Renormalized volume and GJMS Operators in Higher Codimension via the Singular Yamabe Problem

(Joint work with Sri Rama Chandra Kushtagi)

Stephen E. McKeown

UT Dallas

Osaka Workshop in Conformal and CR Geometry
February 21, 2025

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The *conformal infinity* of M is the conformal class $[\varphi^2 g_+|_{T\Sigma}]$ on Σ.
We say g_+ is *Einstein* if $\mathrm{Ric}(g_+) = -n g_+$.

Lemma

Let (M, g_+) *be an AH space with conformal infinity* $(\Sigma^n, [h])$. *Let* $h \in [h]$. *Then for* $\varepsilon > 0$ *small, there is a unique diffeomorphism* $\psi : [0, \varepsilon)_r \times \Sigma \hookrightarrow M$ *onto a collar neighborhood of* Σ *such that* $\psi^* g_+ = \frac{dr^2 + g_r}{r^2}$, *where* g_r *is a one-parameter family of metrics on* Σ *and* $g_0 = h$.

If g_+ is Einstein, then one has the asymptotics

$$g_r = h + r^2 g^{(2)} + r^4 g^{(4)} + (even) + r^n \log(r)\mathcal{A} + r^n g^{(n)} + \dots$$

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Here, $\mathcal{A} = 0$ if n is odd.The tracefree part of $g^{(n)}$ is formally undetermined. The trace is determined, and vanishes if n is odd.

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Let r be a geodesic defining function. One has

$$\mathrm{vol}_{g_+}(\{r > \varepsilon\}) = c_0\varepsilon^{-n} + c_1\varepsilon^{1-n} + \cdots + c_{n-1}\varepsilon^{-1} + \mathcal{E}\log\left(\frac{1}{\varepsilon}\right) + V + o(1)$$

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$$dV_{g_+} = r^{-(1+n)}(1 + r^2 v_2 + r^4 v_4 + \dots)dV_h dr.$$

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Mazzeo (1991) proved polyhomogeneity:

$$u(x) = r + r^2 u^{(2)} + r^3 u^{(3)} + \cdots + r^{n+1} u^{(n+1)} + r^{n+2} \log(r) \mathcal{A} + r^{n+2} u^{(n+2)} + \ldots.$$

Consider

$$\mathrm{vol}_g(\{r > \varepsilon\}) = c_0\varepsilon^{-n} + c_1\varepsilon^{1-n} + \cdots + c_{n-1}\varepsilon^{-1} + \mathcal{E}\log\left(\frac{1}{\varepsilon}\right) + V + o(1)$$

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Theorem (Graham 2017; Gover and Waldron, 2017)

The energy $\mathcal{E}$ is a conformal invariant.

Suppose $\mathcal{F} : (-\varepsilon, \varepsilon) \times \Sigma \hookrightarrow M$, and $X = \frac{d}{ds}\mathcal{F}(s, \cdot)\big|_{s=0} \in \Gamma(\Sigma, N\Sigma)$. Let μ be the inward-pointing unit normal.

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Theorem

$$\frac{d}{ds}\mathcal{E}\bigg|_{s=0} = c_n \int_\Sigma \langle X, \mu\rangle \mathcal{A} dV_h,$$

where $h = \bar{g}|_{T\Sigma}$.

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Recall, near Σ, we may decompose M

$$\begin{aligned} M &\approx [0,\delta)_t \times SN\Sigma \\ &\approx [0,\delta)_t \times \Sigma \times S^{k-1} \text{ (locally)}. \end{aligned}$$

Then u has an expansion

$$u = t + t^2 u^{(2)} + t^3 u^{(3)} + \cdots + t^{n+1} u^{(n+1)} + t^{n+1+\delta} u^{(n+1+\delta)} + t^{n+2} \log(t) \mathcal{A} + t^{n+2} u^{(n+2)} + o(t^{n+2}),$$

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3. $\mathcal{A}$ is locally determined and linear.

Consider the expansion

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Theorem (Kushtagi-M. 2024)

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Lemma (Kushtagi-M. 2024)

The function $\frac{u}{t}$ is smooth up through order t^n. Moreover, if n is odd, then $\mathcal{A} = 0$.

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Theorem (Kushtagi-M., 2024)

If n is even, then

$$\frac{d}{ds}\mathcal{E}\bigg|_{s=0} = c_{n,k} \int_\Sigma \mathcal{A}(X) dV_h.$$

If n is odd, then

$$\frac{d}{ds}V\bigg|_{s=0} = c_{n,k} \int_\Sigma u^{(n+2)}(X) dV_h.$$

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Theorem (Kushtagi-M. 2025)

Let Σ^n be embedded in $(M^{n+k}, \bar{g})$. For $0 \leq j \leq \frac{n}{2}$ (if n is even or $(n,k) \in \mathcal{E}$) or for $j \geq 0$ (if n is odd and $(n,k) \notin \mathcal{E}$), there exists a natural, extrinsically defined differential operator $P_j : C^\infty(\Sigma) \to C^\infty(\Sigma)$ of order $2j$, with the same principal part as Δ_Σ^j, and under conformal change $\tilde{g} = e^{2\omega} g$ satisfying

$$\widetilde{P}_j = e^{(-n/2-j)\omega} P_j e^{(n/2-j)\omega}.$$

This P_j is formally self-adjoint.

(Compare GJMS operators, Gover-Waldron, . . .).

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Impose $F|_\Sigma = f$. Define $P_j f = G|_\Sigma$. Note that one could now consider other expansions and get higher-rank operators.

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$$I_{s,\sigma} = \Delta_{S^{k-1}} + (s(n-s) - \sigma(n-\sigma)).$$

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Proposition (Kushtagi-M. 2025)

The equation

$$\Delta_g U = -n + O(t^{n+1} \log t)$$

has a solution of the form

$$U = \log t + A + Bt^n \log t + O(t^n),$$

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with $A|_\Sigma = 0$*; and* U *is unique mod* $O(t^n)$*.*

We define $Q_{n,k} = c_{n,k} B|_\Sigma$.

Theorem (Kushtagi-M. 2025)

1. *Suppose* $\tilde{g} = e^{2\omega} g$. *Then*

$$e^{n\omega} \widetilde{Q}_{n,k} = Q_{n,k} + P_n \omega.$$

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2. $$\mathcal{E} = a_{n,k}\int_\Sigma Q dV_h$$

Happy Birthday, Kengo!