On a problem of conformal fill in by Poincare Einstein metrics

Sun-Yung Alice Chang , Princeton University
Report of joint works with
Yuxin Ge, University of Toulouse, France

Osaka Workshop on Conformal and CR Geometry
Celebration of 60th birthday of Kengo Hirachi

§1. Conformal fill ins by Einstein manifolds

Given a compact manifold (M^n, h), when is it the boundary of a conformally compact (Poincaré) Einstein manifold (X^{n+1}, g^+) with $\rho^2 g^+|_M = h$, where ρ is a defining function on X? This problem of finding "conformal fill in" is motivated by:

- The AdS/CFT correspondence in quantum gravity (proposed by Maldacena also Witten, around 1998)

- Geometric considerations to study the structure of non-compact asymptotically hyperbolic manifolds.

§1. Conformally compact Einstein manifolds, Definition

- On a manifold X with boundary M, we call ρ a defining function on X, if $\rho > 0$ *on* X, $\rho = 0$ *on* M and $d\rho \neq 0$ on M.

- (X^{n+1}, g^+) is conformally compact if $(\bar{X}^{n+1}, \rho^2 g^+)$ is compact. Denote $h = \rho^2 g^+|_M$, we call $(M^n, [h])$ the conformal infinity of (X^{n+1}, g^+), where $[h]$ denotes the conformal class of metrics of h, i.e. the collection of metrics $\phi^2 h$ for some function ϕ on M.

- If $\mathrm{Ric}[g^+] = -n \; g^+$, we call (X^{n+1}, M^n, g^+) a conformally compact (Poincaré) Einstein (CCE) manifold.

- We remark on a CCE manifold, special ρ (called the geodesic defining function) can be chosen, with $|\nabla_{(r^2 g^+)} r| \equiv 1$ in an nbhd of $M \times (0, \epsilon)$ for some $\epsilon > 0$, so that $r^2 g^+$ is with totally geodesic boundary. We remark as a consequence, all other compactified metrics of g^+ are umbilic.

§1. Examples of CCE manifold

- Example 1.

On $(\mathbb{R}^{n+1}_+, \mathbb{R}^n, g_{\mathbb{H}})$, where $g_{\mathbb{H}} = \frac{dx^2+dy^2}{y^2}$, $x \in \mathbb{R}^n$, $y > 0$. Choose $r = y$, then $(\mathbb{R}^{n+1}_+, dx^2 + dy^2)$ is not compact, but conformal to $g_{\mathbb{H}}$, with conformal infinity $(\mathbb{R}^n, [dx^2])$.

- Example 2.

On $(\mathbb{B}^{n+1}, \mathbb{S}^n, g_{\mathbb{H}})$, where $(\mathbb{B}^{n+1},\ g_{\mathbb{H}} = (\frac{2}{1-|x|^2})^2|dx|^2))$. Choose

$$r := 2\,\frac{1-|x|}{1+|x|},$$

$$g_{\mathbb{H}} = g^+ = r^{-2}\left(dr^2 + (1-\frac{r^2}{4})^2\ g_c\right).$$

with $(\mathbb{S}^n, [g_c])$ as conformal infinity.
We remark that $r = e^{-2d}$, where $d(x) = dist_{g^+}(0,x)$.

§1. Examples of CCE manifold

- Example 3.

On $\mathbb{S}^1(\lambda) \times \mathbb{S}^2$ with the product metric, when $0 < \lambda < \frac{1}{\sqrt{3}}$, there are at least 3 different "conformal fill ins".

(a) One is when X is $(\mathbb{S}^1(\lambda) \times \mathbb{B}^3)$ with the fill in the hyperbolic metric $g^+ = f(x)dt^2 + g_{\mathbb{H}^3(x)}$.

(b) The other two: X is the AdS-Schwarzchild space $(\mathbb{R}^2 \times \mathbb{S}^2, g_m^+)$, where

$$g_m^+ = Vdt^2 + V^{-1}dr^2 + r^2 g_c,$$

$$V = 1 + r^2 - \frac{2m}{r}.$$

It turns out for $\lambda < \frac{1}{\sqrt{3}}$, there are two different choices of m. This is the famous "non-unique fill in" example of Hawking-Page '83.

§1. Examples of CCE manifold

Some recent work of C., Paul Yang and Ruobing Zhang

Theorem

Given $n \geqslant 3$ and $\lambda > 0$, let (X^{n+1}, g^+) be a complete Poincaré-Einstein manifold with conformal infinity $(S^1(\lambda) \times S^{n-1}, [h_\lambda])$ such that $Ric_{g^+} = -ng^+$. If $\sec_{g^+} \leqslant 0$, then X^{n+1} is diffeomorphic to $S^1 \times B^n$ and g has constant sectional curvature -1.

Theorem

Given $n \geqslant 3$, there exists a positive number $\lambda_0(n) >> 1$ such that the following property holds. Let (X^{n+1}, g^+) be a complete Poincaré-Einstein manifold whose conformal infinity is given by $(S^1(\lambda) \times S^{n-1}, [h_\lambda])$. If the normalized product metric h_λ satisfies $\lambda \geqslant \lambda_0$, then X^{n+1} is diffeomorphic to $S^1 \times B^n$ and g is isometric to a hyperbolic metric with constant sectional curvature -1.

§1. Some earlier existence and non-existence results

• "Ambient Metric" of Fefferman-Graham '85. On any compact manifold (M^n, h), h real analytic, there is a CCE metric on some $M^n \times (0, \epsilon)$ of M. Gursky-Székelyhidi '17, extend to smooth h .

• Graham-Lee '91: Any h in some small $C^(2, \alpha)$ neighborhood of h_c on $\mathbb{S}^n$. We remark that the fill in metrics constructed by Graham-Lee g^+ for h all exist in a small nbhd of the Hyperbolic metric, it turns out they are "unique" by a later result of C-Ge-Qing, '21.

• Gursky-Han '17 and Gursky-Han-Stolz '18 constructed many examples of boundary conformal classes that do not allow Poincaré-Einstein extensions on specified manifolds X^{4k} for $k \geqslant 2$.
Theorem (J, Lee '95). On CCE manifolds, if $R(h) > 0$, then $\lambda_1(-\Delta_{g^+}) \geqslant \frac{n^2}{4}$.
Corollary (J.Qing '03) On CCE manifolds, if $R(h) > 0$, then there exists a compactified metric g with $g|_M = h$ and $R(g) > 0$.

§2. Compactness of CCE manifolds – the set-up

- An open question: Does the entire class of metrics $(\mathbb{S}^3, h)$ with positive scalar curvature allow CCE filling in $\mathbb{B}^4$?
The class is path-connected by a result of F. Marques '12.
The index argument for non-existence of Gursky-Han, Gursky-Han-Stolz does not apply.

- We propose to study the "compactness" problem, which hopefully will lead to some degree theory argument for the positive answer to the question above. More precisely, we ask the question:

Given a sequence of $(\mathbb{M}^n, [h_i])$ metrics with positive Yamabe constants, which are conformal infinity of CCE $(\mathbb{X}^{n+1}, g_i^+)$, where $d = n+1$, when would

$$\{[h_i]\} \text{ forms a compact family on } \mathbb{M}^n$$

$$\Longrightarrow \{[g_i]\} \text{ forms a compact family on } X^{n+1}?$$

where g_i is some compactification of $\{g_i^+\}$ with $g_i|_M = h_i$.

§2. Compactness of CCE manifolds – the set up-

The difficulty of the problem lies in the existence of an "non-local" term.
We will illustrate the case on (X^4, M^3, g^+) CCE manifold with (M^3, h) conformal infinity, recall the asymptotic behavior

$$g := \rho^2 g^+ = d\rho^2 + h + g^{(2)}\rho^2 + g^{(3)}\rho^3 + g^{(4)}\rho^4 + \cdots,$$

where $g^{(2)} = -\frac{1}{2}(Ric_h - \frac{1}{4}R_h h)$ determined by h (a local terms), $Tr_h g^{(3)} = 0$, while

$$S_{\alpha,\beta} :=: -\frac{3}{2} g^{(3)}_{\alpha,\beta} = -\frac{\partial}{\partial n}(P(g))_{\alpha,\beta}$$

is a non-local term not determined by h.

We remark that h together with $g^{(3)}$ determines the asymptotic behavior of g. Fefferman-Graham '07, Biquard '08).

§3. Conformal invariants

Yamabe constant

- On (M^n, h), compact closed manifold, $Y(M,[h]) = \inf_{\tilde{h}\in[h]} \frac{\int_M R[\tilde{h}]dvol[\tilde{h}]}{Vol(M,\tilde{h})^{\frac{(n-2)}{n}}}$. We remark $Y(M,[h])$ corresponds to the "isoperimetric constant" of the Sobolev embedding of $W^{1,2}$ into $L^{\frac{2n}{n-2}}$.
- On compact manifold with boundary, there are two such constants. $(X^{n+1}, M^n, \bar{g})$

$$Y_a(X, M, [\bar{g}]) = \inf_{\tilde{g}\in[\bar{g}]} \frac{\int_X R[\tilde{g}]dvol[\tilde{g}] + c_n \int_M H[\tilde{g}|_M]d\sigma[\tilde{g}|_M]}{Vol(X,\tilde{g})^{\frac{(n-1)}{(n+1)}}}$$

$$Y_b(X, M, [\bar{g}]) = \inf_{\tilde{g}\in[\bar{g}]} \frac{\int_X R[\tilde{g}]dvol[\tilde{g}] + c_n \int_M H[\tilde{g}|_M]d\sigma[\tilde{g}|_M]}{Vol(M,\tilde{g}|_M)^{\frac{(n-1)}{n}}}.$$

Y_a and Y_b each corresponds to the (isoperimetric) constants in the Sobolev and Sobolev trace embeddings.

§3. Conformal invariants

• As we have mentioned before, it follows from result of J. Lee '95, and the observation by J. Qing, that on CCE setting, $Y(M, [h]) > 0$ implies that $Y_a(X, M, [g]) \geqslant 0$.

• Combining works of Gursky-Han '17, X. Chen- M. Lai and F. Wang '18, Chang-Ge '21 we established that, there exists some constant C_n, such that

$$Y_a(X, M, [g]) \geqslant C_n Y(M, [h])^{\frac{n}{n+1}}.$$

Recall X. Chen-M. Lai and F. Wang

$$Y_b(X, M, [g]) \geqslant C_n Y(M, [h])^{\frac{1}{2}}$$

§3. Conformal invariants

• Another conformally invariant quantity is Weyl curvature W. $|W|[\tilde{g}] = \rho^{-2}|W|[g]$, if $\tilde{g} = \rho^2 g$. Thus $\int_X |W|^{\frac{n+1}{2}}[g]dv_g$ is a conformal invariant.

• On compact 4-manifold X with boundary M, consider the functional

$$g --> \int_X |W|^2[g]dv_g + 8\int_M W_{n\alpha n\beta}L^{\alpha\beta}$$

Critical metric of (interior variation) is *Bachflat*, i.e. the Bach tensor $B_{ij} = \nabla^l\nabla^k W_{kilj} + \frac{1}{2}Ric^{kl}W_{kilj}$ vanishes. Critical metric for (boundary variation) gives rise to S tensor vanishes. Both Bach tensor and S tensor are pointwise conformal invariants. Einstein metrics are Bach flat, hence so are all metrics in the same conformal class of Einstein metric. Thus in a CCE setting (X, M, g^+), all compactified metrics of $[g^+]$ are Bach flat.

§3. Conformal invariants

• Bach flat metrics on compact metric are well studied in Tian-Viaclovsky '01-'03. In particular, they pointed out Bach flat can be viewed a 4th order system of PDE of elliptic type,

$$\Delta R_{ij} = c\nabla_i\nabla_j R + R_m * Ric,$$

which plays an important role in our estimates of the compactified metrics later. We also remark that for this PDE, the non-local tensor $S = -\frac{3}{2}g^{(3)} = \frac{\partial P}{\partial n}|_M$ is a natural matching boundary condition, where P is the Schouten tensor when the scalar curvature is a constant.

§4. Adapted metrics

• For convenience, we choose $h = h^Y \in [h]$, the Yamabe metric on M. But what is a good choice of the compactified metric $g \in [g^+]$? A first attempt is to choose $g = g^Y$, a Yamabe metric among compactified metrics of g^+. The difficulty of this choice is we do not know how to control the behavior of $g^Y|_M$ in terms of h^Y.

• Instead, following the work of J. Lee, Graham-Zworski, '03 we will make a choice of a special representative metric , which we call scalar flat Adapted metrics on X obtained by solving the Poisson equation $(*)_s$ the boundary metric h with $R(h) > 0$ on M.

$$(*)_s \quad -\Delta_{g^+} v - s(n-s)v = 0, \;\; X^{n+1},$$

with Dirichlet data $f \equiv 1$. Choose $\rho = v^{\frac{1}{n-s}}$ and denote the adapted metric $g* = \rho^2 g^+$.

• Properties of $(*)_s$ has been studied in Fefferman-Graham '02, Chang-Gonzalez '11, Case-Chang '16, F. Wang '21-'22 and S. Lee '23 and others, Lee's metric is the adapted metric when $s = n+1$. In the statement of the theorems below, we choose $s = \frac{n}{2} + 1$, call it the scalar flat adapted metric.

§4. Properties of the scalar flat adapted metric

On $(X, M, g+)$ CCE, for a given metric we have the adapted metric g^*, $g^*|_M = h$, with the key properties:
(1) $R[g*] = 0$ on X.
(2) $R[h] > 0$ on M implies the mean curvature $H > 0$ on M.
(3) Denote $g^* = \rho^2 g^+$, $|\nabla_{g*}\rho| \leqslant 1$.
(4) Gauss Bonnet formula

$$8\pi^2\chi = \int_X (\frac{1}{4}|W|^2 + \frac{1}{24}R^2 - \frac{1}{2}|E|^2) + \oint_M (\frac{4}{3}R[h]H - \frac{2}{27}H^3).$$

Hence Hence under the assumption $R[h] > 0$,

$$\int_X |E|^2 + \oint_M H^3 \leqslant C(\int_X |W|^2 + \oint_M (R[h])^3,)$$

where E denote the traceless Ricci.

Main result, a compactness theorem

Compactness Theorem (C and Yuxin Ge)

Let $\{X, M = \partial X, g_i^+\}$ be a family of 4-dimensional CCE manifolds. g_i is a sequence of adapted metrics. Denote $h_i = g_i|_M$. Assume

1. The boundary metric (M, h_i) is compact in $C^{k,\alpha}$ norm with $k \geqslant 6$; and there exists some positive constant $C_1 > 0$
$$Y(M, [h_i]) \geqslant C_1;$$
2. There exists some positive constant $C_2 > 0$ such that
$$\int_{X^4} |W[g_i]|^2 \leqslant C_2$$
3. $H^2(X, \mathbb{Z}) = H_2(X, \mathbb{Z}) = 0$.

Then, the sequence g_i is compact in $C^{k,\alpha'}$ norm for any $\alpha' \in (0, \alpha)$ up to a diffeomorphism fixing the boundary.

§4. Some ideas in the proof of the compactness theorem

In the case of boundary blow up, for suitably rescaled metrics $\bar{g}_i = \bar{\rho}_i^2 {g_i}^+$ with bounded curvature in C^1 norm.
Recall in the main theorem, we have only assumed that the Weyl curvature is bounded in L^2.

• $\bar{g}_i$ converges to some complete metric g_∞ with the boundary in pointed Gromov-Hausdorff sense and $\bar{\rho}_i \to \rho_\infty$.

• $g_\infty^+ :=: \rho_\infty^{-2} g_\infty$ is Einstein, thus g_∞ is conformal to Einstein and has flat boundary $\mathbb{R}^3$.

• **Decay of Ricci curvature**:

$$|Ric[g_\infty](x)| = o(\frac{1}{d(x,p)^2})$$

for some fixed $p \in X$.

§4. A Liouville type rigidity result

Proposition (Chang-Ge)

Let $(X_\infty, \mathbb{R}^3 = \partial X_\infty, g_\infty^+)$ be AH and $g_\infty = \rho_\infty^2 g_\infty^+$ be a complete metric with boundary and positive injectivity radius. Assume

- the conformal infinity of g_∞^+ is the Euclidean space $\mathbb{R}^3$;
- $R[g_\infty] = 0$ and $|\nabla \rho_\infty|_{g_\infty} \leqslant 1$
- there $\delta > 0$ s.t. $|Ric[g_\infty](x)| = o(\frac{1}{d(x,p)^\delta})$

Then g_∞^+ is the hyperbolic space and g_∞ is the half euclidean space.

§5. An Existence Result

Recall Graham-Lee '91: Any h in a small smooth neighborhood of h_c on $\mathbb{S}^3$ allows a CCE fill in, which are in a small nbhd of the hyperbolic metric on B^4, thus has the small L^2 norm of its Weyl tensor.

On the other hand, we also know the following result;

Lemma: When $n = 3$, on a CCE manifold (X^4, M^3, g^+) if $Y(M, [h]) > 0$, and

$$(*) \int_X |W|^2_{g^+} dv_{g^+} \leqslant cY_a^2$$

for some $c \leqslant \frac{1}{12^2}$, then any metric in some small nbhd of h allows a (unique) CCE fill in.

The natural question we then ask is can one impose conditions on the boundary metric h which will ensure $(*)$ to happen? As an application of our compactness result, we partially answer the question above.

§5. Statement of an existence result

Theorem: (Existence Result)
Let (B^4, S^3) and h be a metric on S^3 with the positive scalar curvature. Assume that h is in C^6 and denote h_c the canonical metric on S^3. Then there exists some (explicit) positive constants a such that

$$\|h - h_c\|_{C^2} + \||h - h_c\|_{W^{4,3/2}} \leqslant a \qquad (1)$$

then we can find a CCE filling-in metric with the conformal infinity $[h]$ which satisfies the (*) condition. Moreover, such solution with the above bound is unique.
Remarks: 1. The size of the constant a is around $\frac{45}{88^3\pi^2}$.
2. Condition (1) can be replaced by

$$||h||_{C^4}||E(h)||_2 \leqslant a' \qquad (2)$$

for some other constant a'.

§6. Outline of proof of the existence theorem

The strategy of proof is as follows: Denote $g = g^*$, and $S = g^{(3)}$ the non-local term, under assumptions of the theorem, apply the Bach equation, we have

$$(\frac{Y_a}{2}-\sqrt{3}\|W\|_2-\frac{2}{\sqrt{3}}\|E\|_2)(\|E\|_4^2+\|W\|_4^2) \leqslant -12\oint\langle \hat{E}, S\rangle+\frac{1}{Y_b}\|\hat{\nabla}\hat{R}\|^2_{L^{3/2}(M)}$$

<u>Step 2</u>: Under the assumption

$(**)$ $\quad(\frac{5}{18}Y_a - c_0(||W||_2 + ||E||_2) > 0)$, we have

$$Y_b(X, M, [g])\|S\|_3 \leqslant 10\|\hat{\nabla}\hat{C}\|_{\frac{3}{2}}.$$

(This is the hard step, which we will supplement later.)

Combine step (1) and (2) we have if

$$(\frac{120}{Y_a}\|\hat{E}\|_{\frac{3}{2}}\|\hat{\nabla}\hat{C}\|_{\frac{3}{2}} + \frac{1}{Y_a}\|\hat{\nabla}\hat{R}\|^2_{L^{\frac{3}{2}}(M)})(vol(h))^{2/3} \leqslant \frac{43Y_a^3}{5\times(88)^3} \qquad (3)$$

then under $(**)$, we have $c_0(||W||_2 + ||E||_2) \leqslant \frac{1}{4}Y_a$.

§6. Outline of proof of the existence theorem

Step 3

Denote $\theta = h - h_c$ and $h_t = (1-t)h_c + th = 1 + t\theta$, under the assumption on the size of $\theta \leqslant a$, one check inequality (3) holds.. We now run a continuity argument connecting h to h_c in S^3. Note for metrics close to h_c, the fill in metric always exists and $||W||_2 + ||E||_2$ tends to zero so $(**)$ condition is always satisfied. Combining the three steps, along this path, under the assumptions of the Existence theorem, (**) is automatic and we reached the estimate in Step 2 and finished the proof of the theorem.

§6. More outline of proof of Step 2 –estimate of S

- Step 2

Estimate of S-tensor: Recall $S = \frac{\partial}{\partial n_g} Ric_g$. To estimate S, we first recall a fact which was used in the work of S. Bando, A. Kasue, H. Nakajima [BKN]'89 to derive ALE decay of sequence of Einstein metrics. In the special case of 4-manifold, if g^+ is an Einstein metric, denote W^+ the Weyl tensor of g^+, then there is a Kato inequality

$$|\nabla_{g^+} W^+|^2 \geqslant \frac{5}{3} |\nabla_{g^+} |W^+||^2$$

From these, one can derive

$$-\triangle_{g^+} |W^+|^{1/3} + \frac{1}{6} R_{g^+} |W^+|^{1/3} \leqslant c|W^+|^{4/3}. \qquad (4)$$

In work of [BKN], when scalar curvature $R_{g^+} = 0$, on a region $\int_A |W^+|^2_{g^+} dv_{g^+}$ is small, [BKN] derive the decay estimate

$$|W^+|^{1/3}(x) \lesssim \frac{1}{|x|^{2^-}} \text{ when } x \in A \text{ and } |x| \to \infty$$

Our Lemma is an application of (4) in conformal Einstein setting.

§6. More outline of proof of Step 2

Lemma 1 Let g^+ be CCE, $g = \rho^2 g^+$ be a compactification, define $U = U_g := \left(\frac{|W|_g}{\rho}\right)^{1/3}$, then

$$-\triangle_g U + \frac{1}{6} R_g U \leqslant c|W|_g U \qquad (5)$$

Lemma 2 Denote $\tilde{r}(x) = \mathrm{dist}_g(x, M)$, $x \in X$, $g = g^*$, then
$|W|_g^2 = e_2 \tilde{r}^2 + e_3 \tilde{r}^3 + O(\tilde{r}^4)$, where
$e_2 = 8|S|^2 + 4|\hat{C}|^2$, $\hat{C}$ is the Cotton tensor on M^3,
$e_3 = -4S_{\alpha\beta}(\hat{\nabla}_\gamma \hat{C}_{\alpha\beta\gamma} + \hat{\nabla}_\gamma \hat{C}_{\beta\alpha\gamma}) + 4H|S|^2 +$ some other lower order terms.

§5. More outline of proof of Step 2

Lemma 3

$$U_g^6 = \frac{|W|_g^2}{\rho_g^2} = \frac{|W|_g^2}{r^2}\frac{r^2}{\rho_g^2}$$

where

$$\rho_g = r - \frac{H}{18}r^2 + O(r^3) \tag{6}$$

$$U_g^6|_{\partial X} = e_2 \tag{7}$$

$$\frac{\partial U_g^6}{\partial r} = \frac{1}{9}He_2 + e_3 \tag{8}$$

§5. More outline of proof of Step 2

We then use the estimates

$$Y_a \left(\int_X U^{12} \right)^{1/2} \leqslant \int_X |\nabla U^3|^2 \tag{9}$$

$$Y_b \left(\oint_{\partial X} U^9 \right)^{1/3} \leqslant \int_X |\nabla U^3|^2 \tag{10}$$

while

$$\begin{aligned} \frac{5}{9} \int_X |\nabla U^3|^2 dv_g &= - \int_X (\triangle_g U) U^5 + \frac{1}{6} \oint_{\partial X} \frac{\partial U^6}{\partial r} \\ &\leqslant c \int_X |W|_g U^6 + \frac{1}{6} \oint_{\partial X} \frac{\partial U^6}{\partial r} \\ &\leqslant c \left(\int_X |W|_g^2 \right)^{1/2} \left(\int_X U^{12} \right)^{1/2} + \frac{1}{6} \oint_{\partial X} \frac{\partial U^6}{\partial r} \end{aligned}$$

§6. More outline of proof of Step 2

Combine (9) and (10) and estimate in (7) and (8) of U_g^6 and $\frac{\partial U_g^6}{\partial r}$ on ∂X, we get

$$\left(\frac{5}{18}Y_a - c\|W\|_2\right)\left(\int_X U^{12}\right)^{1/2} + Y_b\|S\|_3^2 \lesssim \int_X \left|S\hat{\nabla}\hat{C}\right|$$

Thus under the assumption

$$(**) \quad \frac{5}{18}Y_a - c\|W\|_2 > 0$$

we get

$$||S||_3 \leqslant C\|\hat{\nabla}\hat{C}\|_{\frac{3}{2}}$$

Open questions

(1). An open question is on (X^4, M^3, g^+) CCE setting, assuming $Y(M^3, [h]) \geqslant c > 0$, would that imply a L^2 bound of the $W[g^+]$?

(2). In a joint work of C.-Ge-Jin and Qing '23, On (X^{n+1}, M^n, g^+) CCE when $n > 3$, when n is odd, one can replace the role played of Bach tensor by the Obstruction Tensor of Hirachi-Graham '05 and obtained a perturbation compactness theorem when $Y(M^n, [h])$ is close to that of $Y(S^n, h_c)$. In the case when n is even, one needs to use other more complicated method to obtain the same result.

For general (X^{n+1}, M^n, g^+) CCE manifolds when $n > 3$, C-Ge '24 , under the assumption $Y(M, h_i) > c > 0$, h_i a compact family, one can verify for the scalar flat adapted metrics g_i will not have boundary blow up without the additional assumption that $|W|_{g_i} \in L^{\frac{n+1}{2}}$ being uniformly bounded. But so far we have not figured out some topological assumptions to prevent the possible interior blow up.

*C*ongratulations! Kengo, for your fantastic achievement!
May you have many more productive years to come!!

*T*hank you all for your attention !!