Recent developments on para-CR structures

Andreas Čap

University of Vienna
Faculty of Mathematics

Osaka Workshop on Conformal and CR Geometry
February 19, 2025

- I will report on recent results (partly unpublished) which originate in the thesis work of my students M. Wasilewicz and Z. Guo.
- I will focus on the versions for para-CR structures (also known as Legendrean contact structures). For some of these, analogs in CR geometry can be expected. Many of the results have extensions to larger classes of geometries, I'll only partly indicate those.
- The starting point for all the results are relative BGG sequences, so I will review some elements of those, again focusing on the case of para-CR structures.
- The results go in different directions. Some of them are available in arXiv:2405.13614, others in arXiv:2410.10410 and some of them are not published yet.

Contents

Recall that a *para-CR structure* or *Legendrean contact structure* on a smooth manifold M of odd dimension $2n+1$ is given by a contact structure $H \subset TM$ together with a decomposition $H = E \oplus F$ as a direct sum of Legendrean subbundles. So they have rank n and satisfy $[\Gamma(E), \Gamma(E)] \subset \Gamma(H)$ and likewise for F. After complexification, these look the same as partially integrable almost CR structures. Integrability in the CR setting corresponds to involutivity of the distributions $E, F \subset TM$.

There is a homogeneous model, the flag variety $F_{1,n+1}$ of lines in hyperplanes in $\mathbb{R}^{n+2}$. This fits into a double fibration as $\mathbb{R}P^{n+1} \leftarrow F_{1,n+1} \to \mathbb{R}P^{(n+1)*}$ and E and F are the two vertical bundles. Correspondingly, $F_{1,n+1} = SL(n+2, \mathbb{R})/Q$, $Q = P \cap \tilde{P}$.

Again, this is parallel to the homogeneous model $S^{2n+1} = SU(n+1,1)/\tilde{Q}$ of CR structures, but in the CR case, $\tilde{Q}$ is a maximal parabolic.

Relative BGG sequences were first used in the Baston-Eastwood book for the homogeneous model. They observed that the fiber of $F_{1,n+1} \to \mathbb{R}P^{n+1}$ is $P/Q \cong \mathbb{R}P^n$ and then used a "BGG resolution along the fibers". The geometric version of relative BGG's (Č.-Souček, 2017) took a quite different route, however.

- $(M, H = E \oplus F)$ can be equivalently encoded as a Cartan geometry $(\mathcal{G} \to M, \omega)$, where $\mathcal{G}$ has structure group Q.
- Hence $TM = \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$ and $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ corresponds to the subbundle $F \subset TM$.
- Similarly to tractor bundles, one defines *relative tractor bundles* as associated to representations of Q which are restrictions of completely reducible representations of P.
- Any relative tractor bundle $\mathcal{V}M$ carries a canonical *partial connection* $\nabla^{\mathcal{V}}$, which allows differentiation in F-directions only.

If F is involutive, one can simply couple $\nabla^{\mathcal{V}}$ to the exterior derivative to obtain $d^{\mathcal{V}} : \Omega^k_\rho(M, \mathcal{V}M) \to \Omega^{k+1}_\rho(M, \mathcal{V}M)$, (sections of $\Lambda^* F^* \otimes \mathcal{V}M$). For non-involutive F, a more complicated construction for an operator $d^{\mathcal{V}}$ is available.

These bundles are induced by the representations $\Lambda^*(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V}$ of Q and $(\mathfrak{p}/\mathfrak{q})^* \cong \mathfrak{q}_+/\mathfrak{p}_+$, the quotient of the nilradicals of the parabolics. This is a Lie algebra, which allows one to develop an analog of the algebraic input used in the BGG construction. In particular, one has the homology spaces $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ which are representations of Q with trivial action of $\mathfrak{q}_+$. Denote the associated bundles by $\mathcal{H}^{\mathcal{V}}_* M$.

The homology spaces can be described similarly as for usual BGG's via a subset $W^{\mathfrak{q}}_{\mathfrak{p}}$ of the Weyl group of $\mathfrak{g}$ and highest weights of irreducible components can be determined algorithmically. In the para-CR case, each $H_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ is irreducible.

As in the standard BGG machinery, one can "compress" the operators $d^{\mathcal{V}}$ on $\Omega^*_\rho(M, \mathcal{V}M)$ to a sequence of operators $D^{\mathcal{V}}$ acting on $\Gamma(\mathcal{H}^{\mathcal{V}}_* M)$. A nice feature is that both sequences are complexes and locally exact in many non-flat situations. In the para-CR case, this is always true if F is involutive. Two cases (depending on $\mathbb{V}$):

Case 1: Regular and integral infinitesimal character

This happens if the highest weight of $\mathbb{V}$ lies in the Weyl orbit of a dominant integral weight or equivalently if $\mathbb{V} \cong H_\ell(\mathfrak{p}_+, \tilde{\mathbb{V}})$ for a $\mathfrak{g}$-irreducible representation $\tilde{\mathbb{V}}$ and some ℓ. (P-irreducible quotient for $\ell = 0$.)

- All $\mathcal{H}^{\mathcal{V}}_k M$ occur in the the BGG sequence of $\tilde{\mathbb{V}}$. Varying k and $\mathbb{V}$ (via ℓ), they exhaust that BGG pattern.
- Relative BGG operators are components of BGG operators

Hence we get a simplified construction for some BGG operators and identifies subcomplexes in BGG sequences in this case.

Case 2: Singular or non-integral infinitesimal character

For all other choices of $\mathbb{V}$, one obtains bundles and operators that do not show up in any BGG sequence. So these construct "new" operators, which sometimes occur in families, e.g. with variable density weight.

If $(\Omega^*_\rho(M, \mathcal{V}M), d^{\mathcal{V}})$ is a complex (which requires involutivity of F), it is a fine resolution of the sheaf of local sections of $\mathcal{V}M$, which are "parallel" for the tractor connection $\nabla^{\mathcal{V}}$. Since this is just a partial connection, this locally is the pullback of a sheaf on a local leaf space for F. The relative BGG sequence then resolves the same sheaf.

In the works on relative BGGs it was shown that this situation occurs often if F is involutive, e.g. for any $\mathbb{V}$ in the para-CR case. However, beyond the case of correspondence spaces, it was not clear how to describe the resolved sheaves.

The question of the resolved sheaves and looking for analogs of tractor calculus in the relative setting motivated the following developments. Given a para-CR structure (M, E, F), one first looks for a "relative standard tractor bundle". We started experimenting with a quotient of the para-CR standard tractors, but it turns out that there is a much simpler choice.

Proposition

The bundle $TM/F \to M$ is a relative tractor bundle that can be realized as a quotient of the adjoint tractor bundle $\mathcal{A}M$. A choice of contact form induces an isomorphism $TM/F \cong E \oplus TM/H$.

- From this, on obtains essentially all relative tractor bundles by tensorial constructions.
- $\Lambda^{n+1}(TM/F)$ is a density bundle, which inherits a canonical partial (tractor) connection (also for powers and roots). This leads to the families with variable density weights.

For a para-CR structure, a contact form defines a Weyl-structure and hence Weyl connections on the (irreducible) bundles $E \cong H/F$ and TM/H. Using those, one can describe the relative tractor connection "in slots". Alternatively, one can use this as an ad-hoc definition of the relative tractor connection and verify invariance. The relative to adjoint tractors again leads to a simpler description.

Proposition

The relative tractor connection on TM/F is induced by the adjoint tractor connection. If F is involutive, it coincides with the *Bott connection*, i.e. one gets $\nabla^{\mathcal{T}}_{\xi}(\eta + F) = [\xi, \eta] + F$ for $\xi \in \Gamma(F)$ and $\eta \in \mathfrak{X}(M)$. If F is non-involutive, there a similar formula with a correction term involving torsion.

Hence for involutive F, the relative tractor connection and hence the whole relative BGG sequence is independent of the second Legendrean subbundle E!

Turning things around, one can extend the whole relative BGG construction to the situation of a contact manifold (M, H) endowed with a single involutive Legendrean distribution $F \subset H$. This is the main content of the thesis (in preparation) of M. Wasilewicz. (There also is an extension in the non-involutive case – one may drop the requirement that the complementary distribution E is Legendrean.)

The structure $(M, H \supset F)$ considered here is of infinite order and has no local invariants. Still one obtains many differential operators (which can have arbitrarily high order) that are intrinsic to the structure. So this is interesting from the point of view of the general "landscape" of geometric structures.

Similar results for (generalized) path geometries have been obtained in the thesis work of Z. Guo under my direction. In the joint article arXiv:2405.13614 with her and M. Wasilewicz, which has recently appeared in SIGMA, we have extended this to general parabolic geometries, obtaining

- a basic class of relative tractor bundles obtained from the adjoint tractor bundle
- relations of the relative tractor connection to the adjoint tractor connection and, assuming involutivity, to Bott connections
- precise (weak) conditions on the torsion of the geometry that are needed to ensure involutivity and a description of the sheaves resolved by relative BGG sequences as pullbacks from local leaf spaces

This returns to the point of view of "BGG sequences along the fibers" mentioned before. Let us start with the case of a contact manifold (M, H) endowed with an involutive Legendrean distribution $F \subset H$ (for example coming from a para-CR structure).

We saw that TM/F inherits a canonical partial connection (in F-directions), while a choice of contact form θ defines a splitting $TM/F \cong H/F \oplus TM/H$ (and a connection on TM/H). Fixing θ, we obtain a partial connection on H/F and since $F \cong (H/F)^* \otimes TM/F$, we also get a partial connection on F.

If F comes from a para-CR structure, this is just the restriction of the Webster-Tanaka connection determined by θ. The usual formulae for the change of the connections (in F-directions) under a change of contact forms shows that they are related by the obvious analog of a projective change. This extends to the more general setting.

One then shows that there is an analog of the Cartan description of projective structures for this projective class of partial connections on F. The canonical connection on TM/F turns out to be closely related to this description (there is a GL vs. SL issue here). Partial structures provide a vast generalization of that point of view.

Let M be a smooth manifold of dimension $n+k$ and let $F \subset TM$ be an involutive distribution of rank n. The frame bundle of F then is a principal bundle $p : P \to M$ with structure group $GL(n,R)$ that carries an analog $\theta \in \Gamma(L(\widehat{F},\mathbb{R}^n))$ where $\widehat{F} = Tp^{-1}(F) \subset TP$. Now on can consider reductions to a structure group $G_0 \subset GL(n,\mathbb{R})$ (which usually correspond to G_0-structures on n-manifolds). This defines a *partial G_0-structure* on (M,F) with analogous explicit description (for any k). The original G_0-structures occur as the special case $k=0$.

Similarly, for a Lie group G and closed subgroup $H \subset G$ such that $\dim(G/H) = n$, the concept of a Cartan geometry of type (G, H) on n-manifolds naturally extends to a *partial Cartan geometry* on (M, F) for any k. Here one needs a principal H-bundle $\mathcal{G} \to M$ and a partial Cartan connection $\omega \in \Gamma(L(\widehat{F}, \mathfrak{g}))$, which restricts to a linear isomorphism in each point.

In the joint article arXiv:2410.10410 with M. Wasilewicz, we show that for AHS-structures (induced by $|1|$-gradings of simple Lie algebras), there is a categorical equivalence between partial G_0-structures and normal partial Cartan geometries of type (G, P) in any codimension k. (Normality is defined as for AHS-structures.)

In particular, we obtain a Cartan description and hence e.g. tractor calculus associated to a conformal class of bundle metrics on involutive distributions of rank ≥ 3 on manifolds of any dimension. (Note that morphisms are not necessarily local diffeomorphism, even dimensions may be different.)