Recent Developments in Conformal Submanifold Geometry

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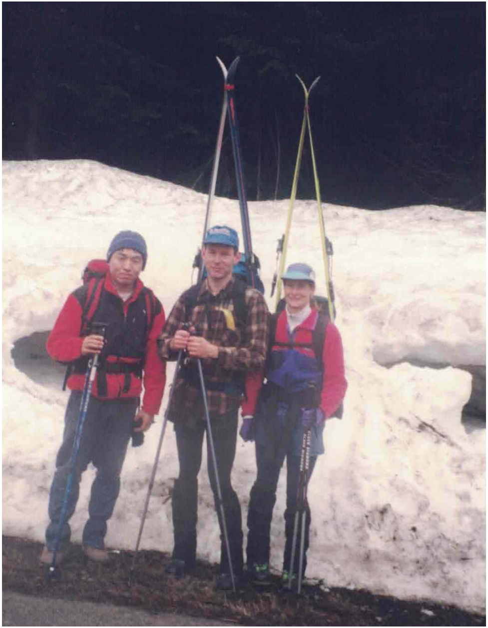
Osaka, December, 1990

Kyoto, December, 1990



Mt. St. Helens, Washington, March, 1998

HAPPY BIRTHDAY

KENGO!!!

Motivation and References

Motivation: Derive a formula for renormalized area analogous to the formula of Chang-Qing-Yang for renormalized volume

Required development of topics in conformal submanifold geometry

References:

1. Case, Graham, Kuo, Tyrrell and Waldron: A Gauss–Bonnet formula for the renormalized area of minimal submanifolds of Poincaré–Einstein manifolds, arXiv:2403.16710, just appeared *Comm. Math. Phys*.
2. Case, Graham and Kuo: Extrinsic GJMS operators for submanifolds, arXiv:2306.11294, to appear *Rev. Mat. Iberoam.*
3. Graham and Kuo: Geodesic normal coordinates and natural tensors for pseudo-Riemannian submanifolds, arXiv:2411.09679.

Chang-Qing-Yang for Renormalized Volume

(X^{n+1}, g_+) PE, n odd. Conformal infinity $(M, \mathfrak{c})$. Choice of $g \in \mathfrak{c}$ determines an r near M by: $(r^2 g_+)|_{TM} = g$, $|dr|_{r^2 g_+} = 1$.

$$\mathrm{Vol}(X \cap \{r > \epsilon\}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \text{(odd powers)} + c_{n-1}\epsilon^{-1} + V + o(1)$$

V is independent of choice of $g \in \mathfrak{c}$. Absolute numerical invariant

Theorem (Chang-Qing-Yang, 2006): Let n be odd.
There is a scalar pointwise conformal invariant $\mathcal{Z}$ of weight $-n$ so that if (X^{n+1}, g_+) is PE, then

$$V = b_n \chi(X) + \int_X \mathcal{Z}_{g_+}\, dv_{g_+}.$$

Three main ingredients:

- Scattering compactification
- Critical GJMS operator and Q-curvature
- Alexakis' decomposition for conformally invariant integrals

Scattering Compactification

Setting: (X^{n+1}, g_+) Poincaré–Einstein, n odd.

Theorem (Fefferman–G, 2002): Let $g \in \mathfrak{c}$ and r be the associated geodesic defining function. There is a unique $v \in C^\infty(\mathring{X})$ solving

$$\Delta_{g_+} v = -n, \qquad v = \log r + a + br^n$$

where a, $b \in C^\infty(X)$ are even modulo $O(r^\infty)$ and $a|_M = 0$.

Moreover, $\int_M b|_M \, dv_g = V$. v is called the scattering potential.

Now $e^v = re^{a+br^n}$ is a defining function. $= r + O(r^3)$.

Set $\widehat{g} := e^{2v} g_+$ Smooth metric on X.

Called the scattering compactification associated to $g \in \mathfrak{c}$.

Critical GJMS Operator and Q-curvature

(N^d, g) conformal manifold, d even

P_d: critical GJMS operator. Conformally covariant Self-adjoint.

$$P_d = (-\Delta_g)^{d/2} + lots, \qquad P_d 1 = 0$$

Q: Branson's Q-curvature. Natural scalar

If $\widehat{g} = e^{2\omega} g$, then

$$e^{d\omega} \widehat{Q} = Q + P_d \omega.$$

Follows that $\int_N Q\, dv_g$ is conformally invariant if N compact.

Factorization for Einstein metrics:

Suppose $\mathrm{Ric}(g) = \lambda(d-1)g$. Then

$$P_d = \prod_{j=1}^{d/2} (-\Delta_g + \lambda c_j), \qquad c_j = (\tfrac{d}{2} + j - 1)(\tfrac{d}{2} - j).$$

and

$$Q_d = \lambda^{d/2}(d-1)!$$

Alexakis Decomposition

d even.

Theorem (Alexakis): Let $\mathcal{I}$ be a scalar invariant of Riemannian d-manifolds such that $\int_N \mathcal{I}\, dv$ is conformally invariant for N compact. Then

$$\mathcal{I} = c\,\mathrm{Pff} + \mathcal{Z}_{\mathcal{I}} + \mathrm{div}\, T,$$

where

- Pff = Pfaffian
- $\mathcal{Z}_{\mathcal{I}}$ is pointwise conformally invariant
- T is a natural vector field.

Can apply to $\mathcal{I} = Q$. The $\mathcal{Z}$ in the CQY Theorem will be a multiple of $\mathcal{Z}_Q$.

The Chang-Qing-Yang Proof

Have (X^{n+1}, g_+) with n odd. Take $d = n+1$. Consider the scattering compactification $\widehat{g} = e^{2v} g_+$. What is $\widehat{Q}$?

$$e^{nv}\widehat{Q} = Q(g_+) + P_{n+1}(g_+)v.$$

But g_+ is Einstein with $\lambda = -1$. So $Q(g_+) = (-1)^{(n+1)/2} n!$.

And $P_{n+1}(g_+)$ is a product of $-\Delta_{g_+} - c_j$. But $\Delta_{g_+} v = -n$.

So $P_{n+1}(g_+)v$ is a constant. Calculate, get $\widehat{Q} = 0$.

Now write down the Alexakis decomposition for Q:

$$Q = c\,\mathrm{Pff} + \mathcal{Z}_Q + \mathrm{div}\, T.$$

Apply to $\widehat{g}$:

$$0 = c\widehat{\mathrm{Pff}} + \widehat{\mathcal{Z}_Q} + \widehat{\mathrm{div}}\,\widehat{T}.$$

Integrate over $(X, dv_{\widehat{g}})$. Last term gives $\int_M \langle \widehat{n}, \widehat{T} \rangle \, dv_g$.

Some work shows $\langle \widehat{n}, \widehat{T} \rangle = c' b|_M$, so $\int_M \langle \widehat{n}, \widehat{T} \rangle \, dv_g = c' V$.

Minimal Submanifolds of Poincaré–Einstein Spaces

Interested in minimal submanifolds $Y^{k+1} \subset (X^{n+1}, g_+)$.

Require $Y \cap M = \Sigma^k$ is an embedded submanifold of M.

Interesting subject in geometry

Plateau problem at infinity: given Σ, find Y.

Much existence and regularity theory, especially for $X = \mathbb{H}^{n+1}$

Anderson (1980's). For $X = \mathbb{H}^{n+1}$, always exists such a Y

Y may have singularities: it is a current

Such Y also arise in the AdS/CFT correspondence

"submanifold observables"

We will consider Y which are regular at infinity.

Renormalized Area

Area of any such Y is infinite. Renormalize similarly to volume.

(X, g_+) PE, $\quad Y^{k+1} \subset X^{n+1}$ minimal, $\quad$ assume k is odd.

Set $h_+ = g_+|_{T\mathring{Y}}$. h_+ is an asymptotically hyperbolic metric on $\mathring{Y}$.

Again choose $g \in \mathfrak{c}$. $\quad r =$ associated geodesic defining function.

Asymptotic expansion for area:

$$\text{Area}(Y \cap \{r > \epsilon\}) = a_0\epsilon^{-k} + a_2\epsilon^{-k+2} + (\text{odd powers}) + a_{k-1}\epsilon^{-1} + A + o(1)$$

Then A is independent of choice of g.

A is the renormalized area of $Y \subset X$.

Physical interpretation of A: entanglement entropy

Low Dimensions: Y^2 and Y^4

Would like a geometric understanding of A, along the lines of the Chang-Qing-Yang formula.

Alexakis-Mazzeo (2010): $Y^2 \subset (X^{n+1}, g_+)$ minimal. Then

$$A = -2\pi\chi(Y) + \int_Y \left(W^T - \tfrac{1}{2}|\mathring{L}|^2 \right) da_{h_+}$$

L: second fund. form of Y with respect to g_+, $\mathring{L}$: trace-free part.

W^T: tangential component of Weyl tensor of g_+ along Y
$W^T = W(e_1, e_2, e_1, e_2)$, $\{e_1, e_2\}$: orthonormal basis for TY.

$\mathring{L}$ and W^T are conformally invariant, integral converges

Aaron Tyrrell derived a similar formula for $Y^4 \subset X^5$.

Integrand is not conformally invariant.

Subtle *ad hoc* analysis required to deduce convergence.

Our Result

Theorem (modulo a conjecture): Let k be odd. There is a scalar pointwise conformal submanifold invariant $\mathcal{W}$ of weight $-k$ so that if (X^{n+1}, g_+) is PE and $Y^{k+1} \subset X$ is minimal, then

$$A = c_k \chi(Y) + \int_Y \mathcal{W}\, da_{h_+}$$

Again, integral converges by conformal invariance.

Definition: A scalar Riemannian submanifold invariant of submanifolds $S^\ell \subset (N^d, g)$ (or natural submanifold scalar) is a linear combination of complete contractions of

$$\pi_1(\nabla^{M_1} \mathrm{Rm}) \otimes \cdots \otimes \pi_p(\nabla^{M_p} \mathrm{Rm}) \otimes \overline{\nabla}^{N_1} L \otimes \cdots \otimes \overline{\nabla}^{N_q} L.$$

$\nabla^{M_i} \mathrm{Rm}$: for g, $\quad \pi_i$: project to TS or NS in each index.

$\overline{\nabla}^{N_j} L$: covariant derivatives of second fundamental form.

Contractions using some pairings of tangential and normal indices.

Minimal Submanifold Scattering Compactification

Proof of Theorem follows same outline as CQY argument, applied on Y. Must develop ingredients for submanifolds.

Scattering compactification is easy. Work intrinsically on Y.

$(\mathring{Y}, h_+)$ is an AH manifold with conformal infinity $(\Sigma, [h])$.

Scattering construction works the same way on a general AH manifold.

Get scattering potential from same equation

$$\Delta_{h_+} v = -k, \quad \text{and get} \quad \int_\Sigma b|_\Sigma = c' A.$$

Scattering compactification same way: $\widehat{h} = e^{2v} h_+$.

For our purposes, need to extend v and $\widehat{h}$ off Y near Σ in a good way.

Extrinsic Submanifold GJMS Operators

General setting: $S^\ell \subset (N^d, g)$. $h = g|_{TS}$ induced metric.

Want submanifold version of GJMS operators with factorization and constant Q property if g is Einstein and S is minimal.

Try intrinsic operators on (S, h). Doesn't work. h not Einstein.

Gover-Waldron have extrinsic operators/Q in hypersurface case from singular Yamabe problem. Higher codimension extension: Kushtagi-McKeown. Also doesn't work.

Need new construction of extrinsic submanifold operators and Q.

Existence Result for Extrinsic GJMS Operators

Theorem: Let $d \geq 3$, $1 \leq \ell \leq d-1$. For the following values of m, there exists an extrinsic GJMS operator $P_{2m} : C^\infty(S) \to C^\infty(S)$.

- $1 \leq m < \infty$ if d and ℓ are both odd,
- $1 \leq m < d/2$ if d is even and ℓ is odd,
- $1 \leq m \leq \ell/2 + 1$ if ℓ is even. (If $m = \ell/2 + 1$ and d is even, also assume $d > \ell + 2$.)

P_{2m} is natural, self-adjoint, leading term $(-\Delta_h)^m$ and

$$\widehat{P}_{2m} = e^{(-\ell/2-m)\,\omega|_\Sigma} \circ P_{2m} \circ e^{(\ell/2-m)\,\omega|_\Sigma}, \qquad \widehat{g} = e^{2\omega} g.$$

Have $P_{2m}1 = \frac{\ell-2m}{2} Q_{2m}$. If ℓ is even, analytically continue to obtain Q_ℓ: extrinsic critical Q-curvature.

$$e^{\ell\,\omega|_\Sigma} \widehat{Q}_\ell = Q_\ell + P_\ell(\omega|_\Sigma).$$

Factorization for Extrinsic Operators

The extrinsic operators satisfy the same factorization for minimal submanifolds of Einstein manifolds that the usual GJMS operators satisfy for Einstein manifolds!

Theorem: Suppose $\mathrm{Ric}(g) = \lambda(d-1)g$ and $S \subset (N, g)$ is minimal. Then

$$P_{2m} = \prod_{j=1}^{m}(-\Delta_h + \lambda c_j), \qquad c_j = (\tfrac{\ell}{2} + j - 1)(\tfrac{\ell}{2} - j).$$

If ℓ is even, then

$$Q_\ell = \lambda^{\ell/2}(\ell - 1)!$$

$P_2,\ Q_2$

Notation: Use local coordinates $z^i = (x^\alpha, u^{\alpha'})$, where $1 \le i \le d$, $1 \le \alpha \le \ell$, $\ell + 1 \le \alpha' \le d$. Always assume $S = \{u = 0\}$ and $\partial_\alpha \perp \partial_{\alpha'}$ on S.

Schouten tensor: $\mathrm{P}_{ij} := \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)} R g_{ij}\right)$

Decomposes into $\mathrm{P}_{\alpha\beta}$, $\mathrm{P}_{\alpha\alpha'}$, $\mathrm{P}_{\alpha'\beta'}$

Second fundamental form: $L(X, Y) := ({}^g\nabla_X Y)^\perp$. Written $L_{\alpha\beta}^{\alpha'}$

Mean curvature vector: $H^{\alpha'} := \frac{1}{\ell} h^{\alpha\beta} L_{\alpha\beta}^{\alpha'}$. Then

$$P_2 = -\Delta_h + \frac{\ell - 2}{2} Q_2, \qquad Q_2 = \mathrm{P}_\alpha{}^\alpha + \frac{\ell}{2}|H|^2.$$

Note Q_2 is constant if g Einstein and S minimal.

Intrinsic operator is the conformal Laplacian

$$\overline{P}_2 = -\Delta_h + \frac{\ell - 2}{2}\overline{Q}_2, \qquad \overline{Q}_2 = \frac{\overline{R}}{2(\ell - 1)}$$

$\overline{R}$ need not be constant.

Construction of Usual GJMS Operators

Original construction used ambient metric. Later reformulated in terms of formal Poincaré metric.

Given (N^d, g), construct formal even Poincaré metric

$g_+ = r^{-2}(dr^2 + g_r)$ on $X = N \times (0, \epsilon)$.

The GJMS operators arise as obstructions to smooth extension as eigenfunctions of Δ_{g_+}.

Given $f \in C^\infty(N)$, search for $F \in C^\infty(X)$ so that $F|_{r=0} = f$ and $u := r^{d/2-m} F$ satisfies

$$\left[\Delta_{h+} + \left((d/2)^2 - m^2\right)\right] u = O(r^\infty).$$

P_{2m} is the obstruction to smooth extension at order $2m$.

Can carry out the same construction replacing g_+ by any AH metric. Get differential operators P_{2m} on N. But they depend on the AH metric, not just on (N, g).

Minimal Submanifold Extension

Our construction associates an AH metric to $S^\ell \subset (N^d, g)$.

First extend the background space to (X, g_+) just as before:
$X = N \times [0, \epsilon)_r$, $g_+ = r^{-2}(dr^2 + g_r)$.

Then search for a submanifold $Y \subset X$ satisfying

- Y is asymptotically minimal with respect to g_+
- $Y \cap N = S$
- Y is smooth and even

If d, ℓ both odd, there exists unique Y to infinite order.
If ℓ even, obstructed at order $\ell + 2$

Let $h_+ =$ metric induced on Y by g_+.

Then (Y, h_+) is AH with conformal infinity $(S, [h])$

Apply usual GJMS construction on (Y, h_+) to get extrinsic operators.

Call them minimal submanifold extrinsic operators because the construction involves the minimal extension of the submanifold.

Proof of Factorization

Can identify explicitly the Poincaré metric for an Einstein metric.

If $\mathrm{Ric}(g) = \lambda(d-1)g$, then

$$g_+ = r^{-2}\big(dr^2 + g_r\big), \qquad g_r = \big(1 - \tfrac{1}{4}\lambda r^2\big)^2 g$$

GJMS algorithm reduces to a recursion for polynomials in Δ. Dual Hahn polynomials. Gives factorization for usual GJMS operators.

Key observation: If $S \subset (N, g)$ is minimal, then the minimal extension is $Y = S \times \mathbb{R}_r$. Uses that g_+ is a warped product. So

$$h_+ = r^{-2}\big(dr^2 + h_r\big), \qquad h_r = \big(1 - \tfrac{1}{4}\lambda r^2\big)^2 h.$$

So induced metric has the same form. Get same recursion.

Induced metric h on S is not Einstein, induced metric h_+ on Y is not asymptotically Einstein, but still get the same recursion and the same factorization.

Conjectured Alexakis Decomposition for Submanifolds

Conjecture: Fix ℓ even and $d > \ell$. Let $\mathcal{J}$ be a natural scalar invariant of ℓ-dimensional submanifolds S of d-dimensional Riemannian manifolds (N, g). Suppose that for compact $S \subset N$, $\int_S \mathcal{J}\, dv_h$ is invariant under conformal rescaling of g. Then

$$\mathcal{J} = c\,\mathrm{Pff}_h + \mathcal{W} + \mathrm{div}_h\, T, \qquad \text{where}$$

Pff_h = Pfaffian for the induced metric

$\mathcal{W}$ is a scalar pointwise conformal submanifold invariant of wt. $-\ell$.

T is a submanifold natural vector field.

Chang–Qing–Yang proof then goes through. Apply Conjecture to critical extrinsic Q for scattering compactification. Integrate, etc

Only need conjecture for $\mathcal{J}$ = critical extrinsic Q. Have identified decomposition explicitly for Q for $\ell = 2, 4$. Recovers Alexakis–Mazzeo and gives generalization of Tyrrell for $Y^4 \subset (X^{n+1}, g_+)$, with a conformally invariant integrand.

Conjecture is true for $\ell = 2, 4$

We identify some new submanifold conformal invariants of weight -4 in general dimension and codimension. One of them enters into the decomposition for Q when $\ell = 4$ and thus into the formula for renormalized area.

Theorem: The conjecture is true for $\ell = 2, 4$.

The proof is by brute force. But there are very many terms to consider for $\ell = 4$. Write down all natural submanifold scalars of homogeneity -4 and throw out all you can that are conformally invariant or divergences, using obvious ones and our new invariants. After that are left with 33 terms. Use Gauss-Codazzi to rewrite. Then calculate conformal change of a linear combination of all 33 terms and integrate and show only possibility is Pfaffian.

Case, Khaitan, Lin, Tyrrell and Yuan have recently given a new proof of CQY result that avoids Alexakis' Theorem and explicitly identifies the conformal invariant. arXiv:2404.11319. Can this method be extended to the submanifold case?