Canonical connections of geometric structures

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- We will restrict our discussion to a special type of geometric structures, called G-structures. They include most of the interesting examples and their structure theory is well developed.

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- Let $\mathbb{F}_x M := \mathrm{Isom}(V, T_x M)$ be the set of all frames at x. The $\mathrm{GL}(V)$-principal bundle

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- For a complex Lie subgroup $G \subset \mathrm{GL}(V)$, a G-principal subbundle $\mathcal{G} \subset \mathbb{F}M$ is called a G-structure with the structure group G on M.

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- Conversely, a G-structure on M with the structure group $\mathrm{O}(V, \sigma) \subset \mathrm{GL}(V)$ determines a Riemannian metric on M.

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- Such a G-structure is called a pseudo-product structure on M if the two distributions $\mathcal{U}$ and $\mathcal{W}$ are integrable. In other words, it is a pair of transversally intersecting foliations on M.

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- Such a G-structure is also called
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  - an absolute parallelism on M.
- By fixing a basis of V, an absolute parallelism can be represented as a collection of vector fields $\vec{v}_1, \ldots, \vec{v}_n$ on M which gives a basis of $T_x M$ at every $x \in M$.

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- We say that two G-structures $\mathcal{G} \subset \mathbb{F}M$ and $\widetilde{\mathcal{G}} \subset \mathbb{F}\widetilde{M}$ with the same structure group $G \subset \mathrm{GL}(V)$ are equivalent if there exists a biholomorphic map $\varphi : M \to \widetilde{M}$ such that the induced map $\varphi_* : \mathbb{F}M \to \mathbb{F}\widetilde{M}$ sends $\mathcal{G}$ to $\widetilde{\mathcal{G}}$.

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- We say that they are locally equivalent if there are open subsets $O \subset M$ and $\widetilde{O} \subset \widetilde{M}$ such that the G-structures obtained by restrictions

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- **Basic Problem** Develop methods to check local equivalences of G-structures.

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- Two absolute parallelisms $\vec{v}_1, \ldots, \vec{v}_n$ on M and $\widetilde{\vec{v}}_1, \ldots \widetilde{\vec{v}}_n$ on $\widetilde{M}$ are equivalent iff there exists a biholomorphic map $\varphi : M \to \widetilde{M}$ such that

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- $c^i_{jk} \equiv 0$ for all i, j, k iff there exist local coordinates $z^1, \ldots, z^n$ such that $\vec{v}_i = \frac{\partial}{\partial z^i}, i = 1, \ldots, n$.
- $c^i_{jk} \equiv \mathrm{constant}$ for all i, j, k iff $\vec{v}_1, \ldots, \vec{v}_n$ are locally equivalent to a basis of left-invariant fields (Maurer-Cartan absolute parallelism) on an n-dimensional Lie group.

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$$\Longleftrightarrow$$

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$$\mathcal{P} = \mathcal{G}^{(k)} \to \mathcal{G}^{(k-1)} \to \cdots \to \mathcal{G}^{(1)} \to \mathcal{G}^{(0)} \to M,$$
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- If this happens, we call the pair $(\mathcal{P}, \theta)$ a Cartan connection on M. In this case, the torsion function of θ can be regarded as a tensor field (= curvature tensor of the G-structure) on M in a suitable sense.

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- The condition that $G \subset \mathrm{GL}(V)$ is **of finite type** means $\mathfrak{g}^{(k)} = 0$ for some positive integer k. Thus the above procedure, called **Cartan prolongations**, terminates at the k-th step.

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- Thus if we can **find a natural splitting** $T\mathcal{G} = T^{\mathrm{vert}} \oplus \mathcal{H}$, we **obtain a natural absolute parallelism** $\theta^{\mathrm{vert}} \oplus \theta^{\mathcal{H}}$ on $\mathcal{G}$ by
$$\theta_h^{\mathrm{vert}} \oplus \theta^{\mathcal{H}_h} : \mathfrak{g} \oplus V \to T_h^{\mathrm{vert}} \oplus \mathcal{H}_h = T_h\mathcal{G}.$$

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- The value at $x \in M$ of the torsion function

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- Thus we can define the torsion of a horizontal subspace:

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$\Downarrow$

its **torsion** $c(\mathcal{H}_h) \;\in \mathrm{Hom}(\wedge^2 V, V)$.

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$$\begin{array}{ccc} \text{variation } \mathcal{H}_{h,\varepsilon} \text{ of } \mathcal{H}_h & \Leftrightarrow & \varepsilon \in \mathrm{Hom}(V, \mathfrak{g}) \\ \text{torsion} \downarrow & & \downarrow \partial \\ c(\mathcal{H}_{h,\epsilon}) - c(\mathcal{H}_h) & = & \partial\varepsilon \in \mathrm{Hom}(\wedge^2 V, V) \\ \Uparrow & & \cup \\ \textbf{change in torsions} & \in & \mathrm{Im}(\partial) \end{array}$$

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- If $\mathfrak{g}^{(1)} = 0$, we have a canonical splitting $T\mathcal{G} = T^{\mathrm{vert}} \oplus \mathcal{H}$ with $c(\mathcal{H}) \in \mathbf{W}$, which gives a canonical absolute parallelism θ on $\mathcal{G}$.

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- The successive definitions of $\mathfrak{g}^{(i)} \subset \mathrm{Hom}(V, \mathfrak{g}^{(i-1)})$ and $\mathcal{G}^{(i)} \to \mathcal{G}^{(i-1)}$ are similar, although more complicated.

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If $\mathfrak{g} \subset \mathfrak{gl}(V)$ *is an* **irreducible representation**, *then it is* **of finite type** *unless*

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- **Example.** The conformal orthogonal group $\mathrm{CO}(V, \sigma)$ has $\mathfrak{g}^{(1)} \neq 0, \mathfrak{g}^{(2)} = 0$. A conformal structure has a natural Cartan connection on $\mathcal{G}^{(1)}$.

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- This means that **there are many interesting examples of geometric structures to which Cartan's Theorem cannot be applied.**

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- The key point is to **replace the commutative vector space** V in our discussion of G-structures **by a noncommutative nilpotent Lie algebra** arising from the successive Lie brackets.

Tanaka's refinement of Cartan prolongation

- For G-structure determined by conditions on a distribution, Noboru Tanaka discovered that if the underlying distribution is bracket-generating, namely, successive Lie brackets of sections of the distributions span TM, then we can refine Cartan's method.
- The key point is to **replace the commutative vector space** V in our discussion of G-structures **by a noncommutative nilpotent Lie algebra** arising from the successive Lie brackets.
- We need to change the definitions of $\mathfrak{g}^{(i)}$, incorporating the nilpotent structure. Then many examples satisfy $\mathfrak{g}^{(k)} = 0$ for some positive integer k in this new sense, even when $\mathfrak{g}$ is of infinite type in the previous sense.