \frac{i(w^2 + z_2^2)}{2} \right), \tag{6}$$

where $z = (z_1, z_2)$.

Geometric rank of sphere/hyperquadric maps

For each point p, there are CR automorphisms ψ and γ such that $\gamma(0) = p$ and $\tilde{H} := \psi^{-1} \circ H \circ \gamma$ has the following form: $\tilde{H} = (F^k)\colon \mathbb{H}_\ell^{2n+1} \to \mathbb{H}_\ell^{2N+1}$, where

$$F^k = \begin{cases} z_\alpha + \frac{i}{2} a_\alpha(z) w + O_{wt}(4) & k = \alpha \in \{1, 2, \dots, n\} \\ \phi_l^{(2)}(z) + O_{wt}(3), & k = l \in \{n+1, \dots, N\} \\ w + O_{wt}(5) & k = N+1 \end{cases} \tag{7}$$

with

$$\langle \bar{z}, a(z) \rangle_\ell \|z\|_\ell^2 = \|\phi^{(2)}(z)\|_\tau^2.$$

Here $\tau = \ell' - \ell \geq 0$ is the signature difference. (In the case $\tau < 0$, we should consider side reversing maps, which is similar.) Moreover, $a(z) = zA(p)$ for some Hermitian matrix A. The rank of A is then an invariant of the spherically equivalent class of the map, which is called the *geometric rank* of H at p (Huang 1999, Huang–Lu–Tang–Xiao 2000).

Theorem (Huang 1999)

A CR sphere map of vanishing geometric rank is equivalent to a linear map.

Thus,

$$\mathrm{Rk}_p(F) = 0, \quad \forall p \approx 0 \Rightarrow F \sim (z, 0, w).$$

Semi-linearity of maps of low rank (Huang 2003).
Hyperquadrics in $\mathbb{C}^{n+1}$:

$$\Im(z_{n+1}) - \sum_{k=1}^{n} \epsilon_k |z_k|^2 = 0, \quad \epsilon_k \in \{-1, 1\}.$$

Theorem (Huang–Lu–Tang–Xiao 2020)

A CR hyperquadric map of vanishing geometric rank is the restriction of an isometry between canonical (indefinite) Kähler metric of the generalized balls.

Sphere maps with lower geometric ranks have been studied in a lot of works.

CR Ahlfors derivative

Let M and N be Levi-nondegenerate real hypersurfaces. We choose two pseudohermitian structures θ and η on M and N, respectively. Associated to each transversal CR map $F\colon M \to N$ is a tensor $\mathcal{A}(F)$, called the CR Ahlfors derivative of F (Lamel–S. 2021). This tensor has interesting property that are similar to those of the Ahlfors derivative of maps from curves. For a chain of CR immersions $(M,\theta) \xrightarrow{F} (N,\eta) \xrightarrow{G} (P,\zeta)$, it holds that

$$\mathcal{A}(G \circ F) = \mathcal{A}(F) + F^* \mathcal{A}(G). \tag{8}$$

In particular, if $\mathcal{A}(G) = 0$, then $\mathcal{A}(G \circ F) = \mathcal{A}(F)$. This is an invariant property, which holds if G is a CR automorphisms of a sphere or a hyperquadrics. Thus, the CR Ahlfors derivative gives rise to an invariant of spherically equivalent classes of CR maps between spheres/hyperquadrics.

- For u smooth on M:

$$\mathcal{H}_\theta(u) = \operatorname{Sym} \nabla\nabla u - \partial_b u \otimes \partial_b u - \bar{\partial}_b u \otimes \bar{\partial}_b u + \frac{1}{2}|\bar{\partial}_b u|^2 L_\theta. \tag{9}$$

- For CR submanifold $N \subset M$, define

$$\nu(X, Y) = 2\,\langle \operatorname{Sym} \mathrm{I\!I}(X, Y), \mu\rangle - \langle X, Y\rangle |\mu|^2, \tag{10}$$

where μ is "$(1,0)$-mean curvature", $\operatorname{Sym} \mathrm{I\!I}(X, Y)$ is the symmetrized pseudo-Hermitian second fundamental form.

Definition (Lamel–S. 2021)

Let (M^{2n+1}, θ) and (N^{2d+1}, η) be strictly pseudoconvex pseudohermitian manifolds and let $F \colon M \to N$ be a CR immersion. Let u be the smooth function on M such that $F^*\eta = e^u\theta$. We define the CR Ahlfors derivative (or CR Ahlfors tensor) of F to be

$$\mathcal{A}(F) := \mathcal{H}_\theta(u) + F^*\left(\nu^N_{F(M)}\right) + \frac{1}{2}F^*(J_\Theta L_\Theta) - \frac{1}{2}J_\theta L_\theta. \tag{11}$$

where $J_\theta = R_\theta/(n(n+1))$ and $J_\Theta = R_\Theta/(d(d+1))$ are the normalized Webster scalar curvatures on M and N, respectively, and L_Θ and L_θ are the corresponding Levi forms.

For conformal immersion, the Ahlfors tensor was constructed by Stowe in 2015.

A chain rule for the CR Ahlfors tensor:

Theorem (Lamel–S. 2021)

For CR immersions $F\colon (M,\theta) \to (N,\eta)$ *and* $G\colon (N,\eta) \to (P,\zeta)$, *we have*

$$\mathcal{A}(G \circ F) = \mathcal{A}(F) + F^*\mathcal{A}(G). \tag{12}$$

This is a CR counterpart of Denis Stowe's result in 2015 for conformal immersions. CR case is simpler.

Corollary (Lamel-S. 2021)

Let (M, θ) *be a strictly pseudoconvex pseudohermitian manifold.*

(i) *Suppose that* $F\colon M \to \mathbb{S}^{2N+1}$ *is a CR immersion and* $\phi\colon \mathbb{S}^{2N+1} \to \mathbb{S}^{2N'+1}$ *(*$N' \geq N$*) is a totally geodesic embedding, then*

$$\mathcal{A}(F) = \mathcal{A}(\phi \circ F). \tag{13}$$

In particular, if F *and* G *are left spherical equivalent CR maps from* M *into* $\mathbb{S}^{2N+1}$*, then*

$$\mathcal{A}(F) = \mathcal{A}(G). \tag{14}$$

(ii) *Suppose that* $G\colon (\mathbb{S}^{2n+1}, \Theta) \to (M, \theta)$ *is a CR immersion and* $\gamma\colon N \to \mathbb{S}^{2n+1}$ *is a totally geodesic embedding, then*

$$\gamma^* \mathcal{A}(G) = \mathcal{A}(G \circ \gamma). \tag{15}$$

Theorem (Lamel–S. 2021)

Let $F\colon (M,\theta) \to (\mathbb{S}^{2d+1},\Theta)$ *be a CR immersion. If* $\mathcal{A}(F) = 0$*, then* M *is CR spherical and* F *is spherically equivalent to the linear mapping.*

Vanishing of CR Ahlfors tensor $\Rightarrow$ vanishing of second fundamental form (similar to Riemannian case but not true for "positive signature" case).

Geometric rank vs. CR Ahlfors tensor:

$$\mathrm{Rk}(F)_p = \mathrm{Rk}(\mathcal{A}_{\alpha\bar\beta}|_p)$$

Why? Explicit formula for the CR Ahlfors tensor.

Theorem (Lamel–S. 2021)

$$\mathcal{A}_{\alpha\bar\beta}(F) = D^{\rho}_{\alpha\bar\beta}\log Q - \frac{1}{2}\left(P_{\widetilde\rho}\log J(\widetilde\rho) - P_{\rho}\log J(\rho)\right)h_{\alpha\bar\beta}. \tag{16}$$

$D^{\rho}_{\alpha\bar\beta}$ and P are second order differential operator:

$$D^{\rho}_{\alpha\bar\beta}u = (i\partial\bar\partial u)(Z_{\alpha}, Z_{\bar\beta}).$$

Related to holomorphic isometries and extendable to other situations.

An example

A generalization of a hypersurface in $\mathbb{C}^3$ of Levi signature $\ell' = 1$, first appears in Winkelmann classification of homogeneous complex manifolds: Winkelmann hypersurfaces:

$$\rho_{\mathcal{W}_{\ell'}^{2n'+3}} := \Im(w + \bar{z}_{n'}\zeta) - |z_{n'}|^4 - \sum_{k=1}^{n'-1} \epsilon'_k |z_k|^2 = 0, \quad n' \geq 1, \tag{17}$$

where $(z_1, \ldots, z_{n'}, \zeta, w)$ are holomorphic coordinates in $\mathbb{C}^{n'+2}$ and $\epsilon'_k = -1$ for $1 \leq k \leq \ell' - 1$ (there is no such k if $\ell' = 1$) and $\epsilon'_k = 1$ for $k = \ell', \ldots n' - 1$, has exactly $(n' + 1)^2 + 4$ independent symmetries.
This model plays an important role in the study of homogeneous real hypersurfaces (Doubrov–Medvedev–The 2021, Kruglikov 2016, ...).

Embeddability into the hyperquadric:

$$\Phi(z_1, \ldots, z_{n'}, \zeta, w) = \left(z_1, \ldots, z_{n'-1}, z_n^2, \frac{z_{n'} + i\zeta}{2}, \frac{z_{n'} - i\zeta}{2}, w \right),$$

sends $\mathcal{W}$ into $\mathbb{H}$. This map is of vanishing geometric rank.

Example

The map

$$I(z_1, z_2, \ldots, w) = \left(z_1, \ldots, z_{n-1}, \sqrt{1+z_n} - 1, 4i\left(\sqrt{1+z_n} - 1 - z_n\right) w\right), \qquad (18)$$

send a hyperquadric into the Winkelmann hypersurface which is of vanishing geometric rank.

Example

For $\epsilon \in \{-1, 1\}$, the quadratic map $R\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ defined by

$$\begin{aligned} & R(z_1, z_2, \ldots, z_n, w) \\ & \quad = ((1+\epsilon z_n)z_1, \ldots, (1+\epsilon z_n)z_{n-1}, z_n, w(\epsilon + z_n) - iz_n(1+2\epsilon z_n), w(1+\epsilon z_n)) \end{aligned} \tag{19}$$

sends $\mathbb{H}_\ell^{2n+1}$ into $\mathcal{W}_{\ell+1}^{2n+3}$. Along $\mathbb{H}_\ell^{2n+1}$, it is CR transversal precisely when $z_n \neq -\epsilon$. Furthermore, it has full rank precisely when $z_n \neq -\epsilon$. Moreover, since

$$\rho_{\mathcal{W}_{\ell+1}^{2n+3}} \circ R = |\epsilon + z_n|^2 \rho_{\mathbb{H}_\ell^{2n+1}},$$

we see that R is a local isometric embedding of $\mathcal{U}^{n+1} \setminus \{z_n = -\epsilon\}$ into Ω_+. Clearly,

$$\mathcal{A}(R)_{\alpha\bar\beta} = \langle i\partial\bar\partial \log |\epsilon + z_n|^2, Z_\alpha \wedge Z_{\bar\beta} \rangle = 0,$$

i.e., R has vanishing geometric rank.

Based on Huang–Lu–Tang–Xiao's result in 2020:

Theorem (Reiter–S. 2024)

Let $U \subset \mathbb{H}_{\ell}^{2n+1}$ be an open subset and $H\colon U \to \mathcal{W}_{\ell'}^{2n'+3}$. Let $\mathcal{A}_{\alpha\bar{\beta}}(H)$ be the Hermitian part of the CR Ahlfors tensor of H with respect to "standard" pseudo-Hermitian structures of the source and target. Then $\mathcal{A}_{\alpha\bar{\beta}}(H) = 0$ on U if and only if H extends to a local isometric embedding of the indefinite Kähler metrics.

Geometric rank of CR maps into the tube over the light cone

CR maps from the Heisenberg model into the $\mathcal{X}$ model can be normalized:

$$\begin{aligned}
\widetilde{f}(z,w) &= z + \frac{i}{2}\alpha z w + \nu w^2 + O(3), \\
\widetilde{\phi}(z,w) &= \lambda w + \alpha z^2 + \mu z w + \sigma w^2 + O(3), \\
\widetilde{g}(z,w) &= w + O(3),
\end{aligned} \tag{20}$$

Similarly to Huang's definition: The coefficient α give the geometric rank of H at the origin:

$$\alpha = 0 \Rightarrow rank(H) = 0,$$

otherwise

$$\alpha \neq 0 \Rightarrow rank(H) = 1,$$

Let U be an open neighborhood of a point p in $\mathbb{C}^3$, $p \in \mathbb{H}^5$, and let $H\colon U \to \mathbb{C}^4$ be a holomorphic map such that $H(U \cap \mathbb{H}^5) \subset \mathcal{X}$. Then there exist automorphisms ϕ and ψ of $\mathbb{H}^5$ and $\mathcal{X}$, respectively, which satisfy $\psi(p) = 0$, $\gamma(H(p)) = 0$, and

$$\gamma \circ H \circ \psi^{-1} = (f, \phi, g),$$

where f, ϕ, and g take the following form

$$\begin{cases} f = z + \frac{i}{2} w(zA_{\alpha,\beta}) + w^2 \nu + O(3), \\ \phi = \lambda w + zA_{\alpha,\beta} z^t + wz\mu^t + \sigma w^2 + O(3), \\ g = w + O(3), \end{cases} \tag{21}$$

where

$$A_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \in \mathrm{Mat}(2 \times 2; \mathbb{R}), \quad \nu = (\nu_1, \nu_2) \in \mathbb{C}^2, \quad \mu = (\mu_1, \mu_2) \in \mathbb{C}^2.$$

Moreover, the rank of $A_{\alpha,\beta}$ does not depend on the pair (γ, ψ) satisfying the conditions above.

Geometric rank via normalization.

Definition (Reiter–S. 2024)

The rank of the matrix A is called the *geometric rank* of H at p, and denoted by $\operatorname{rank}(H)(p)$.

Definition (Reiter–S. 2024)

Let M and M' be real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, defined by ρ and ρ', respectively. Suppose that $H\colon U \to \mathbb{C}^{N+1}$ is a holomorphic map such that $H(U \cap M) \subset M'$. Assume that $V \subset U$ is an open subset with $V \cap M \neq \emptyset$ and $Q\colon V \to \mathbb{R}$ is a positive real-valued function satisfying

$$\rho'(H(z), \overline{H(z)}) = Q(z, \bar{z})\, \rho(z, \bar{z}), \quad z \in V \subset U. \tag{22}$$

Then we define a tensor $\mathcal{A}'(H)$ associated to H on $V \cap M$ as follows:

$$\mathcal{A}'(H)(Z, \overline{W}) = (i\partial\bar{\partial} \log Q)(Z, \overline{W}), \quad Z, W \in T^{(1,0)}M. \tag{23}$$

Let U be an open neighborhood of a point p in $\mathbb{C}^3$, $p \in \mathbb{H}^5$, and let $H\colon U \to \mathbb{C}^4$ be a holomorphic map such that $H(U \cap \mathbb{H}^5) \subset \mathcal{X}$. Then $\operatorname{rank}(H)(p) = \operatorname{rank} \mathcal{A}'(H)(p)$. Similarly for the ball and type IV models: Applying Huang–Lu–Tang–Xiao's result mentioned earlier.

Theorem (Reiter–S. 2024)

Let U be an open neigborhood of a point $p \in \mathbb{S}^{2n+1}$ and H a holomorphic map from U into $\mathbb{C}^m$. Assume that $U \cap \mathbb{B}^{n+1}$ is connected, $H(U \cap \mathbb{B}^{n+1}) \subset D_m^{\mathrm{IV}}$, and $H(U \cap \mathbb{S}^{2n+1}) \subset \mathcal{R} = \partial D_m^{\mathrm{IV}}$. Then the following are equivalent:

1. *H is transversal at p and $\mathcal{A}(H) = 0$ on an open neigborhood of p in $\mathbb{S}^{2n+1}$.*
2. *H is an isometric embedding from $U \cap \mathbb{B}^{n+1}$ into D_m^{IV}.*

This was used to prove the classification of CR maps from the sphere into the tube over the future light cone.

Thank you!